Hydrodynamics
Blasius solution

The Prandtl equation for the boundary layer on a flat plate (no outer pressure gradient) in streamfunction formulation writes

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^3\psi}{\partial y^3} \tag{1}$$

The self-similar variable is $\eta = \frac{y}{\sqrt{x}}$. My "mistake" was to look for a solution of the form $\psi(x,y) = yg(\eta)$ instead of a solution in the form $\psi(x,y) = \sqrt{x}f(\eta)$. I put "mistake" in quotes, because there is no mistake, since both forms are self-similar and $f = \eta g$, but the Blasius way $\psi(x,y) = \sqrt{x}f(\eta)$ gives a far more handy final ODE. We will use the derivative of a product and the composed derivative repeatedly. We note derivatives of f by f', f'' and f'''. We will need

$$\frac{\partial\eta}{\partial y} = \frac{1}{\sqrt{x}} \tag{2}$$

and

$$\frac{\partial\eta}{\partial x} = -\frac{y}{2x\sqrt{x}} = -\frac{\eta}{2x}. \tag{3}$$

Let us now express the x and y derivatives

$$\frac{\partial\psi}{\partial y} = \frac{\partial(\sqrt{x}f)}{\partial y} = \sqrt{x}\frac{\partial\eta}{\partial y}f' = f', \tag{4}$$

$$\frac{\partial\psi}{\partial x} = \frac{1}{2\sqrt{x}}f + \sqrt{x}\frac{\partial\eta}{\partial x}f' = \frac{f}{2\sqrt{x}} - \frac{y}{2x}f' = \frac{1}{2\sqrt{x}}(f - \eta f'), \tag{5}$$

as well as the cross derivative

$$\frac{\partial^2\psi}{\partial y\partial x} = \frac{\partial f'}{\partial x} = -f''\frac{y}{2x\sqrt{x}} = -\frac{1}{2x}\eta f''. \tag{6}$$

We can check our calculations by taking the y derivative of the x derivative

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial(\frac{1}{2\sqrt{x}}(f - \eta f'))}{\partial y} = \frac{1}{2\sqrt{x}}(\frac{f'}{\sqrt{x}} - \frac{f'}{\sqrt{x}} - \eta f''\frac{1}{\sqrt{x}}) = -\frac{1}{2x}\eta f'' \tag{7}$$

We also have

$$\frac{\partial^2\psi}{\partial y^2} = \frac{1}{\sqrt{x}}f'', \tag{8}$$

and

$$\frac{\partial^3\psi}{\partial y^3} = \frac{1}{x}f''', \tag{9}$$

so that

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} = -\frac{1}{2x}\eta f'' f', \tag{10}$$

and

$$-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = -\frac{1}{2\sqrt{x}}(f - \eta f')\frac{1}{\sqrt{x}}f''. \tag{11}$$

Prandtl equations (1) therefore become

$$-\frac{1}{2x}\eta f''f' - \frac{1}{2x}\eta f f'' + \frac{1}{2x}\eta f''f' = \frac{1}{x}f''' \tag{12}$$

which simplifies into Blasius' equation

$$\eta f f'' + 2f''' = 0. \tag{13}$$

The boundary conditions on $\psi(x,y) = \sqrt{x}f(\eta)$ in $y = 0$ and $y \to \infty$ translate into boundary conditions for $f(\eta)$. From this point of view, this choice is much better than the one I did in class.

$$f(0) = f'(0) = 0; f(\eta \to \infty) \to \eta. \tag{14}$$

This ODE (third order, 3 boundary conditions, OK!) has to be solved numerically.