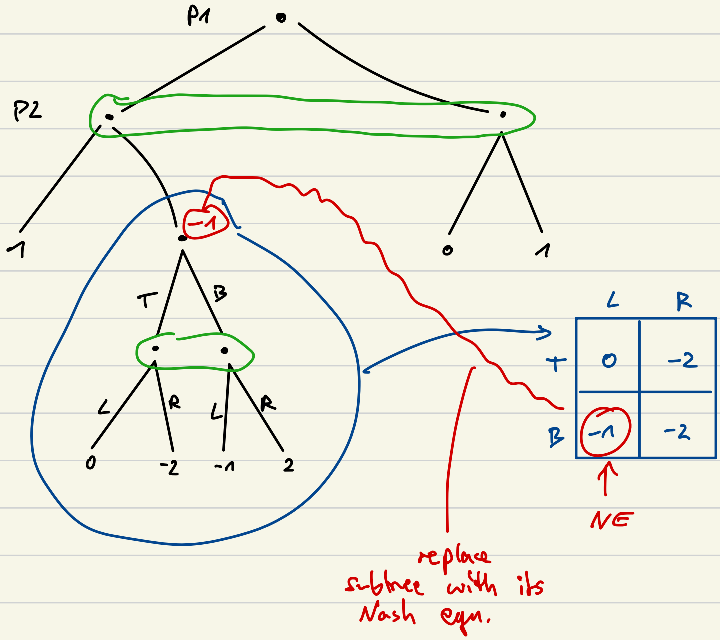
LECTURE 5

PART 1: EXTENSIVE FORM GAMES

Example for backward induction:

Consider the zero-sum game : (player 1 : min, player 2 : max)



	L	R
T	0	-2
B	-1	-2

We can conclude that the overall value of the game is 0.

Subgame Perfect Equilibria

Consider a feedback game (i.e. inf. sets don't span over stages) with K stages. Let I/J be the set of all inf. sets of player 1 / player 2.

Feedback game $\Rightarrow$ $I = \bigcup_{i=1}^{K} I_i$ where I_i are inf. set at stage i

Define $\gamma : I \to A$ with $\gamma = \{\gamma_1, \dots, \gamma_K\}$
where $\gamma_i : I_i \to A$

and similarly define $\sigma : J \to A$ for player 2.

A pair of strategies (γ^*, σ^*) is a subgame perfect eq. if for all stages $t = 1, \dots, K$ and every $\gamma_1, \gamma_2, \dots, \gamma_{t-1}$ and $\sigma_1, \sigma_2, \dots, \sigma_{t-1}$ prior to stage t, for all γ_t, σ_t, we have

$$J(\gamma_{1:t-1}, \gamma_t^*, \gamma_{t+1:K}^*, \sigma_{1:t-1}, \sigma_t, \sigma_{t+1:K}^*)$$

$$\leq J(\gamma_{1:t-1}, \gamma_t^*, \gamma_{t+1:K}^*, \sigma_{1:t-1}, \sigma_t^*, \sigma_{t+1:K}^*)$$

$$\leq J(\gamma_{1:t-1}, \gamma_t, \gamma_{t+1:K}^*, \sigma_{1:t-1}, \sigma_t^*, \sigma_{t+1:K}^*)$$

where we use the notation $\gamma_{1:t-1} := (\gamma_1, \dots, \gamma_{t-1})$, meaning

$$J(\gamma_{1:t-1}, \gamma_t^*, \gamma_{t+1:K}^*, \sigma_{1:t-1}, \sigma_t^*, \sigma_{t+1:K}^*) :=$$

$$J(\gamma_1, \gamma_2, \dots, \gamma_{t-1}, \gamma_t^*, \gamma_{t+1}^*, \dots, \gamma_K^*, \sigma_1, \sigma_2, \dots, \sigma_{t-1}, \sigma_t^*, \sigma_{t+1}^*, \dots, \sigma_K^*)$$

Nash Equilibria

(γ^*, σ^*) is a Nash Equ. if $\forall \gamma_{1:K}, \sigma_{1:K}$ we have

$$J(\gamma_{1:K}^*, \sigma_{1:K}) \leq J(\gamma_{1:K}^*, \sigma_{1:K}^*) \leq J(\gamma_{1:K}, \sigma_{1:K}^*)$$

Observe that: (see Lemma 7.1 in Hespanha book)

subgame perfect equilibrium
$\Downarrow$
Nash equilibrium

Proof idea:

For stage $t=1$ by definition of subgame perfect equilibrium,

$$\forall \gamma, \quad J(\gamma_1^*, \gamma_{2:K}^*, \sigma^*) \leq J(\gamma_1, \gamma_{2:K}^*, \sigma^*)$$

$$\leq J(\gamma_1, \gamma_2, \gamma_{3:K}^*, \sigma^*)$$

by applying def. for stage 2

$$\vdots$$

$$\leq J(\gamma_{1:K}, \sigma^*)$$

PART 2: Review (for Quiz)

1 General-sum games with finite action spaces

Ex 1 (see Thm. 10.1 in Hespanha book)

