Lecture 5

Networked control: coordination among agents
Graph theory

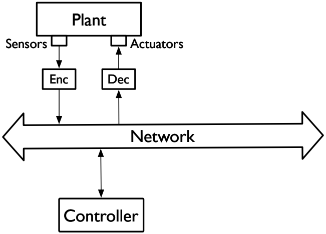
Giancarlo Ferrari Trecate[1]

[1]Dependable Control and Decision Group
École Polytechnique Fédérale de Lausanne (EPFL), Switzerland
giancarlo.ferraritrecate@epfl.ch

Timetable and Course Schedule (tentative)

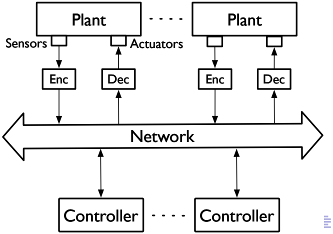
Part 1: Challenges (Week 1-6) - mostly 1 plant, 1 controller setting

- Review of LTI systems
- Review of Linear Matrix Inequalities (LMIs)
- Control networks and NCS
- Impact of delays
- Impact of packet drops



Part 2: Opportunities (Week 7-14) - multiple systems

- Coordination: motivating examples
- Elements of graph and matrix theory
- Discrete-time consensus
- Continuous-time consensus

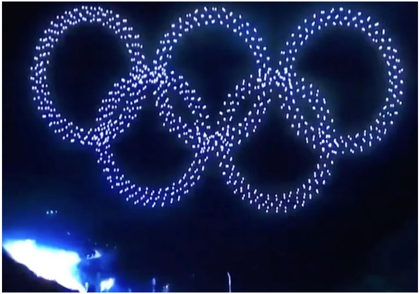


Literature

- Opportunities in NCSs

  - Francesco Bullo, Lecture notes on network systems, 2017. Available online at: http://motion.me.ucsb.edu/book-lns/
    From now on, this is called "THE textbook"
  - F. Garin and L. Schenato, "A Survey on Distributed Estimation and Control Applications Using Linear Consensus Algorithms," in Networked Control Systems, Springer London, pp.75-107, 2010.

Opportunities: coordination among agents



Drone show at the 2020 Olympic games

Swarm of mobile robots

Wishes

- Partial communication (limited transmission power)
- Distributed control
- Self-organizing for performing tasks

Coordination in nature

Social behavior: creatures cluster in large moving formations

School of fish

Swarm of flying birds

- Partial communication
- No centralized control
- Global emergent behavior

Outline

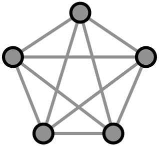
- Coordination: motivating problems in engineering and beyond (Textbook, 1.1-1.3)
- Modeling communication: graph theory (Textbook, Ch. 3)

Social influence networks - De Groot model

- n individuals, each with an estimate $p_i(0) \in \mathbb{R}$ of a common parameter
- individuals exchange information (communication network !)
- each individual i talks to all others and revises his estimate as

$$p_i^+ = \sum_{j=1}^{n} a_{ij} p_j$$

- $a_{ij} \geq 0$ are influence weights
- $\sum_{j=1}^{n} a_{ij} = 1$ (local averaging behavior)



Collective model

Set $p = \begin{bmatrix} p_1 & \dots & p_n \end{bmatrix}^T$

$$p^+ = Ap, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

- DT LTI system
- A is *row-stochastic*, which means
  - non-negative entries
  - elements of each row sum up to 1

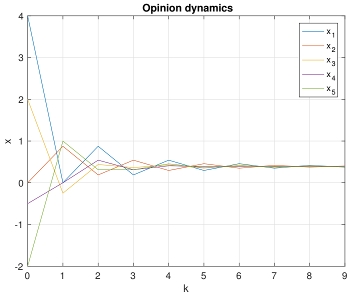
Problem

Will the team achieve *consensus*, i.e.

$$\exists \bar{p} \in \mathbb{R} : p_i(k) \to \bar{p} \text{ as } k \to +\infty, \forall i = 1, \dots, n \ ?$$

Simulations - De Groot model

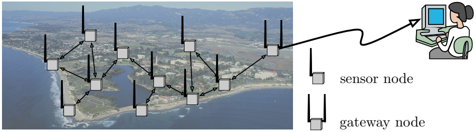
$$p^+ = Ap, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$



Problems

- Properties of the weights a_{ij} for achieving consensus ?
- Can we predict the consensus value ?

