Networked Control Systems (ME-427) - Exercise session 4

Prof. G. Ferrari Trecate

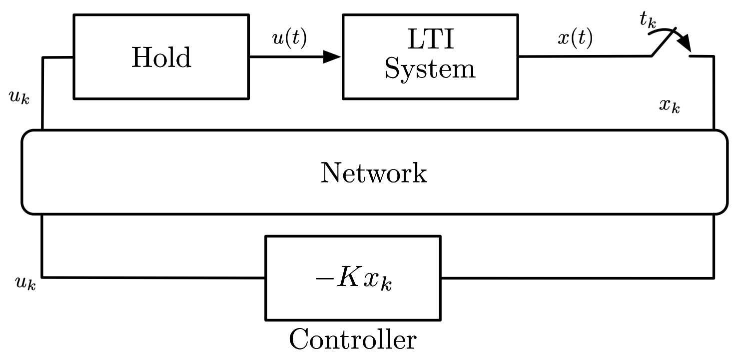

Figure 1: Networked control system

1. **Nonuniform sampling intervals.** Consider the NCS in figure 1, where the LTI system and the sampler are given by

$$\begin{aligned} \dot{x} &= \bar{A}x + \bar{B}u \\ x_k &= x(t_k) \end{aligned} \tag{1}$$

The MAC protocol can produce time-varying sampling intervals $T_k = t_{k+1} - t_k$. We will analyze the effect of nonuniform sampling on stability. The discrete-time model of the system is

$$x_{k+1} = A_k x + B_k u, \tag{2}$$

where $A_k = e^{\bar{A}T_k}$, $B_k = \Gamma(T_k)B$, and $\Gamma(T_k) = \int_0^t e^{\bar{A}s}ds$. Hence, the closed-loop NCS model is

$$x_{k+1} = (A_k - B_k K)x_k \tag{3}$$

(a) Consider $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 6 \end{bmatrix}$, and assume $T_k \in \{h_1, h_2\}$, $h_1 = 0.18$, $h_2 = 0.54$.

i. Using MatLab, check the stability of (3) for uniform sampling intervals $T_k = h_1$ and $T_k = h_2$, $\forall k \geq 0$.

Hint: e^{At} is `expm(A*T)` in Matlab. Similarly $\Gamma(T)$ is obtained as

```
Gamma = (@(X) (expm(A*X)));
GammaT = integral(Gamma,0,T,'ArrayValued',true);
```

ii. Using the periodic sequence

$$T_k = \begin{cases} h_1 & \text{if } k \text{ is even} \\ h_2 & \text{if } k \text{ is odd} \end{cases}$$

simulate the system for $k = 0, 1, \ldots, 100$ from $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Analyze asymptotic stability by studying the model relating x_k to x_{k+2}, $k = 0, 2, 4, \ldots$

Solution: The solution code is posted on moodle. Setting $A^{[1]} = e^{\bar{A}h_1}$, $B^{[1]} = \Gamma(h_1)B$, $A^{[2]} = e^{\bar{A}h_2}$, $B^{[2]} = \Gamma(h_2)B$ One has

$$x_{k+2} = \begin{cases} \overbrace{(A^{[2]} - B^{[2]}K)(A^{[1]} - B^{[1]}K)}^{F^{[1]}}\, x_k & k \text{ even} \\ \underbrace{(A^{[1]} - B^{[1]}K)(A^{[2]} - B^{[2]}K)}_{F^{[2]}}\, x_k & k \text{ odd} \end{cases} \tag{4}$$

Using MatLab, one can check that neither $F^{[1]}$ nor $F^{[2]}$ are Schur matrices.

(b) Consider $A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $K = -\begin{bmatrix} 1 & 0 \end{bmatrix}$ and assume $T_k \in \{h_3, h_4\}$, $h_3 = 3.950$, $h_4 = 2.126$.

i. Check instability of (3) for uniform sampling intervals $T_k = h_3$ and $T_k = h_4$, $\forall k \geq 0$

ii. Using the periodic sequence

$$T_k = \begin{cases} h_3 & \text{if } k \text{ is even} \\ h_4 & \text{if } k \text{ is odd} \end{cases}$$

simulate the system for $k = 0, 1, \ldots, 40$ from $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Analyze asymptotic stability studying the model relating x_k to x_{k+2}, $k = 0, 2, 4, \ldots$

Solution: As for the previous point, by replacing h_1 and h_2 with h_3 and h_4, respectively, one builds matrices $F^{[3]}$ and $F^{[4]}$, which are similar to $F^{[1]}$ and $F^{[2]}$ appearing in (4). But in this case, they are both Schur.

(c) From points (1a) and (1b) check which of the following statements about the matrix $A = A_1 \cdot A_2$, where $A_1,\ A_2 \in \mathbb{R}^{n \times n}$ are true

i. If A_1 and A_2 are Schur, then A is Schur.

ii. If A_1 and A_2 are not Schur, then A cannot be Schur.

Solution: None is true.

(d) Considering the setting in point (1b), find a Lyapunov function $V(x) = x^T P x$ for the discrete-time NCS model relating x_k to x_{k+2}, **for** k **odd**. Is $V(x_k)$, $k = 0, 1, \ldots$, monotonically decreasing, at least for k large enough ? Is this property necessary for the asymptotic stability of the NCS?

Hint: In MatLab, use the command `dlyap` for solving a Lyapunov equation and compute P. Then, plot the values $V(x_k)$ obtained using the states x_k, $k = 0, 1, \ldots$ simulated in point (1b).

Solution: The solution code (on moodle) shows that $V(x_k)$, $k = 0, 1, \ldots$ decreases every two steps, but it is never monotonically decreasing. However $V(x_k) \to 0$ as $k \to +\infty$, which is the essential property for asymptotic stability.

2. **Review of pole placement.** Consider the autonomous LTI system

$$\dot{x}_1 = 2x_1 + 3x_2 + u$$
$$\dot{x}_2 = -x_1 + 4x_2$$

(a) Discretize it with sampling period $T = 0.01$ seconds so as to obtain the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(k+1) = \hat{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(k) + \hat{B}u(k) \quad . \tag{5}$$

(b) Design the state-feedback controller $u(k) = -Kx(k)$ so that the closed-loop dynamics has eigenvalues in $\{0.3, 0.8\}$.

Hint: Use the command `'place'` for computing K. Type `'help place'` to learn how it works.

Solution: See the MatLab code on moodle.