Networked Control Systems (ME-427)- Exercise session 8

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1. **Bounded evolution for averaging systems** [Textbook E1.1]. Given a matrix $A \in \mathbb{R}^{n\times n}$ with non-negative entries and unit row sums, consider the *averaging model*

$$x(k+1) = Ax(k). \tag{1}$$

Show that, for all initial conditions $x(0)$ and times k,

$$\min_i x_i(0) \leq \min_i x_i(k) \leq \min_i x_i(k+1) \leq \max_i x_i(k+1) \leq \max_i x_i(k) \leq \max_i x_i(0),$$

where i takes values in $\{1,\dots,n\}$. Note that these inequalities imply that all evolutions of the averaging model are bounded.

Hint: Start with the maximum, show that $\max_i (Ax)_i \leq (\star)\cdot \max_h x_h$ for a suitable $(\star)$.

Solution: For the maximum, let us compute:

$$\max_i (Ax)_i = \max_i \sum_{j=1}^n a_{ij}x_j \leq \max_i \sum_{j=1}^n a_{ij}(\max_h x_h) \leq \left(\max_i \sum_{j=1}^n a_{ij}\right)(\max_h x_h) = 1\cdot(\max_i x_i).$$

Similarly for the minimum,

$$\min_i (Ax)_i = \min_i \sum_{j=1}^n a_{ij}x_j \geq \min_i \sum_{j=1}^n a_{ij}(\min_h x_h) \geq \left(\min_i \sum_{j=1}^n a_{ij}\right)(\min_h x_h) = 1\cdot(\min_i x_i).$$

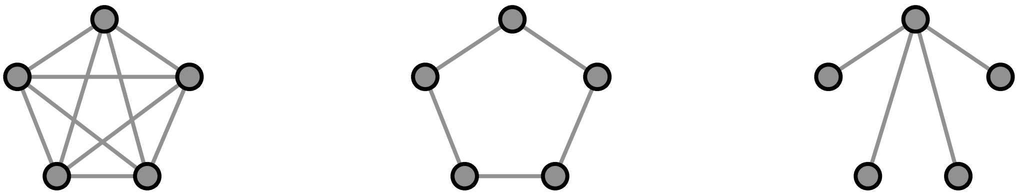
2. **Simulating the average dynamics** [Textbook E1.3]. Simulate in Matlab the linear averaging algorithm

$$x_i^+ := \text{average}(x_i, \{x_j, \text{for all neighbor nodes } j\}). \tag{2}$$

Set $n = 5$, select the initial state equal to $(1,-1,1,-1,1)$, and use the following undirected unweighted graphs (depicted in figure):

(a) the complete graph,
(b) the ring graph, and
(c) the star graph with node 1 as center.

Which value do all nodes converge to? Is it equal to the average of the initial values?



(a) Complete graph (b) Ring graph (c) Star graph

Solution: According to the MatLab file available on Moodle, the three linear averaging algorithms converge to, respectively, 0.2, 0.2, and 0.38. Only in the first two cases the algorithms converges to the average of the initial values. As we will see later in the lectures, this is due to the fact that, in these cases, the collective dynamics $x^+ = Ax$ uses a doubly-stochastic matrix A.

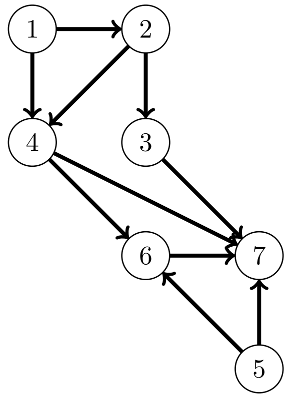


Figure 2: Graph for problem 3

3. **Directed spanning trees.** Draw if possible a directed spanning tree of the digraph in Figure 2.

   Solution. There is none.

4. **Acyclic digraphs** [Textbook E3.1] Let G be an acyclic digraph with nodes $\{1, \ldots, n\}$. Show that G contains at least one sink, i.e., a vertex without out-neighbors and at least one source, i.e., a vertex without in-neighbors.

   Hint: By contradiction assume that G has no sinks and show it must have a cycle.

   Solution: Assume by contradiction that G contains no sinks, i.e., each node of G has at least one out-neighbor. Pick any node v_1 in G and, because each node has at least one out-neighbor, note that there exists at least one infinite sequence $\{v_i\}_{i \in \mathbb{N}}$ with the property that v_{i+1} is an out-neighbor of v_i. Such sequence is a directed path in G of infinite length and, therefore, at least one node of G appears at least twice. The existence of this cycle contradicts the hypothesis that G is acyclic. The same argument, applied to the reverse digraph, shows the existence of at least one source.

5. **Properties of trees.** Consider an undirected graph G with $n > 1$ nodes(recall that self-loops are not allowed in undirected graphs). Show that if G is a tree, then every pair of distinct nodes of G is connected by a unique path.

   Hint: By contradiction ...

   Solution: Assume by contradiction that nodes v and u are connected by two paths: $v, v_1, v_2, \ldots, v_k, u$ and $v, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_k, u$. This implies that $v, v_1, \ldots, v_k, u, \tilde{v}_k, \ldots, \tilde{v}_1, v$ is a cycle. This contradicts the assumption that G is acyclic.

6. **Properties of directed spanning trees.** Assume $G = (V, E)$ is a directed spanning tree. Show that each node has at most one in-neighbor.

   Solution: The root r has no in-neighbors. Let $x \in V$ be different from the root r. Then, since the path from r to x is unique, x has at most one in-neighbor.

7. Check if the following statements are true or false. *T* *F*

   (a) The diagraph $G = (V, E), \ V = \{1\}, \ E = \phi$ is a directed tree.

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   Solution: TRUE. Set the root $r = 1$. Since there is no other node in V, requirements for a directed tree are fulfilled.

   (b) A directed tree has no self-loops

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   Solution: TRUE. Otherwise, the node v with self-loop can be reached from r through two different paths $r, \ldots, v$ and $r, \ldots, v, v$.

(c) A digraph contains at most a directed spanning tree

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Solution: FALSE. Here is a counterexample