Averaging in wireless sensor networks



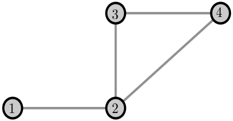
- n spatially distributed devices, each measuring the same environmental variable (temperature, light,...)
- devices exchange information over a communication network
- the operator wants to receive a single average measurement

Distributed algorithm

Sensor i computes

$$x_i^+ = \text{average}(x_i, x_j \mid j \sim i)$$

Example: $x_1^+ = \frac{x_1+x_2}{2}$, $x_2^+ = \frac{x_1+x_2+x_3+x_4}{4}$



$j \sim i \stackrel{\text{def}}{=}$ j is a neighbor of $i \stackrel{\text{def}}{=}$ the edge (i, j) exists

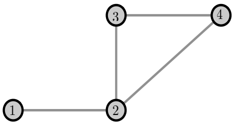
Collective model for the graph in the figure

Set $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$

$$x^+ = Ax$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

A is again row-stochastic



$j \sim i \overset{\text{def}}{=} j$ is a neighbor of $i \overset{\text{def}}{=}$ the edge (i, j) exists

Problem

Will the sensors achieve *average consensus*, i.e.

$$x_i(k) \rightarrow \text{average}(x_i(0), i = 1, \dots, n) \text{ as } k \rightarrow +\infty, \forall i = 1, \dots, n \text{ ?}$$

Remark: communication among sensors is just partial (e.g. 1 not connected to 4)

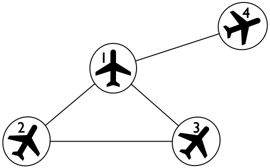
Alignment in teams of moving agents

Set of n moving agents

- Dynamics of agent i:

$$v_i^+ = v_i + u_i \quad v_i(0) = \tilde{v}_i$$

- Velocity : $v_i(k) \in \mathbb{R}^2$
  Control input : $u_i(k) \in \mathbb{R}^2$



Communication network topology: undirected connected graph $G = (V, E)$. Nodes $V = \{1, ..., n\}$, edges $E \subset V^2$.

Partial communication network used for computing the control law

$$u_i = \text{average}(v_i, v_j | \ j \sim i) - v_i$$

Collective model

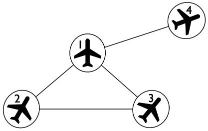
Set $v = \begin{bmatrix} v_1^T & \dots & v_n^T \end{bmatrix}^T$

$$v^+ = Av$$

A composed by 2×2 blocks

$$A_{ij} = \begin{cases} \frac{1}{(\#\text{ neighbors to } i)+1} I_{2\times 2} \text{ if } j \sim i \\ 0_{2\times 2} \text{ otherwise} \end{cases}$$

A is row-stochastic



Problems

- Will the agents converge to a formation (all agent moving with the same velocity), i.e.

$$\exists \bar{v} \in \mathbb{R}^2 : v_i(k) \to \bar{v} \text{ as } k \to +\infty, \forall i = 1, \dots, n \text{ ?}$$

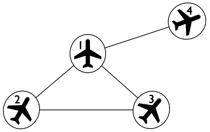
- Is $\bar{v} = \text{average}(v_i(0), \forall i = 1, \dots, n)$?

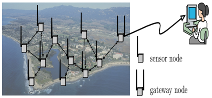
Common features to all examples

- The communication topology is captured by a graph G
- DT LTI collective dynamics

$$x^+ = Ax$$

where the structure of A depends on G





Problem

Is it possible to study consensus by analyzing the graph G ?

Next: elements of

- graph theory
- *algebraic* graph theory = how to relate graph and matrix properties

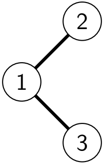
Graph theory

Basic definitions

Undirected graph

An *undirected* graph is a pair $G = (V, E)$ consisting of a finite set of *vertices (or nodes)* $V = \{1, 2, \dots, n\}$ and a set $E \subset V \times V$ of *unordered* pairs called *edges (or arcs)*

Example: $V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3)\}$



Undirected edges : $(1, 2) = (2, 1)$, $(1, 3) = (3, 1)$

Important convention

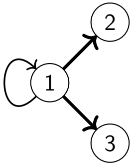
Self-loops (v, v) are NOT allowed in undirected graphs

Basic definitions

Directed graph

A graph G is *directed* (or digraph) if all pairs in E are *ordered*

Example: $V = \{1, 2, 3\}$, $E = \{(1, 1), (1, 2), (1, 3)\}$



Ordered edges: $(1, 2) \neq (2, 1)$, $(1, 3) \neq (3, 1)$

Remark

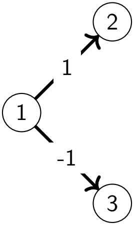
Self-loops (v, v) are allowed in directed graphs

