Problem 1. Communication game

Consider a communication game with N players. Each player $i \in \{1, 2, \dots, N\}$, chooses its transmission power $x^i \in [0, \bar{x}^i] \subset \mathbb{R}$. Let x be the concatenation of all players' strategies. The cost each player has to pay for x^i is $J^i(x) = \lambda(x^i) - \log\left(1 + \gamma^i(x)\right)$, with $\gamma^i(x) = \frac{h^i x^i}{1+\sum_{j\neq i} h^j x^j}$, $h^i > 0$. Here, the first term is the price for transmitting the power and the second term is the signal to interference noise ratio (SINR) of the transmission power.

(a) Show that the game is potential by deriving an exact potential function for this game.

Solution. Note that

$$J^i(x^i, x^{-i}) - J^i(y^i, x^{-i}) = \lambda^i(x^i) - \lambda(y^i) - \log(\frac{1 + \gamma^i(x^i, x^{-i})}{1 + \gamma^i(y^i, x^{-i})})$$
$$= \lambda^i(x^i) - \lambda(y^i) - \log(\frac{1 + \sum_j h^i x^j}{1 + \sum_{j \neq i} h^i x^i + h^i y^i})$$
$$= \lambda^i(x^i) - \log(1 + \sum_i h^i x^i) - (\lambda^i(y^i) - \log(1 + \sum_{j \neq i} h^i x^i + h^i y^i)).$$

So, by defining $P(x) = \sum_{i=1}^N \lambda^i(x^i) - \log(1 + \sum_{i=1}^N h^i x^i)$ we get the desired result.

(b) Let $\lambda(x) = 0$, $\forall x \in \mathbb{R}$. Does a Nash equilibrium exist? Design the pricing scheme $\lambda : \mathbb{R} \to \mathbb{R}$ to ensure a unique Nash equilibrium. Justify your answer.

Solution. First, the strategy sets $[0, \bar{x}^i] \subset \mathbb{R}$ are compact convex and the cost function J^i is continuous in all variables. To ensure existence of the Nash equilibrium, we need the cost function J^i to be convex in x^i. Since J^i is continuously differentiable in x^i, it is convex if its second derivative with respect to x^i is non-negative. The first derivative of $J^i(x)$ with respect to x^i is

$$\nabla_{x^i} J^i(x) = \lambda'(x^i) - \frac{1}{1 + \gamma^i(x)} \frac{h^i}{1+\sum_{j\neq i} h^j x^j}$$
$$= \lambda'(x^i) - \frac{h^i}{1 + S}, \quad S = \sum_{j=1}^N h^j x^j,$$

The second derivative of J^i with respect to x^i is given by

$$\nabla^2_{x^i} J^i(x) = \lambda''(x^i) + \frac{(h^i)^2}{(1 + S)^2}.$$

It can be seen that for any convex λ a Nash equilibrium exists.

To ensure uniqueness of the Nash equilibrium a sufficient condition is strict monotonicity of the game mapping. The game mapping is

$$F(x) = (\nabla_{x^1}\theta^1(x); \nabla_{x^2}\theta^2(x); \dots; \nabla_{x^N}\theta^N(x)) \in \mathbb{R}^N.$$

Since F is continuously differentiable it is strictly monotone if and only if $JF(x) \in \mathbb{R}^{N\times N}$ is positive definite.

$$JF(x) = \begin{pmatrix} \lambda''(x^1) & 0 & \cdots & 0 \\ 0 & \lambda''(x^2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda''(x^N) \end{pmatrix} + \frac{1}{(1 + S)^2} \begin{pmatrix} h^1h^1 & h^1h^2 & \cdots & h^1h^N \\ h^2h^1 & h^2h^2 & \cdots & h^2h^N \\ \vdots & \vdots & \ddots & \vdots \\ h^Nh^1 & h^Nh^2 & \cdots & h^Nh^N \end{pmatrix}.$$

The second matrix above is symmetric and thus positive semi-definite. To ensure positive definiteness of JF we need λ to be a strictly convex function.

