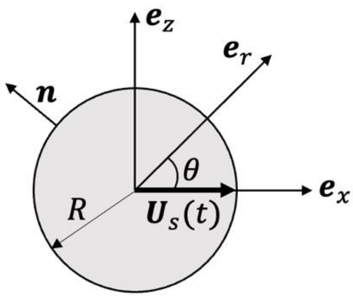
Exercises – Serie 3 – Added mass

Exercise 1

Consider a sphere moving with velocity $\boldsymbol{U}_s(t) = U_s(t)\boldsymbol{e}_x$ in a quiescent fluid. The fluid domain is unbounded.



In this case, the velocity potential is given by:

$$\phi = U_s(t)\frac{R^3}{2r^2}\cos\theta$$

a) Give the expression of the velocity field $\boldsymbol{u}(r,\theta,t)$
Potential flow is defined as an incompressible ($\nabla \cdot \boldsymbol{u} = 0$), irrotational ($\nabla \times \boldsymbol{u} = \boldsymbol{0}$) and inviscid ($\mu = 0$) flow.

Since the flow is irrotational we have $\boldsymbol{u} = \nabla\phi$.

$$\boldsymbol{u} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\frac{\partial\phi}{\partial r} = -U_s\frac{R^3}{r^3}\cos\theta \qquad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_s\frac{R^3}{2r^3}\sin\theta \qquad \frac{\partial\phi}{\partial\varphi} = 0$$

The velocity field is then:

$$\boldsymbol{u} = -U_s\frac{R^3}{r^3}\left(\cos\theta, \frac{1}{2}\sin\theta, 0\right)$$

b) Give the expression of the pressure field $p(r,\theta,t)$ using the unsteady Bernoulli equation $\frac{p}{\rho} + \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 = \mathrm{C}(t)$.

From the Bernoulli equation we have:

$$p = \mathrm{C}(t) - \rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right)$$

$$\frac{\partial\phi}{\partial t} = \dot{U}_s\frac{R^3}{2r^2}\cos\theta$$

$$|\nabla\phi|^2 = U_s^2\frac{R^6}{r^6}\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right)$$

The pressure field is therefore:

$$p(r,\theta,t) = C(t) - \rho\left[\dot{U}_s \frac{R^3}{2r^2}\cos\theta + U_s^2 \frac{R^6}{2r^6}\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right)\right]$$

Note: Bernoulli equation can be retrieved from the Navier-stokes equations. For an incompressible and inviscid flow, and neglecting gravity, we have:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla\boldsymbol{u}\right) = -\nabla p$$

$$\nabla \cdot \boldsymbol{u} = \boldsymbol{0}$$

Using the vector identity:

$$\boldsymbol{u}.\nabla\boldsymbol{u} = \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})$$

The Navier-Stokes equations can be rewritten as:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})\right) = -\nabla p$$

And since the flow is irrotational we have:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla|\boldsymbol{u}|^2\right) = -\nabla p$$

Then, using $\boldsymbol{u} = \nabla\phi$ it comes:

$$\rho\left(\frac{\partial \nabla\phi}{\partial t} + \frac{1}{2}\nabla|\nabla\phi|^2\right) = -\nabla p \Longrightarrow \nabla\left[p + \rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right)\right] = 0$$

And therefore, the quantity inside the gradient should be a function of time only, leading to the integration constant $C(t)$.

$$p + \rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right) = C(t)$$

c) Find the force $F_x(t) = \boldsymbol{F}(t) \cdot \boldsymbol{e}_x$ exerted by the flow on the sphere

Since the flow is considered inviscid, the stress tensor reads $\boldsymbol{\sigma} = -p\boldsymbol{I}$.
The force exerted by the flow on the sphere is then:

$$\boldsymbol{F}(t) = \iint_S \boldsymbol{\sigma}\boldsymbol{n}dS = -\iint_S p(R,\theta,t)\boldsymbol{n}dS$$

The x-component of the force is:

$$F_x(t) = \boldsymbol{F}(t) \cdot \boldsymbol{e}_x = -\iint_S p(R,\theta,t)\boldsymbol{n} \cdot \boldsymbol{e}_x dS$$

With

$$\boldsymbol{n} = \boldsymbol{e}_r = \sin\theta\cos\varphi\,\boldsymbol{e}_y + \sin\theta\sin\varphi\boldsymbol{e}_z + \cos\theta\,\boldsymbol{e}_x \Longrightarrow \boldsymbol{n} \cdot \boldsymbol{e}_x = \cos\theta$$

And

$$dS = R^2 \sin\theta \, d\theta d\varphi$$

$$F_x(t) = -\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} p(R,\theta,t)\cos\theta \, R^2 \sin\theta d\theta d\varphi = -2\pi R^2 \int_0^{\pi} p(R,\theta,t)\cos\theta\sin\theta d\theta$$