Basic definitions

Weighted graph

If a function $w : E \to \mathbb{R}$ is specified, the graph/digraph is called *weighted*

Example: $V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3)\}$ $w(1, 2) = 1$ and $w(1, 3) = -1$



Remarks

- Notation overload: $w_{ij} = w(i, j)$
- An unweighted graph is assimilated to a weighted graph with $w_{ij} = 1$, $\forall (i, j) \in E$

Undirected graphs

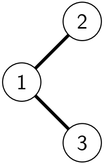
Undirected graphs: neighbors of a node

Let $G = (V, E)$ be an undirected graph

- Nodes u and v are neighbors if $(u, v) \in E$
  - Set of neighbors of u: $\mathcal{N}(u) = \{v \in V : (u, v) \in E\}$
  - Degree of u:

$$d(u) = \sum_{j \in \mathcal{N}(u)} w_{uj}$$

In unweighted graphs $d(u)$ counts the number of neighbors of u

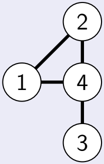


Undirected graphs: subgraphs

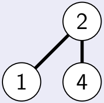
Subgraph

The graph $H = (U, F)$ is a *subgraph* of $G = (V, E)$ if $U \subseteq V$, $F \subseteq E$ and edges in F connect only nodes in U

Graph $G = (V, E)$



Graph $H = (U, F)$



$H = (U, F)$ is a subgraph of G because $U = \{1, 2, 4\} \subseteq V$,
$F = \{(1, 2), (2, 4)\} \subseteq E$ and edges in F connects only nodes in U.

Spanning subgraph

A subgraph is spanning if its node set is V

Connectivity in *undirected* graphs

Path

A sequence of arcs $e_1 e_2 \cdots e_k$ such that

$$e_1 = (v_1, v_2), e_2 = (v_2, v_3), \ldots, e_k = (v_k, v_{k+1})$$

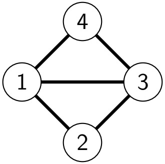
is a *path* from v_1 to v_{k+1}.

Notation: $v_1 v_2 \cdots v_{k+1}$

Path classification

- A path is *simple* if it does not pass through the same *vertex* twice (with the exception of the starting node, when it coincides with the end node)
- A *cycle* is a SIMPLE path with $v_{k+1} = v_1$ and crossing at least 3 distinct nodes

Connectivity in *undirected* graphs



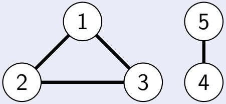
- The paths 1234 and 12341 are simple
- The path 1231 is a cycle
- The path 121 is not a cycle. Also 12312341 is not a cycle

Connectivity in *undirected* graphs

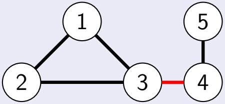
Connectivity and completeness

A node v_2 is *connected* to v_1 if there is a path from v_1 to v_2
- A graph is *connected* if all pairs of distinct vertices are connected
- A graph $G = (V, E)$ is *complete* if, for all pairs of distinct vertices, there is an edge connecting them

Disconnected graph



Connected graph



Remark

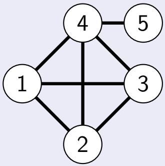
If a graph is disconnected, then it is composed of multiple connected subgraphs, called *connected components*

Connectivity in *undirected* graphs

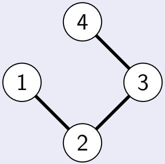
Tree

A tree of the undirected graph $G = (V, E)$ is a connected acyclic subgraph. A tree is *spanning* if it contains n nodes, where $n = |V|$.

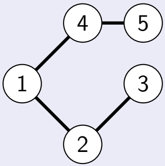
Graph G



Tree



Spanning tree

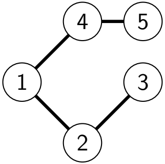


Connectivity in *undirected* graphs

Tree theorem

Let T be an undirected graph. The following conditions are equivalent:

- T is a tree
- T is connected and has n nodes and $n-1$ edges
- Every pair of nodes of T is connected by a unique simple path



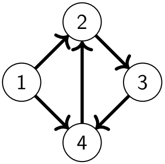
Directed graphs

Digraphs: neighbors of a node

Let $G = (V, E)$ be a digraph and $(u, v) \in E$

- u is an *in-neighbor* of v and v is an *out-neighbor* of u
- $\mathcal{N}^{in}(u)$ and $\mathcal{N}^{out}(u)$ are the sets of in/out neighbors of u
- the in-degree and out-degree of u are defined as

$$d^{in}(u) = \sum_{j \in \mathcal{N}^{in}(u)} w_{ju} \quad d^{out}(u) = \sum_{j \in \mathcal{N}^{out}(u)} w_{uj}$$