(c) Consider the game above with $N = 2$. Let $\lambda(x) = \frac{c}{2}x^2$, where $c > \max\{h^1, h^2\}$. Each player wants to compute her Nash equilibrium strategy $x^i \in [0, \bar{x}^i]$ but only has local information. In particular, she knows her parameters $\bar{x}^i, h^i$ but not those of the other player. At each time instance t, players simultaneously can choose a transmission power $x_t = [x_t^1, x_t^2]$ and each player can then measure her SINR γ_t^i. Derive an iterative algorithm for each player, $x_{t+1}^i = g^i(x_t^i, \gamma_t^i)$ using only her local information that converges to a Nash equilibrium. Provide sufficient conditions for convergence of your algorithm.

Hint 1: Given $C \subset \mathbb{R}^2$ closed and bounded, a contractive operator $f_1 : C \to C$, and a non-expansive operator $f_2 : C \to C$, then the composition $f_2 \circ f_1 : C \to C$ is contractive.

Hint 2: Let $f : C \to C$, where $C \subset \mathbb{R}^2$ is closed and bounded. Let the Jacobian of f be

$$Jf(x) := \left(\frac{\partial f_i}{\partial x_j}\right)(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$

If $\max\{|a(x)| + |b(x)|, |c(x)| + |d(x)|\} < 1$, $\forall x \in C$, then f is contractive in C.

Solution. Since λ is strictly convex, this game has a unique Nash equilibrium.

Consider the skewed projection algorithm $x_{t+1} = \Pi_{K,A}(x_t - A^{-1}F(x_t))$, where $K = [0, \bar{x}^1] \times [0, \bar{x}^2]$ and $A^{-1} = \text{diag}[\alpha^1, \alpha^2] \in \mathbb{R}^{2\times 2}$, with $\alpha^i > 0$, for $i = 1, 2$. Then, each row of this algorithm can be written as

$$x_{t+1}^i = \Pi_{K^i, 1/\alpha^i}\left(x_t^i - \alpha^i \nabla_{x^i} J^i(x_t)\right).$$

This holds because for $A \in \mathcal{S}_{++}^n$ and $K \subset \mathbb{R}^n$ a closed convex set, $\Pi_{K,A} : \mathbb{R}^n \to K$ is defined as

$$\begin{aligned} \Pi_{K,A}(x) := \arg\min_y \quad & \frac{1}{2}(y - x)^T A (y - x) \\ \text{s.t.} \quad & y \in K. \end{aligned}$$

Now, the objective above for diagonal A decouples as $\frac{1}{2}\sum_{i=1}^N (y^i - x^i)^T \alpha^i (y^i - x^i)$ and the constraint sets are also decoupled $K = K^1 \times K^2 \cdots \times K^N$. Hence, the above can be written as N decoupled optimization problems: Furthermore, notice that in this case $\Pi_{K^i,1/\alpha^i}\left(x_t^i - \alpha^i \nabla_{x^i} J^i(x_t)\right) = \Pi_{K^i}\left(x_t^i - \alpha^i \nabla_{x^i} J^i(x_t)\right)$.

To ensure convergence to the Nash equilibrium, then each player needs to choose her corresponding α^i so that the mapping $f : x \mapsto x - A^{-1}F(x)$ is a contraction. Then, since projection is a non-expansive operator, the mapping $g : x \mapsto \Pi_{K,A}(x_t - A^{-1}F(x_t))$ is also a contraction. Furthermore, we know that the fixed point of the operator g, given as $x = \Pi_{K,A}(x - A^{-1}F(x))$, is the Nash equilibrium of the game. Hence, by Banach fixed point theorem, the algorithm converges to the fixed point and consequently to the unique Nash equilibrium.

Notice that $\nabla_{x^i} J^i(x) = cx^i - \frac{h^i}{1+S}$, where $S = \sum_{j=1}^2 h^j x^j$ So, each player needs to be able to compute S to implement the algorithm. From the local data h^i, x^i, and the SINR γ^i, we have $S = \frac{h^i x_t^i}{\gamma_t^i} - 1 + h^i x_t^i$.