The pressure on the surface of the sphere $r = R$ is:

$$p(R,\theta,t) = C(t) - \rho\left[\dot{U}_s \frac{R}{2}\cos\theta + U_s^2 \frac{1}{2}\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right)\right]$$

The contribution of the constant $C(t)$ cancels in the integration since:

$$\int_0^{\pi} C(t)\cos\theta\sin\theta d\theta = C(t)\int_0^{\pi}\cos\theta\sin\theta d\theta = 0$$

Thus, we have:

$$F_x(t) = 2\pi R^2 \int_0^{\pi} \rho\left[\dot{U}_s \frac{R}{2}\cos\theta + U_s^2 \frac{1}{2}\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right)\right]\cos\theta\sin\theta d\theta$$

$$F_x(t) = 2\pi R^2 \rho\left[\dot{U}_s \frac{R}{2}\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta + U_s^2\frac{1}{2}\int_0^{\pi}\cos^3\theta\sin\theta\, d\theta + U_s^2\frac{1}{8}\int_0^{\pi}\sin^3\theta\cos\theta\, d\theta\right]$$

And since:

$$\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta = \frac{2}{3} \qquad \int_0^{\pi}\cos^3\theta\sin\theta\, d\theta = 0 \qquad \int_0^{\pi}\sin^3\theta\cos\theta\, d\theta = 0$$

The force is consequently:

$$F_x(t) = \rho\frac{2}{3}\pi R^3 \dot{U}_s$$

d) Deduce the expression of the added mass m_a?

$$m_a = \rho\frac{2}{3}\pi R^3 = \frac{1}{2}\rho V_s$$

The added mass in this case (unbounded domain) is one half the mass of fluid displaced by the sphere.

Hints:

The gradient operator in spherical coordinates is given by $\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$

The surface element on a sphere of radius R in spherical coordinates is $dS = R^2\sin\theta\, d\theta d\varphi$

Exercise 2

Consider a sphere of radius $R = 5\ cm$ made of steel (ρ_{steel} = $8000\ kg/m^3$). The sphere is supported by a spring with constant $K = 200\ N/m$ and excited by a sinusoidal force of amplitude A which forces the sphere to oscillate around its equilibrium position at $x = 0$. The sphere is then placed in a water tank and the same force is applied to it.

a) Find the ratio of the natural frequencies of both systems.
b) We replace the ball with one made of POM plastic (ρ_{POM} = $1410\ kg/m^3$). Recompute the ratio of the natural frequencies.

(Note: We ignore the viscous damping, this approximation is acceptable for small amplitude oscillations in which inertial terms (accelerations) dominate.

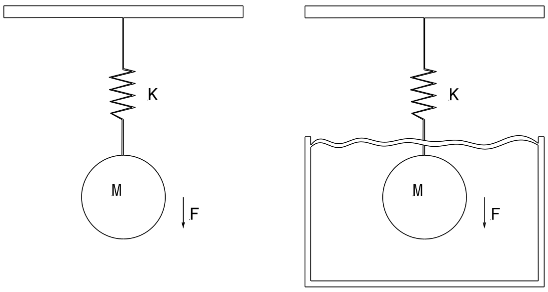


Figure 1: Spring-mass system in the two configurations.

In the first case, the system oscillates in air (a medium where added mass effects can be neglected $\rho_{air} \ll \rho_{steel}$). The equation of motion then reads:

$$m\ddot{x} + kx = A\,sin(\omega t) + F_{buoyancy} - mg$$

And the natural frequency of the system is therefore given by:

$$\omega_{0,1} = \sqrt{\frac{k}{m}}$$

In the second case, however, added mass effects cannot be neglected. From the previous exercise, we have found the expression of the force exerted by the fluid on a accelerated sphere:

$$F_f = \rho\frac{2}{3}\pi R^3\ddot{x} = m_a\ddot{x}$$

The equation of motion therefore now reads:

$$m\ddot{x} + kx = A\sin(\omega t) + F_{buoyancy} - mg - F_f$$

$$m\ddot{x} + kx = A\sin(\omega t) + F_{buoyancy} - mg - m_a\ddot{x}$$

$$(m + m_a)\ddot{x} + kx = A\sin(\omega t) + F_{buoyancy} - mg$$

And the natural frequency of the second system is given by:

$$\omega_{0,2} = \sqrt{\frac{k}{m + m_A}}$$

The ratio of the two frequencies is finally given by:

$$r = \frac{\omega_{0,1}}{\omega_{0,2}} = \sqrt{\frac{m + m_A}{m}} = \sqrt{\frac{{}^4/_3\,\rho_s + {}^2/_3\,\rho_f}{{}^4/_3\,\rho_s}} = \sqrt{1 + \frac{\rho_f}{2\rho_s}}$$

For a ball made of steel we find $r = 1.03$ and for a ball made of POM we find $r = 1.16$.

