

Hydrodynamics
Blasius solution

The Prandtl equation for the boundary layer on a flat plate (no outer pressure gradient) in streamfunction formulation writes

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^3 \psi}{\partial y^3} \quad (1)$$

Possible self-similar variables are $\eta = \frac{y}{\sqrt{x}}$ and $\psi(x, y) = \sqrt{x}f(\eta)$. We will use the derivative of a product and the composed derivative repeatedly. We note derivatives of f by f' , f'' and f''' . We will need

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{x}} \quad (2)$$

and

$$\frac{\partial \eta}{\partial x} = -\frac{y}{2x\sqrt{x}} = -\frac{\eta}{2x}. \quad (3)$$

Let us now express the x and y derivatives

$$\frac{\partial \psi}{\partial y} = \frac{\partial(\sqrt{x}f)}{\partial y} = \sqrt{x} \frac{\partial \eta}{\partial y} f' = f', \quad (4)$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{2\sqrt{x}}f + \sqrt{x} \frac{\partial \eta}{\partial x} f' = \frac{f}{2\sqrt{x}} - \frac{y}{2x}f' = \frac{1}{2\sqrt{x}}(f - \eta f'), \quad (5)$$

as well as the cross derivative

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial f'}{\partial x} = -f'' \frac{y}{2x\sqrt{x}} = -\frac{1}{2x}\eta f''. \quad (6)$$

We can check our calculations by taking the y derivative of the x derivative

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial(\frac{1}{2\sqrt{x}}(f - \eta f'))}{\partial y} = \frac{1}{2\sqrt{x}}\left(\frac{f'}{\sqrt{x}} - \frac{f'}{\sqrt{x}} - \eta f'' \frac{1}{\sqrt{x}}\right) = -\frac{1}{2x}\eta f'' \quad (7)$$

We also have

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{1}{\sqrt{x}}f'', \quad (8)$$

and

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{1}{x}f''', \quad (9)$$

so that

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{2x}\eta f'' f', \quad (10)$$

and

$$-\frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = -\frac{1}{2\sqrt{x}}(f - \eta f') \frac{1}{\sqrt{x}}f''. \quad (11)$$

Prandtl equations (1) therefore become

$$-\frac{1}{2x}\eta f''f' - \frac{1}{2x}ff'' + \frac{1}{2x}\eta f''f' = \frac{1}{x}f''' \quad (12)$$

which simplifies into Blasius' equation

$$ff'' + 2f''' = 0. \quad (13)$$

The boundary conditions on $\psi(x, y) = \sqrt{x}f(\eta)$ in $y = 0$ and $y \rightarrow \infty$ translate into boundary conditions for $f(\eta)$. From this point of view, this choice is much better than the one I did in class.

$$f(0) = f'(0) = 0; f(\eta \rightarrow \infty) \rightarrow \eta. \quad (14)$$

This ODE (third order, 3 boundary conditions, OK!) has to be solved numerically.