Next, the parameter α^i needs to be chosen such that f is contractive. A sufficient condition for f being a contraction is that an induced norm of $Jf(x)$ be strictly less than one $x \in K$. Notice that

$$Jf(x) = \begin{pmatrix} 1 - \alpha^1(c + \frac{(h^1)^2}{(1+S)^2}) & -\alpha^1(\frac{h^1 h^2}{(1+S)^2}) \\ -\alpha^2(\frac{h^1 h^2}{(1+S)^2}) & 1 - \alpha^2(c + \frac{(h^2)^2}{(1+S)^2}) \end{pmatrix}$$

Consider the induced 1-norm $\|Jf(x)\|_1$. It is the maximum row sum of $|Jf(x)|$. Now, by ensuring that for all $x \in K$

$$\begin{aligned} |1 - \alpha^1(c + \frac{(h^1)^2}{(1+S)^2})| + |-\alpha^1(\frac{h^1 h^2}{(1+S)^2})| &< 1 \\ |1 - \alpha^2(c + \frac{(h^2)^2}{(1+S)^2})| + |-\alpha^2(\frac{h^2 h^2}{(1+S)^2})| &< 1, \end{aligned}$$

we can ensure $\|Jf(x)\|_1 < 1$ and hence f is a contraction in the 1-norm. We consider two cases for the sign of the first argument of the absolute value. If the argument is positive, the conditions for convergence is satisfied by using the fact that $c > \bar{h}$. If the argument is negative, then we use the fact that $S > 0$ and is increasing with x, together with $0 < h^i < \bar{h}$, and we find a sufficient condition on α^i as follows: $0 < \alpha^i < \frac{2}{\bar{h}^2+c}$.

(d) Consider a game Γ with N players. Player i's strategy space is $K^i \subset \mathbb{R}^n$ and her cost function is $J^i : \mathbb{R}^N \to \mathbb{R}$. Let x be the concatenation of all players' strategies in $K = K^1 \times \cdots \times K^N$. A game is an *exact potential game* if there exists a function $P : \mathbb{R}^N \to \mathbb{R}$ for which, for all $x \in K$, for any player i,

$$J^i(x^i, x^{-i}) - J^i(y^i, x^{-i}) = P(x^i, x^{-i}) - P(y^i, x^{-i}), \ \forall y^i \in K^i.$$

i) Show that if $x \in K$ minimises P then it is a Nash equilibriumof the game Γ.
ii) Under which conditions is the converse statement also true?

Solution. i) Consider a minimiser $x \in K$ of P. Then,

$$P(x^i, x^{-i}) \leq P(y^i, y^{-i}), \ \forall y^i \in K^i, y^{-i} \in K^{-i}.$$

Given i, let $y^{-i} = x^{-i}$. Then, we have

$$P(x^i, x^{-i}) \leq P(y^i, x^{-i}), \ \forall y^i \in K^i.$$

From definition of exact potential game, it follows that for every i we have

$$J^i(x^i, x^{-i}) - J^i(y^i, x^{-i}) \leq 0, \ \forall y^i \in K^i.$$

Hence, x is a Nash equilibrium of the game.

ii) The converse statement is in general not true. However, for a game with a convex potential function, we have that $\frac{\partial P}{\partial x} = F_\Gamma$, where F_Γ is the game map, and an $x^\star$ is an optimizer of $P(.)$ if and only if $(y-x)^T F_\Gamma(x) \ \geq 0$ for all $y \in K$. This is equivalent condition for being the Nash equilibrium of the game .