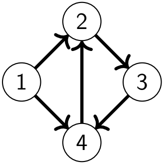


Digraphs: neighbors of a subset of nodes

Let $G = (V, E)$ be a digraph and $S \subseteq V$. The set of

- out-neighbors of S is $\mathcal{N}^{out}(S) = \{j \in V \backslash S : (i, j) \in E \text{ for some } i \in S\}$
- in-neighbors of S is $\mathcal{N}^{in}(S) = \{i \in V \backslash S : (i, j) \in E \text{ for some } j \in S\}$

The set $V \backslash S$ is the difference of sets V and S, collecting all vertices in V that are not in S



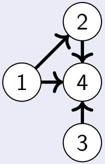
$$\mathcal{N}^{out}(\{1, 2, 3\}) = \{4\}$$
$$\mathcal{N}^{in}(\{1, 2, 3\}) = \{4\}$$

Directed graphs: subgraphs

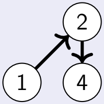
Subgraph

The digraph $H = (U, F)$ is a *subgraph* of the digraph $G = (V, E)$ if $U \subseteq V$, $F \subseteq E$ and edges in F connect only nodes in U

Graph $G = (V, E)$



Graph $H = (U, F)$



$H = (U, F)$ is a subgraph of G because $U = \{1, 2, 4\} \subseteq V$, $F = \{(1, 2), (2, 4)\} \subseteq E$ and edges in F connects only nodes in U.

Spanning subgraph

A subgraph is spanning if its node set is V

Connectivity in *digraphs*

Sources and sinks

- A source is a node v with no in-neighbors
- A sink is a node v with no out-neighbors

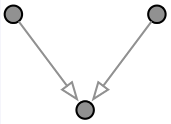
Path and cycles

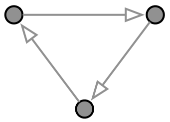
- A path $v_1 \dots v_k$ is defined as for undirected graphs
- As for undirected graphs, a path is *simple* if it does not pass through the same *vertex* twice (with the exception of the starting node, when it coincides with the end node)
- A *cycle* is a SIMPLE path starting and ending in the same node
- Remember:
  - self-loops are (v, v) are allowed in digraphs
  - (v, v) and $(v, u)(u, v)$ are cycles in digraphs

Connectivity in *digraphs*

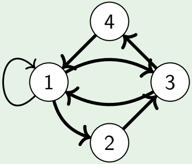
Remark

Every acyclic digraph has always at least one source and one sink





Example - simple paths and cycles



- The path 1234 is simple
- The paths 1231, 131, and 11 are cycles

Connectivity in *digraphs*

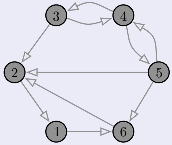
Directed tree

A directed tree is an acyclic digraph where there is a node r (called the root) such that any other node $v \in V$ can be reached from r through one and only one path

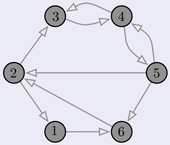
Weak/strong connectivity

- G is *strongly connected* if for any $u, v \in V$, $u \neq v$ there is a path from u to v
- G is weakly connected if its undirected version is connected

Weakly connected G



Strongly connected G



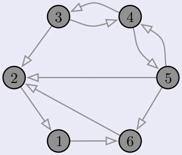
Examples

Connectivity in *digraphs*

Global reachable node and directed spanning tree

- G has a *globally reachable node* if there is $g \in V$ connected to any other node $v \in V$ (i.e. there is a path from v to g)
- G has a *directed spanning tree* if it contains a directed tree comprising all nodes in V

G with a globally reachable node (node 2)



Connectivity in *digraphs*

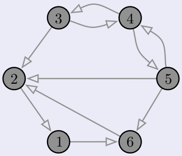
Reverse graph

Let $G = (V, E)$ be a digraph. Its reverse graph $G^{rev}(V, E^{rev})$ is given by the edge set $E^{rev} = \{(i, j) \text{ if } (j, i) \in E\}$

Remark

G has a globally reachable node, if and only if G^{rev} includes a directed spanning tree

G with a globally reachable node (node 2)



Take home messages and open problems

- Graph theory allows one to easily model the topology of a control network
- Graphs define connectivity properties
- Any algebraic characterization of graph connectivity properties ?
- How graph connectivity relates to the achievement of consensus in the examples ?