Hydrodynamics
Blasius solution, Drag force on a plate and boundary layer thickness

1. Blasius solution

The Prandtl equation for the boundary layer on a flat plate (no outer pressure gradient) in streamfunction formulation writes

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^3\psi}{\partial y^3} \tag{1}$$

Possible self-similar variables are $\eta = \frac{y}{\sqrt{x}}$ and $\psi(x,y) = \sqrt{x}f(\eta)$. We will use the derivative of a product and the composed derivative repeatedly. We note derivatives of f by f', f'' and f'''. We will need

$$\frac{\partial\eta}{\partial y} = \frac{1}{\sqrt{x}} \tag{2}$$

and

$$\frac{\partial\eta}{\partial x} = -\frac{y}{2x\sqrt{x}} = -\frac{\eta}{2x}. \tag{3}$$

Let us now express the x and y derivatives

$$\frac{\partial\psi}{\partial y} = \frac{\partial(\sqrt{x}f)}{\partial y} = \sqrt{x}\frac{\partial\eta}{\partial y}f' = f', \tag{4}$$

$$\frac{\partial\psi}{\partial x} = \frac{1}{2\sqrt{x}}f + \sqrt{x}\frac{\partial\eta}{\partial x}f' = \frac{f}{2\sqrt{x}} - \frac{y}{2x}f' = \frac{1}{2\sqrt{x}}(f - \eta f'), \tag{5}$$

as well as the cross derivative

$$\frac{\partial^2\psi}{\partial y\partial x} = \frac{\partial f'}{\partial x} = -f''\frac{y}{2x\sqrt{x}} = -\frac{1}{2x}\eta f''. \tag{6}$$

We can check our calculations by taking the y derivative of the x derivative

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial(\frac{1}{2\sqrt{x}}(f-\eta f'))}{\partial y} = \frac{1}{2\sqrt{x}}(\frac{f'}{\sqrt{x}} - \frac{f'}{\sqrt{x}} - \eta f''\frac{1}{\sqrt{x}}) = -\frac{1}{2x}\eta f'' \tag{7}$$

We also have

$$\frac{\partial^2\psi}{\partial y^2} = \frac{1}{\sqrt{x}}f'', \tag{8}$$

and

$$\frac{\partial^3\psi}{\partial y^3} = \frac{1}{x}f''', \tag{9}$$

so that

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} = -\frac{1}{2x}\eta f'' f', \tag{10}$$

and

$$-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = -\frac{1}{2\sqrt{x}}(f-\eta f')\frac{1}{\sqrt{x}}f''. \tag{11}$$

Prandtl equations (1) therefore become

$$-\frac{1}{2x}\eta f''f' - \frac{1}{2x}ff'' + \frac{1}{2x}\eta f''f' = \frac{1}{x}f''' \tag{12}$$

which simplifies into Blasius' equation

$$ff'' + 2f''' = 0. \tag{13}$$

The boundary conditions on $\psi(x,y) = \sqrt{x}f(\eta)$ in $y = 0$ and $y \to \infty$ translate into boundary conditions for $f(\eta)$. From this point of view, this choice is much better than the one I did in class.

$$f(0) = f'(0) = 0; f(\eta \to \infty) \to \eta. \tag{14}$$

This ODE (third order, 3 boundary conditions, OK!) has to be solved numerically. The solution for $u = f'$ is depicted in figure 1.

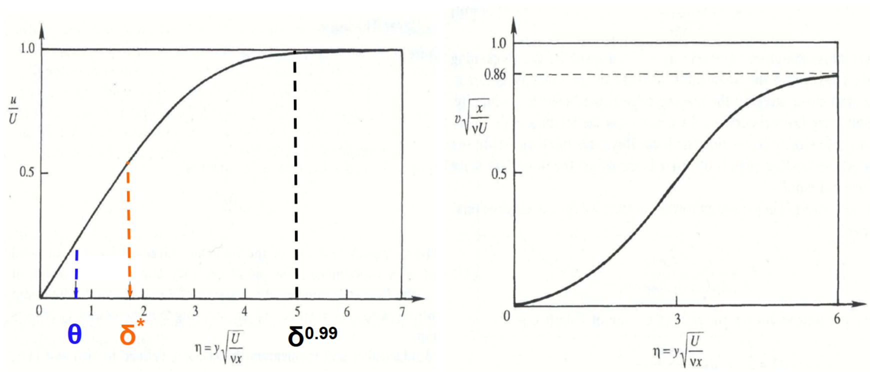

Figure 1: Self-similar solution $u = f'$ and $\sqrt{x}v = \frac{1}{2}(f - \eta f')$ of the Blasius equation.