Problem 2. Finite horizon Cournot game

Consider an extension of the Cournot game that we discussed in class as follows. We have $N = 10$ players, which in this case are consumers rather than producers. The decision space of each player is denoted by

$$K^i = \{x \in \mathbb{R}^3 \,|\, 0 \leq x \leq 1, \mathbf{1}^T x = 1\},$$

where $\mathbf{1} \in \mathbb{R}^3$ is the unit vector. The interpretation of the inequalities in the constraint set above is that the consumption of each player has a minimum of 0 and a maximum of 1 at each time instance $t = 1, 2, 3$ and the total consumption over the three time periods should also not exceed 1. The cost function of the player i' is given by

$$J^i(x) = p\left(\sum_{j=1}^{N} x^j\right)^T x^i,$$

where $p : \mathbb{R}^3 \to \mathbb{R}^3$ is a price function. Consider the case in which $p(x) = x + c$, with $c \in \mathbb{R}^3$.

a) Derive the game map. (1 point)

Solution:
The game map is

$$\begin{bmatrix} \nabla_{x^1} J^1(x) \\ \dots \\ \nabla_{x^N} J^N(x) \end{bmatrix} = M(x),$$

where the the term $\nabla_{x^i} J^i(x)$ is given by

$$\nabla_{x^i} J^i(x) = \frac{\partial}{\partial x^i}\left[p\left(\sum_{j=1}^N x^j\right)^\top x^i\right] = \frac{\partial}{\partial x^i}\left[\left(\sum_{j=1}^N x^j + c\right)^\top x^i\right]$$
$$= x^i + \sum_{j=1}^N x^j + c = 2x^i + \sum_{j\neq i} x^j + c.$$

In the above, the first equality is given by definition of cost, the second one by definition of price and the last is algebra. Note that $\nabla_{x^i} J^i(x) : \mathbb{R}^{Nn} \to \mathbb{R}^n$, with $N = 10$ and $n = 3$.
Let's try to write it in a more compact form. The actions of player i are

$$x^i = \begin{bmatrix} x^i_1 \\ x^i_2 \\ x^i_3 \end{bmatrix} \in \mathcal{K}^i.$$

Then, for player 1 we can its gradient with respect to its actions as

$$\nabla_{x^1} J^1(x) = \begin{bmatrix} c^1 \\ c^2 \\ c^3 \end{bmatrix} + \left[\begin{array}{ccc|ccc|ccc|c|ccc} 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & \dots & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^N \end{bmatrix}$$
$$= c + \left(\begin{bmatrix} 2 & 1 & 1 & \dots & 1 \end{bmatrix} \otimes I_3\right) x.$$

We conclude that

$$\begin{bmatrix} \nabla_{x^1} J^1(x) \\ \dots \\ \nabla_{x^N} J^N(x) \end{bmatrix} = \mathbf{1}_N \otimes c + \left(\begin{bmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2 \end{bmatrix} \otimes I_3\right)(\mathbf{1}_N \otimes x).$$

Above. $\mathbf{1}_N$ is a vector containing N ones.
Notice that from symmetry, each player actions will be the same.

b) Show that there exists a unique Nash equilibrium for this game. (1 point)

Solution:
From the derivation of the game map above and Hint 1, we conclude that the game map is strictly monotone (see Exercise 5 in Lecture notes 'Convex games I').

c) Show that this is a potential game. (1 point)

Solution:
We notice that the game map derived in part (a) is $M(x) = Bx + \mathbf{1}_N \otimes c$, where $B \in \mathbb{R}^{3N\times 3N}$ is given in the hint. This game map has a Jacobian given by $JM(x) = B$. Since B is symmetric, there exists a single function whose gradient is $M(x)$. In this case, it is easy to see that $\Phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top B\mathbf{x} + (\mathbf{1}_N \otimes c)^\top x + \alpha$, where α is any arbitrary constant. This function $P(x)$ is the potential function of the game. In particular, note that the first- order optimality for finding $\min_{x\in\mathcal{K}} P(x)$ is exactly the same as the optimality criteria for finding games Nash equilibria.

d) Design a projection-based algorithm, $x_{t+1} = \Pi_K(x_t - \gamma F_\Gamma(x_t))$, as discussed in the lecture notes, that converges to the Nash equilibrium $x^\star \in K$. Plot the iterates $\|x_t - x^*\|_2$. (3 points)

