Networked Control Systems (ME-427)- Exercise session 7

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1. **Analysis of deterministic packet dropouts.** Consider the NCS with sensor-collocated control

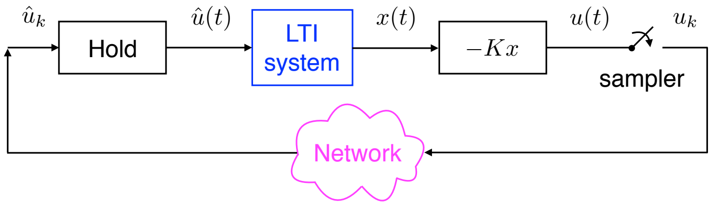

Figure 1: NCS with sensor-collocated control

The LTI system dynamics is

$$\dot{x} = x + u, \tag{1}$$

the control gain is $K = 1.5$ and the sampling interval is uniform and equal to $T = 0.1s$. We assume a network not affected by delays but lossy, where the binary variable θ_k is zero if there is a packet drop at time k. Moreover, packet loss occur at the asymptotic rate r given by

$$r = \lim_{N\to\infty} \frac{1}{N} \sum_{k=k_0}^{k_0+N+1} (1-\theta_k), \qquad \forall k_0 \in \mathbb{N}.$$

Download from moodle the simulator for the NCS in Figure 1

`Pdrop_estimation_scalar_system.m`

The first section of the `.m` file, configured for running an experiment defined by $t_0 = 0$, $x(0) = 1$, and $\Gamma = 0.2$, produces a plot of the state $x(t_k)$ at sampling times t_k, $k = 0, 1, \ldots$

Remark: Since the binary sequences θ_k are extracted randomly over a finite simulation horizon, it can happen that $\theta_k = 0$ occurs at a rate different from r. The simulator displays in the MatLab command windows the difference, in order to let the user assess if a simulation is acceptable or not.

The second section is devoted to the estimation of the maximal dropout rate that guarantees exponential stability of the system. The code is just partially complete. Given a grid of values for β_0 and β_1, it computes the pairs (β_0, β_1) for which exponential stability can be guaranteed. However, the code for estimating the maximal dropout rate is missing.

(a) Familiarize yourself with the simulator. Run it for $r = 0.2$, $r = 0.4$ and $r = 0.7$.

(b) Fill in the missing code for computing an estimate $\bar{r}$ of the maximal dropout rate.

(c) Try to improve the value of $\bar{r}$ by changing the grid values for β_0 and β_1.

Solution: For parts (a), (b), and (c), the solutions are provided in

`Pdrop_estimation_scalar_system_solved.m`

file.

(d) Through simulations, check how conservative is $\bar{r}$. Can you motivate the discrepancy?

Solution: The critical value of r for which unstable behavior appears in several simulations is $r_{max} = 0.95$. The discrepancy is mainly because of two reasons:

- Through simulations, we can check only a limited number of packet loss sequences $\theta_k \in \{0, 1\}$. Instead, if $r < \bar{r}$, exponential stability is guaranteed for ALL sequences θ_k.
- The LMI conditions used for computing $r < \bar{r}$ are just sufficient but not necessary. Therefore, $\bar{r}$ provides a conservative estimate of the maximum dropout rate that can be tolerated.

2. **Stochastic dropouts.** Consider the NCS with sensor-collocated control in Figure 2

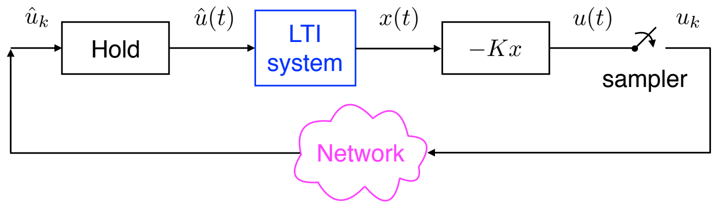

Figure 2: NCS with sensor-collocated control

(a) Assume that packet dropout happens with probability $p = 0.4$, that there is no delay and that sampling is uniform with period $T = 0.1$.
The LTI system dynamics is

$$\dot{x} = 0.5x + 2u$$

Find all values of the control gain $K \in \mathbb{R}$ such that the average NCS model seen in the lectures is asymptotically stable.

