Networked Control Systems (ME-427)- Exercise session 12

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1. **A sample DeGroot panel.** [Textbook E5.1] A conversation between 5 panelists is modeled according to the DeGroot model by an averaging system $x^+ = Ax$, where

$$A = \begin{bmatrix} 0.15 & 0.15 & 0.1 & 0.2 & 0.4 \\ 0 & 0.55 & 0 & 0 & 0.45 \\ 0.3 & 0.05 & 0.05 & 0 & 0.6 \\ 0 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \end{bmatrix}.$$

Assuming that the panel has sufficiently long deliberations, answer the following:

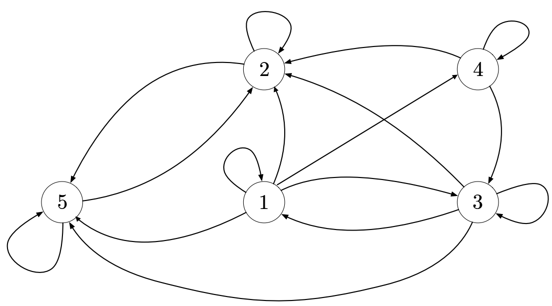
(a) Draw the condensation of the associated digraph.

(b) Do the panelists finally agree on a common decision?

(c) In the event of agreement, does the initial opinion of any panelists get rejected? If so, which ones?

(d) Assume the panelists' initial opinions are their self-appraisals (i.e., the self-weights $a_{11}, \ldots, a_{55}$) and compute the final opinion (Hint: use MatLab for computing relevant eigenvectors).

Solution: The digraph G associated to A is



- Nodes 2, 5 are globally reachable.
- No path from 5 to 4 $\Rightarrow$ G is not strongly connected.

(a) The condensation digraph has two nodes: $\{1, 3, 4\}$ with a directed link to $\{2, 5\}$.

(b) The condensation digraph corresponding to A has a globally reachable node induced by $\{2, 5\}$. This subgraph is aperiodic and strongly connected. Therefore, from the theorem seen in the lectures about consensus in presence of a globally reachable node, the panelists reach to an agreement (i.e. consensus is achieved).

(c) The final decision depends entirely on nodes 2 and 5. The views of nodes 1, 3, and 4 get rejected.

(d) Since A is stochastic, from the theorem seen in the lecture,

$$x(k) \to (w^T x(0))\mathbb{1}_5 \text{ as } k \to \infty,$$

where w is the left eigenvectors of A verifying $w^T \mathbb{1}_5 = 1$. Moreover, we know that $w^T = [0 \;\; \star \;\; 0 \;\; 0 \;\; \star]$, where $\star$ are the nonzero elements. Using MatLab for computing the right eigenvector of A^T, one obtains $w^T = [0 \; 0.4 \; 0 \; 0 \; 0.6]$ and, for the initial opinion $x(0) = [0.15 \; 0.55 \; 0.05 \; 0.5 \; 0.7]$, the agreed decision $(w^T x(0))\mathbb{1}_5 = 0.64 \cdot \mathbb{1}_5$.

2. **A stubborn agent.** [Textbook E5.5] Pick $\alpha \in]0,1[$, and consider the discrete-time averaging algorithm

$$x_1(k+1) = x_1(k),$$
$$x_2(k+1) = \alpha x_1(k) + (1-\alpha)x_2(k).$$

Perform the following tasks:

(a) compute the matrix A representing this algorithm and verify it is row-stochastic,

(b) compute the eigenvalues and eigenvectors of A,

(c) draw the directed graph G representing this algorithm and discuss its connectivity properties,

(d) compute the condensation digraph of G,

(e) compute the final value of this algorithm as a function of the initial values invoking the theorem on consensus with globally reachable nodes.

Solution: The digraph G associated to A is

(a) We have

$$A = \begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix}$$

Clearly, A is non-negative and has row-sums equal to 1.

(b) The eigenvalues are read off the diagonal of A, because A is lower triangular. Therefore, $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha$.

From row-stochasticity, we also know that the eigenvector corresponding to λ_1 is $\mathbb{1}_2$. A direct computation also shows that $[0,\ 1]^T$ is a right eigenvector for the eigenvalue λ_2.

(c)

(d)

(e) Note that G is aperiodic with a globally reachable node. Moreover, we compute the left eigenvector of A corresponding to eigenvalue λ_1 to be $[1,\ 0]$. Hence, by the Theorem on consensus with globally reachable nodes the final consensus value is identical to the initial condition of the first agent.

3. **The equal-neighbor row-stochastic matrix for weighted directed graphs.** [Textbook E5.3] Let G be a weighted digraph with n nodes, weighted adjacency matrix A and weighted out-degree matrix D_{out}. Define the *equal-neighbor matrix*

$$A_{\text{equal-neighbor}} = (I_n + D_{\text{out}})^{-1}(I_n + A).$$

Show that

(a) $A_{\text{equal-neighbor}}$ is row-stochastic;

(b) $A_{\text{equal-neighbor}}$ is primitive if and only if G is strongly connected; and

(c) $A_{\text{equal-neighbor}}$ is doubly-stochastic if G is weight-balanced and the weighted degree is constant for all nodes (i.e., $D_{\text{out}} = D_{\text{in}} = dI_n$ for some $d \in \mathbb{R}_{>0}$).