2. Drag on a plate

We next calculate the shear force applied on a plate of length L and width W. This will be the drag exerted on the flat, perfectly streamlined, plate. The viscous force is actually not only the shear force, but can be more generally written

$$\tau_x = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{15}$$

where all quantities are dimensional. However, it is clear that the first term (the shear force) is ϵ^{-2} times larger than the second. Moving to dimensionless quantities, the total force (per unit length) on the plate therefore writes

$$f = \frac{\mu U_\infty L}{\delta}\int_0^1 \left.\frac{\partial \tilde{u}}{\partial \tilde{y}}\right|_{\tilde{y}=0} d\tilde{x}, \tag{16}$$

where δ is the characteristic boundary layer thickness $\delta = \frac{L}{\sqrt{Re_L}} = \sqrt{\frac{\rho\mu L}{U_\infty}}$. Since $\tilde{u} = \frac{\partial\tilde{\psi}}{\partial\tilde{y}} = f'$, $\frac{\partial\tilde{u}}{\partial\tilde{y}} = \tilde{x}^{-1/2}f''$. Given that $\tilde{y} = 0$ corresponds to $\eta = 0$ (the wall is also in 0 in terms fo sefl-similar variable η) and that the numerical solution shows $f''(0) = 0.332$, the force (per unit length) on the plate writes

$$f = \sqrt{\mu\rho U_\infty^3 L}0.332\int_0^1 \tilde{x}^{-1/2}d\tilde{x} = \sqrt{\mu\rho U_\infty^3 L}0.664\,\tilde{x}^{1/2}\Big|_0^1. \tag{17}$$

The total force on the plate is therefore

$$F = 0.664W\sqrt{\mu\rho U_\infty^3 L}. \tag{18}$$

It is common to recast the expression for the viscous shear stress τ_x by making it dimensionless with the typical pressure gauge $\frac{1}{2}\rho U_\infty^2$, such that

$$Cf_x = \sqrt{\frac{\mu\rho U_\infty^3}{L}}\frac{2}{\rho U_\infty^2}0.332\tilde{x}^{-1/2} = \sqrt{\frac{\mu}{LU_\infty\rho}}1.328\tilde{x}^{-1/2} = 0.664Re_x^{-1/2} \tag{19}$$

This shows that the dimensional and nondimensional shear stress diminishes along the plate as the boundary layer thickens.

One can also show that $F = 1.328Re^{-1/2}\frac{1}{2}\rho U_\infty^2 LW$ such that the dimensionless drag $CF_x = 1.328Re^{-1/2}$.

3. Boundary layer thickness

We have found the scaling of the boundary layer at any station x, $\delta = \frac{L}{\sqrt{Re_L}} = \sqrt{\frac{\rho\mu L}{U_\infty}}$. However this is only a scaling, not a definition of the thickness. There are actually at least three commonly used definitions:

- The 0.99 thickness, $\delta_{0.99}$, which is the distance from the wall at which the velocity in the boundary layer reaches 99% of its far-field value. Using the numerically found solution of the Blasius equation, one finds that $\delta_{0.99} \approx 5\delta$.

- The displacement thickness, δ^*, which is the distance above the plate at which a fictitious slippery wall should be placed for the same flux to flow uniformly above this plate than the flux flowing about the real no-slip wall. It is therefore given by the following integral relation

$$\int_0^\infty u dy = \int_{\delta^*}^\infty U_\infty dy, \tag{20}$$

therefore, adding $\int_0^{\delta^*} dy$ on both sides, one finds

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy. \tag{21}$$

With the Blasius solution, one finds $\delta^* \approx 1.73\delta$.

- The momentum thickness, Θ^*, which is the distance above the plate at which a fictitious slippery wall should be placed for the same momentum to flow uniformly above this plate than the mometum flowing about the real no-slip wall. Following a similar reasoning, one finds

$$\Theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \tag{22}$$

and $\Theta \approx 0.66\delta$.