Networked Control Systems (ME-427) - Exercise session 2

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1. **Solving LMIs with Yalmip (guided exercise)**
   Consider the discrete-time LTI system

$$x^+ = Ax, \qquad A = \begin{bmatrix} 0.9053 & 0.0928 \\ 0.0098 & 0.9512 \end{bmatrix}$$

Find $P = P^T \in \mathbb{R}^{2\times 2}$ such that

$$\begin{cases} A^T P A - P \leq -Q \\ P > 0 \end{cases} \tag{1}$$

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Note that, for square matrices M_1 and M_2 of the same size, $M_1 \leq M_2$ means that $M_1 - M_2 \leq 0$.

Workflow for solving LMIs (1)

- Define LMI unknowns `P = sdpvar(n,n)`. Square matrices are symmetric by default.
- Define LMI constraints `Li`, $i = 1, \ldots, N$.
- Combine constraints by concatenation `L = [L1; L2]`.
- Solve for unknowns

```
infosol = optimize(L)
```

- Extract and check solutions

```
Psol = double(P)
```

  - `infosol.info` should be `'Succesfully solved'`;
  - double-check the solution fulfills the constraints.

Other useful commands

- For LMI
  - Equality operator: ==
  - Inequality operators: <=, >=
    `P>=0` means P positive semi-definite. Warning: Yalmip does not support strict inequalities.
- Solver options
  - `ops = sdpsetting('solver', 'mosek') %set the solver`
  - `ops = sdpsetting(ops, 'verbose', 0) %suppresses the output`

Exercise (1) is solved by the following MatLab code

```
% Define the system
A = [0.9053 0.0928; 0.0098 0.9512];

% set yalmip options
ops = sdpsettings('solver','mosek');
ops = sdpsettings(ops,'verbose',0);

% Define unknowns, parameters and constraints

P = sdpvar(2,2); % Unknown 2x2 symmetric matrix
Q = 1 * eye(2,2);
CONS1=[A'*P*A-P<=-Q]; % Constraint 1
CONS2=[P>=0]; % Constraint 2
CONS =[CONS1 ,CONS2]; % Combine all constraints

% Solving for P
infosol = optimize(CONS,[],ops);

Psol = double(P) % Converts to standard matrix format

% Check if the solution is OK
infosol.info

% Double-check the solution verifies the constraints

% Psol must be >0. Chech that all eigenvalues of Psol are > 0
eigP=eig(Psol)

% the matrix -(A'*Psol*A-Psol+Q) must be >0. Chech that all its eigenvalues are > 0
eigcons0=eig(-(A'*Psol*A-Psol+Q))
```

2. **LMI analysis**
   Answer to the following question about exercise (1). Use MatLab for computing eigenvalues.

   (a) Analyzing only the matrix A, do you expect feasibility of (1), without solving it?
   Is (1) feasible for other matrices $Q \in \mathbb{R}^{2\times 2}$?

   Hint: Use the Lyapunov equation $A^T PA - P = -Q$ and Lyapunov theory for showing feasibility.

   Solution: Matrix A is Schur as

   $$\text{abs}(\text{eig}(A)) = \begin{bmatrix} 0.8904 \\ 0.9661 \end{bmatrix}.$$

   Hence, the Lyapunov equation has a unique solution $\bar{P} = \bar{P}^T > 0$. This solution verifies (1) and hence LMIs are feasible.
   From Lyapunov theory, the above argument applies for all $Q > 0$.

   (b) For solving (1) with Yalmip, we have replaced $P > 0$ with $P \geq 0$. Prove that, in spite of this change, any solution to $A^T PA - P \leq -I$ verifies $P > 0$.

   Hint: Show first that $M = -(I + A^T PA)$ is negative definite. Then, show that $-P \leq M$ implies $P > 0$.

   Solution: One has, $\forall x \in \mathbb{R}^2$

   $$-x^T(I + A^T PA)x = -||x||^2 - y^T Py$$

where $y = Ax$. Note that, since $P \geq 0$,

$$x \neq 0 \Rightarrow -||x||^2 - \underbrace{y^T P y}_{\leq 0} < 0$$

and this shows that $M < 0$. Since $-P \leq M$, $\forall x \in \mathbb{R}^2$ one has

$$x^T(-P)x \leq x^T M x$$

hence, $x \neq 0 \Rightarrow x^T(-P)x < 0$, i.e. $-P < 0$.

3. **Solving a Lyapunov equation**
   Consider the discrete-time LTI system

$$x^+ = Ax, \qquad A = \frac{1}{10}\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$$

Solve by hand the Lyapunov equation

$$A^T P A - P = -Q \tag{2}$$

where $P = P^T \in \mathbb{R}^{2\times 2}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Hint: Since P is symmetric, set $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$.

Solution: One has,

$$\frac{1}{100}\begin{bmatrix} 4 & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\frac{1}{100}\begin{bmatrix} 4 & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 4p_{11} + 3p_{12} & 5p_{12} \\ 4p_{12} + 3p_{22} & 5p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} (0.16-1)p_{11} + 0.24p_{12} + 0.09p_{22} & (0.2-1)p_{12} + 0.15p_{22} \\ (0.2-1)p_{12} + 0.15p_{22} & (0.25-1)p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{cases} -0.84p_{11} + 0.24p_{12} + 0.09p_{22} = -1 \\ -0.8p_{12} + 0.15p_{22} = 0 \\ -0.8p_{12} + 0.15p_{22} = 0 \\ -0.75p_{22} = -1 \end{cases}$$

Remark: the third equation is redundant because matrices $A^T PA - P$ and Q are symmetric.
It follows that

$$\begin{cases} p_{22} = \dfrac{1}{0.75} = 1.3333 \\[2ex] p_{12} = \dfrac{-0.15p_{22}}{-0.8} = 0.25 \\[2ex] p_{11} \dfrac{-1 - 0.24p_{12} - 0.09p_{22}}{-0.84} = 1.4048 \end{cases}$$

resulting in

$$P = \begin{bmatrix} 1.4048 & 0.25 \\ 0.25 & 1.3333 \end{bmatrix}.$$

Since $\mathrm{Spec}(P) = \{1.1165, 1.6216\}$, he matrix P is positive-definite.