Solution:
From the derivation of the game map in part **(a)** we know that if the step size satisfies $0 < \gamma < \frac{2\mu}{L^2}$, where μ is the strong monotonicity constant and L is the Lipschitz constant, then the algorith converges.
Now, our game map, denoted by $M : \mathbb{R}^{NT} \to \mathbb{R}^{NT}$, is affine. So

$$(M(x) - M(y))^\top (x - y) = (Bx - By)^\top (x - y) = (x - y)^\top B(x - y) \geq \lambda_{min}(B)\|x - y\|_2^2,$$

where $\lambda_{min}(B) = 1$.
Furthermore,

$$\|M(x) - M(y)\|_2^2 = (x - y)^\top B^\top B(x - y) \leq \lambda_{max}(B^\top B)\|x - y\|_2^2,$$

where $\lambda_{max}(B^\top B) = (N+1)^2$. Hence, $0 < \gamma < \frac{2\mu}{(N+1)^2}$. As N becomes larger, the maximum step size becomes smaller, and this can slow down the convergence rate.

Now, consider the welfare cost function $J(x) := \sum_{i=1}^{N} J^i(x)$.

e) Compute the solution of the optimization problem $\min_{x \in K} J(x)$ and denote the solution by x^o. (1 point)

Solution:
Consider

$$\frac{\partial}{\partial x^i}\left(J(x) := \sum_{h=1}^{N} J^h(x)\right) = \frac{\partial}{\partial x^i}\left(\sum_{h=1}^{N} p\left(\sum_{j=1}^{N} x^j\right)^\top x^h\right) = \frac{\partial}{\partial x^i}\left(\sum_{h=1}^{N}\left(\sum_{j=1}^{N} x^j + c\right)^\top x^h\right) = 2\sum_{j=1}^{N} x^j + c$$

In particular, the social welfare function is not the same as the potential function. Hence, it is not surprising that in general its optimizer will be different than the Nash equilibrium of the game, and that its optimal value will be different than its value achieved at the Nash equilibrium.

f) Compute the Price of Anarchy, PoA := $\frac{J(x^\star)}{J(x^o)}$. (1 point)

Solution:
For the computation of the price of anarchy, we need to pick a value $c \in \mathbb{R}^3$. Sample code is provided.

Repeat the previous step for a few increasing values of N, such as $N = 100, 300, 500, 800, 1000$.

g) Determine the range of step size γ, as a function of N to ensure the convergence of the algorithm. (1 point)

Solution:
See solution of point **d**.

h) Plot the price of anarchy as a function of N. Do you observe any pattern in the price of anarchy as a function of N? (1 point)

Solution:

You may use the following facts. Please note although not required for this assignment, you should be able to verify these facts.

For any $m \in \mathbb{N}$, let $I_m \in \mathbb{R}^{m \times m}$ denote the identity matrix.

Hint (eigenvalue identity 1). Consider a matrix $B \in \mathbb{R}^{TN \times TN}$ defined as

$$B = \begin{pmatrix} 2I_T & I_T & \cdots & I_T \\ I_T & 2I_T & \cdots & I_T \\ \vdots & \vdots & \ddots & \vdots \\ I_T & I_T & \cdots & 2I_T \end{pmatrix} = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \otimes I_T,$$

where there are N block columns in the first matrix above, $\otimes$ denotes the Kronecker matrix product and the second matrix lives in $\mathbb{R}^{N \times N}$. The matrix B has an eigenvalue at 1 with multiplicity $T(N-1)$ and an eigenvalue at $N+1$ with multiplicity T.

Hint (eigenvalue identity 2). Given $M \in \mathbb{R}^{m \times m}$, the set of eigenvalues of the matrix $I_m - M$ are $\{1 - \lambda_i(M)\}_{i=1}^{m}$ where $\lambda_i(M)$ correspond to the eigenvalues of M.