Hint: First, for any $v \in \mathbb{R}^n$ with non-zero entries, it is easy to see $\text{diag}(v)^{-1}v = \mathbb{1}_n$, where $\text{diag}(v)$ is the diagonal matrix with the elements of v on the main diagonal. Note also that, by definition, $D_{\text{out}} + I_n = \text{diag}((A + I_n)\mathbb{1}_n)$ and $D_{\text{in}} + I_n = \text{diag}((A + I_n)^T \mathbb{1}_n)$.

Solution: One has

$$\left((D_{\text{out}} + I_n)^{-1}(A + I_n)\right) \mathbb{1}_n = \text{diag}((A + I_n)\mathbb{1}_n)^{-1} \left((A + I_n)\mathbb{1}_n\right) = \mathbb{1}_n,$$

which proves statement (a). To prove statement (b), note that, besides self-loops, G and the weighted digraph associated with $A_{\text{equal-neighbor}}$ have the same edges. Also note that the weighted digraph associated with $A_{\text{equal-neighbor}}$ is aperiodic by design. Finally, if $D_{\text{out}} = D_{\text{in}} = dI_n$ for some $d \in \mathbb{R}_{>0}$, then

$$\begin{aligned}\left((D_{\text{out}} + I_n)^{-1}(A + I_n)\right)^T \mathbb{1}_n &= \frac{1}{d+1} \left((A + I_n)^T \mathbb{1}_n\right) \\ &= (D_{\text{in}} + I)^{-1} \left((A + I_n)^T \mathbb{1}_n\right) \\ &= \text{diag}((A + I_n)^T \mathbb{1}_n)^{-1} \left((A + I_n)^T \mathbb{1}_n\right) = \mathbb{1}_n.\end{aligned}$$

where the last inequality follows from the last part of the hint. This concludes the proof of statement (c).

4. **Reversible primitive row-stochastic matrices.** [Textbook E5.4] Let A be a primitive row-stochastic $n \times n$ matrix and w be its left dominant eigenvector (i.e. the left eigenvector associated with the dominant eigenvalue). The matrix A is *reversible* if

$$w_i A_{ij} = A_{ji} w_j, \quad \text{for all } i, j \in \{1, \dots, n\}, \tag{1}$$

or, equivalently

$$\text{diag}(w)A = A^T \text{diag}(w).$$

Prove the following statements:

(a) if A is reversible, then its associated digraph is undirected, that is, if (i, j) is an edge, then so is (j, i)

(b) if A is reversible, then $\text{diag}(w)^{1/2} \cdot A \cdot \text{diag}(w)^{-1/2}$ is symmetric and, hence, A has n real eigenvalues and n eigenvectors. Recall that, for $w = (w_1, \dots, w_n) > 0$, the following definitions hold: $\text{diag}(w)^{1/2} = \text{diag}(\sqrt{w_1}, ..., \sqrt{w_n})$ and $\text{diag}(w)^{-1/2} = \text{diag}(1/\sqrt{w_1}, ..., 1/\sqrt{w_n})$.

(c) If A is an equal-neighbor matrix for an unweighted undirected graph, then A is reversible. Using MatLab, verify this statement for the equal-neighbor matrix associated to the undirected graph $G = (V, E)$, $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (2, 3), (3, 4), (4, 2)\}$.

Solution:

(a) Since A is primitive, the Perron-Frobenius Theorem implies that its left dominant eigenvector w is entry-wise positive. But then equation (1) implies that $A_{ij} > 0$ if and only if $A_{ji} > 0$.

(b) Define $B = \text{diag}(w)^{1/2} A(w)^{-1/2}$ and note that $B_{ij} = \frac{\sqrt{w_i}}{\sqrt{w_j}} A_{ij}$. In turn, from equation (1), we compute

$$B_{ij} = \frac{\sqrt{w_i}}{\sqrt{w_j}} \frac{A_{ji} w_j}{w_i} = \frac{\sqrt{w_j}}{\sqrt{w_i}} A_{ji} = B_{ji}.$$

This establishes that B is symmetric so that B has n real eigenvalues and n orthogonal eigenvectors. Since B and A are similar, and a similarity transformation does not alter the eigenvalues, A has real eigenvalues as well. Moreover, each eigenvector v of B gives rise to an eigenvector $\text{diag}(w)^{-1/2}v$ for A because

$$Bv = \lambda v \Rightarrow A\text{diag}(w)^{-1/2}v = \lambda \text{diag}(w)^{-1/2}v.$$

(c) For the solution, see MatLab file on moodle.