Consider a 2-pl. general-sum game with costs $A, B \in \mathbb{R}^{m \times n}$.
To find mixed-strategy NE $y^* \in \mathcal{Y}$, $z^* \in \mathcal{Z}$ we can solve the quadratic optimization problem

quadratic objective fct.

$$\min_{y \in \mathbb{R}^m,\, z \in \mathbb{R}^n,\, q, p \in \mathbb{R}} \quad y^T (A+B) z - p - q$$

linear constraints

coordinate-wise

$$\text{s.t.} \quad Az \geq \mathbb{1} p, \quad \mathbb{1}^T z = 1, \quad z \geq 0$$
$$B^T y \geq q \cdot \mathbb{1}, \quad \mathbb{1}^T y = 1, \quad y \geq 0$$

To get an idea for why this is hard to solve, we want to show that the objective function is <u>non-convex</u>.

<u>Recall:</u> For $f: \mathbb{R}^d \to \mathbb{R}$, $f(x) = x^T Q x$,

$$\boxed{f \text{ is convex} \iff \text{all eigenvalues of } Q + Q^T \text{ are } \geq 0}$$

How to show this?

↳ let $x_1, x_2 \in \mathbb{R}^n$, $\lambda \in [0,1]$; we need to show that

$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \iff$ all ev.'s of $Q+Q^T$ are ≥ 0

↳ some algebra ...

$$(\lambda x_1 + (1-\lambda) x_2)^T Q (\lambda x_1 + (1-\lambda) x_2)$$
$$= \lambda x_1^T Q x_1 + \dots$$
$$\vdots$$

more calculations yield: $0 \leq (x_1 - x_2)^T \left(\frac{Q + Q^T}{2}\right)(x_1 - x_2) \quad \forall x_1, x_2$

$$\iff 0 \leq y^T \left(\frac{Q + Q^T}{2}\right) y, \quad \forall y$$

$\iff$ all ev.'s of $Q + Q^T$ are ≥ 0

In our case we want to show non-convexity:

$$f: \mathbb{R}^{n+m} \to \mathbb{R}$$
$$f(y,z) := y^T (A + B) z = \begin{bmatrix} y \\ z \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & \frac{A+B}{2} \\ \frac{(A+B)^T}{2} & 0 \end{bmatrix}}_{:= Q} \begin{bmatrix} y \\ z \end{bmatrix}$$

For any z, set $y = (A+B)z$. Then $z^T(A+B)^T(A+B)z \geq 0$.
But if we instead pick $y = -(A+B)z$, then $-z^T(A+B)^T(A+B)z \leq 0$.

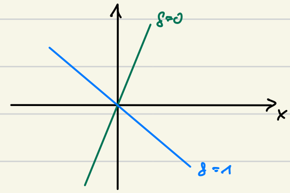
$\Rightarrow$ eigenvalues of $Q+Q^T$ <u>not</u> all ≥ 0

$\Rightarrow$ non-convexity.

② Review of 0-sum games

<u>Ex 2:</u> Consider $f : [-1,1] \times \{0,1\} \to \mathbb{R}$

$$f(x,\delta) := \begin{cases} 2x, & \text{if } \delta = 0 \\ -x, & \text{if } \delta = 1. \end{cases}$$

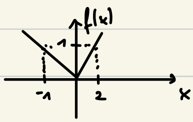


Does it hold that

$$\min_{x\in[-1,1]} \max_{\delta\in\{0,1\}} f(x,\delta) = \max_{\delta\in\{0,1\}} \min_{x\in[-1,1]} f(x,\delta)$$

(<u>Recall:</u> We know that LHS $\geq$ RHS from inf. sup. theorem.)

LHS: $\max_{\delta\in\{0,1\}} f(x,\delta)$ looks like



RHS: $\min_{x\in[-1,1]} f(x,\delta) = \begin{cases} -2, & \text{if } \delta = 0 \\ 1, & \text{if } \delta = 1 \end{cases}$

We conclude that LHS $\neq$ RHS!
(recall that equality is only guaranteed to hold in certain cases, e.g. convex-concave objective)

③ Potential games

<u>Ex 3</u>

Let $J_i : T_1 \times T_2 \times \dots \times T_N \to \mathbb{R}$. Suppose the game with payoffs $\{J_i\}_{i=1}^N$ admits a potential function.

TRUE or FALSE?

1) $\gamma^* \in \Gamma$ is a NE $\Rightarrow$ γ^* is a maximizer of the potential function

→ FALSE (e.g. stag hunt: only one of NEs maximizes potential function)

2) $\gamma^* \in \Gamma$ is maximizer of potential $\Rightarrow$ γ^* is NE

→ TRUE (proven in lecture)

3) best reply dynamics converges $\Rightarrow$ game is potential

→ FALSE (we proved "$\Leftarrow$" direction; as counterexample consider

$$\begin{bmatrix} (1,0) & (2,0) \\ (2,0) & (0,1) \end{bmatrix} \text{— NE}$$

(players are maximizers)

↓

best reply converges to NE but game is not potential → verify this!)