Hint: Recall the Jury's criterion: $\lambda^2 + \alpha\lambda + \beta$ is Schur if and only if

$$\begin{cases} \beta > -\alpha - 1 \\ \beta > \alpha - 1 \\ \beta < 1 \end{cases}$$

Solution: NCS model. Setting $\hat{u}_k = \hat{y}_k$ (to match the notation used in the lectures) and $z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$, the NCS dynamics is

$$z_{k+1} = \bar{A}z_k + \bar{B}\hat{v}_k, \qquad v_k = \begin{bmatrix} -K & -1 \end{bmatrix} z_k, \qquad \hat{v}_k = \Delta_k v_k$$

$$\bar{A} = \begin{bmatrix} e^{AT} - (1-p)\Gamma(T-\tau)BK & e^{A(T-\tau)}\Gamma(\tau)B + p\Gamma(T-\tau)B \\ -(1-p)K & p \end{bmatrix}$$

where $\Gamma(t) = \int_0^t e^{As} ds$.
For $A = 0.5$, $T = 0.1$, $\tau = 0$, $B = 2$ and $p = 0.4$ one has

$$e^{AT} = 1.0513, \qquad \Gamma(0.5) = 0.1025$$

and

$$\begin{aligned} \bar{A} &= \begin{bmatrix} 1.0513 - 0.6 \cdot 0.1025 \cdot 2K & 0.4 \cdot 0.1025 \cdot 2 \\ -0.6K & 0.4 \end{bmatrix} \\ &= \begin{bmatrix} 1.0513 - 0.123K & 0.082 \\ -0.6K & 0.4 \end{bmatrix} \end{aligned}$$

Characteristic polynomial

$$\begin{aligned}\phi(\lambda) &= \det(\lambda I - \bar{A}) = \\ &= \det \begin{bmatrix} \lambda - 1.0513 + 0.123K & -0.082 \\ 0.6K & \lambda - 0.4 \end{bmatrix} = \\ &= \lambda^2 + (-0.4 - 1.0513 + 0.123K)\lambda + 0.4205\end{aligned}$$

A necessary condition for applying the theorem seen in the lectures is that $\bar{A}$ is Schur. From Jury's conditions, we have

$$\begin{cases} 0.4205 > -(-1.4513 + 0.123K) - 1 \\ 0.4205 > (-1.4513 + 0.123K) - 1 \\ 0.4205 < 1 \end{cases} \rightarrow \begin{cases} 0.4205 > 0.4513 - 0.123K \\ 0.4205 > -2.4513 + 0.123K \end{cases}$$

$$\begin{cases} 0.123K > 0.0308 \\ 2.8718 > 0.123K \end{cases} \rightarrow \begin{cases} K > 0.2504 \\ K < 23.348 \end{cases}$$

(b) For $K = 3$ check if the LMIs guaranteeing mean-square stability of the NCS are feasible. Starting from $x_0 = 10$, simulate 100 trajectories u_k on the time interval $[0, 5]$ sec for different stochastic dropouts. Using these simulations, compute and plot the sample estimate of $\rho\left(\mathbb{E}(x_k x_k^T)\right)$ where $\rho(\cdot)$ denotes the spectral radius.
Hints:

i. Download from moodle

`Pdrop_stoch.m`, `barA_Compute.m` and `simNCS_packet_drop.m`.

The latter is a simulator of an NCS in presence of dropouts.

ii. Fill in the missing code in the first file. The part that must be written is the definition of the LMIs seen in the lectures. The simulator will solve them and produce all necessary plots.

Solution: see the code `Pdrop_stoch_solved.m`.