Exercise 3

If a 3D body has a characteristic length in one direction that is considerably longer than its length in the other two directions, the *slender body* approximation can be used to formulate the added mass associated with its motion. The idea behind this is to consider the body as a longitudinal stack of thin sections, each having an easily computed added mass, and to integrate the effects of those sections along the length of the body to find the total added mass (c.f. Figure 2 and Figure 3).

By using this approximation, it is possible to estimate the added mass tensor for a 6 degrees of freedom body, m_{ij} , with i, j = $1, 2, 3, 4, 5, 6$, moving in a fluid as follow:

$$\vec{U}(t) = (U_1, U_2, U_3, \omega_1, \omega_2, \omega_3) = (U_1, U_2, U_3, U_4, U_5, U_6)$$

A good way to think about those components m_{ij} is to consider them as the mass associated with a force/moment in the i^{th} direction due to a unit acceleration in the j^{th} direction. Note, however, that the added mass coefficients related to the x_1 axis (c.f. Figure 2) cannot be obtained with the slender body approach. Additionally, the tensor is symmetric: $m_{ij} = m_{ji}$. Given the symmetries of the 3D body depicted on Figure 2, the added mass tensor is given by:

$$\boldsymbol{m} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{pmatrix} = \begin{pmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{66} \end{pmatrix}$$

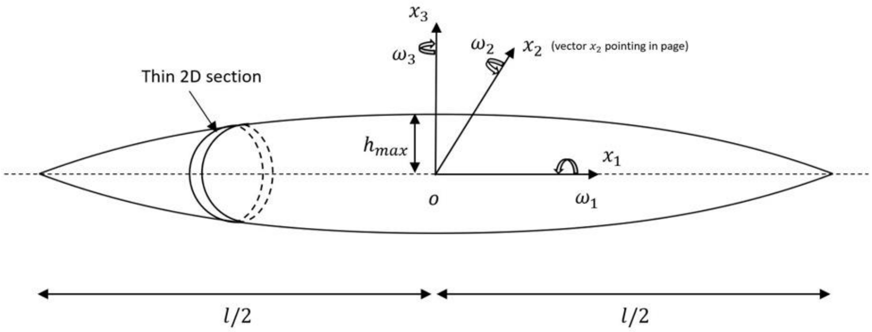
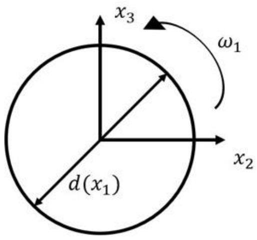


Figure 2: Slender-body approximation. 3D body made of circular 2D sections.



$$a_{22} = a_{33} = \rho\pi\left(\frac{d(x_1)}{2}\right)^2$$

$$a_{44} = 0$$

Figure 3: Added mass coefficients of a 2D circular section.

a) Assuming a solid with the shape $d(x_1) = h_{max}\left(1 - \frac{x_1^2}{l^2/4}\right)$. Calculate the different added masses as a function of l and h_{max}.

$$m_{22} = \int_{-\frac{l}{2}}^{\frac{l}{2}} a_{22} dx_1 \quad m_{33} = \int_{-\frac{l}{2}}^{\frac{l}{2}} a_{33} dx_1 \quad m_{44} = \int_{-\frac{l}{2}}^{\frac{l}{2}} a_{44} dx_1$$

$$m_{55} = \int_{-\frac{l}{2}}^{\frac{l}{2}} x_1^2 a_{33} dx_1 \quad m_{66} = \int_{-\frac{l}{2}}^{\frac{l}{2}} x_1^2 a_{22} dx_1$$

The 2D added mass coefficients are given by:

$$a_{22} = a_{33} = \rho\pi\left(\frac{d(x_1)}{2}\right)^2$$

$$a_{44} = 0$$

The different added masses are:

$$m_{22} = \int_{-l/2}^{l/2} \frac{\rho\pi}{4} d(x_1)^2 dx_1 = \frac{\rho\pi}{4} h_{max}^2 \int_{-l/2}^{l/2} \left(1 - \frac{x_1^2}{l^2/4}\right)^2 dx_1$$

$$= \frac{\rho\pi}{4} h_{max}^2 \int_{-l/2}^{l/2} \left(1 - \frac{2x_1^2}{l^2/4} + \frac{x_1^4}{l^4/16}\right) dx_1 = \frac{2}{15}\rho\pi h_{max}^2 l$$

$$m_{33} = m_{22}$$

$$m_{44} = 0$$

$$m_{55} = \int_{-l/2}^{l/2} x_1^2 \frac{\rho\pi}{4} d(x_1)^2 dx_1 = \frac{\rho\pi}{4} h_{max}^2 \int_{-l/2}^{l/2} x_1^2 \left(1 - \frac{x_1^2}{l^2/4}\right)^2 dx_1$$

$$= \frac{\rho\pi}{4} h_{max}^2 \int_{-l/2}^{l/2} \left(x_1^2 - \frac{2x_1^4}{l^2/4} + \frac{x_1^6}{l^4/16}\right) dx_1 = \frac{1}{210}\rho\pi h_{max}^2 l^3$$

$$m_{66} = m_{55}$$

The effects of added mass on a body can be represented by forces and moments acting on it. In the case of a body evolving in an unbounded and inviscid fluid, they can be evaluated as follow:

$$F_j = -\dot{U}_i m_{ij} - \epsilon_{jkl} U_i \omega_k m_{li}$$

$$M_j = -\dot{U}_i m_{j+3,i} - \epsilon_{jkl} U_i \omega_k m_{l+3,i} - \epsilon_{jkl} U_k U_i m_{li}$$

where $i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$ and ϵ is the Levi-Cavita symbol with:

$$\epsilon_{jkl} = \begin{cases} 0 & if\ any\ j,k,l\ are\ equal \\ 1 & if\ j,k,l\ are\ in\ cyclic\ order \\ -1 & if\ j,k,l\ are\ in\ anti-cyclic\ order \end{cases}$$

Consider the 3D object moving in the fluid as follow:

$$\vec{U} = (U_1, U_2, U_3, U_4, U_5, U_6) = (U\cos(\alpha), 0, -U\sin(\alpha), 0,0,0)$$
$$\dot{\vec{U}} = \left(\dot{U}\,cos(\alpha), 0, -\dot{U}\,sin(\alpha), 0,0,0\right)$$

b) Calculate the forces and the moments in terms of m_{ij}.

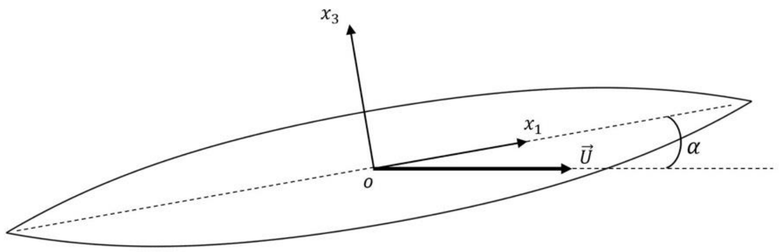


Figure 4: UAV moving in an inviscid and unbounded fluid.

For the forces:

$$U_4 = U_5 = U_6 = 0 \quad \Leftrightarrow \quad \omega_1 = \omega_2 = \omega_3 = 0$$
$$F_j = -\dot{U}_i m_{ij} - \epsilon_{jkl} U_i \omega_k m_{li} = -\dot{U}_i m_{ij}$$

So we have:

$$F_1 = -\dot{U}\cos(\alpha)\, m_{11}$$
$$F_2 = 0$$
$$F_3 = \dot{U}\sin(\alpha)\, m_{33}$$

For the moments, we can make the same simplifications as for the forces:

$$M_j = -\dot{U}_i m_{j+3,i} - \epsilon_{jkl} U_i \omega_k m_{l+3,i} - \epsilon_{jkl} U_k U_i m_{li} = -\dot{U}_i m_{j+3,i} - \epsilon_{jkl} U_k U_i m_{li}$$

Which yields:

$$M_1 = 0$$
$$M_2 = -U^2 \cos(\alpha)\sin(\alpha)\,(m_{33} - m_{11})$$
$$M_3 = 0$$

The moment M_2 is known as the Munk moment. It arises from the asymmetric location of the stagnation points, where the flow decelerates at the front of the body (the pressure increases) and accelerates at the back (the pressure decreases). This moment has a destabilizing effect on the body as it tends to turn it perpendicular to the flow. The Munk moment only appears in its full form in the inviscid fluid case. In the viscous case, the flow around the body is modified and so is the value of the moment.

c) Given $l = 1\,m, h_{max} = 0.1\,m, U = 5\,m/s, \dot{U} = 1\,m/s^2$ and $\alpha = 5°$, compute the various forces and moments derived in (b). The added mass coefficient m_{11} can be approximated in this specific case as follow:

$$m_{11} \approx 0.057 \frac{4}{3}\pi\rho\frac{l}{2}h_{max}^2$$

Given the numerical data and considering that the body is moving in water, we have:

$$m_{11} = 1.194\ kg$$

$$m_{33} = 4.189\ kg$$

And therefore,

$$F_1 = -1.189\ N$$

$$F_3 = 0.365\ N$$

$$M_2 = -6.501\ Nm$$

d) What becomes the moment M_2 if we consider a sphere instead of the body depicted on Figure 2?

If we consider a sphere, we have $m_{11} = m_{33}$ (due to the symmetry of the body). In that case, $M_2 = 0$.