Networked Control Systems (ME-427)- Exercise session 10

Prof. G. Ferrari Trecate

1. Consider the linear averaging algorithm

$$x_i^+ := \text{average}(x_i, \{x_j, \text{for all neighbor nodes } j\}). \tag{1}$$

Set $n = 5$, select the initial state equal to $(1, -1, 1, -1, 1)$, and use the following undirected unweighted graphs (depicted in figure):

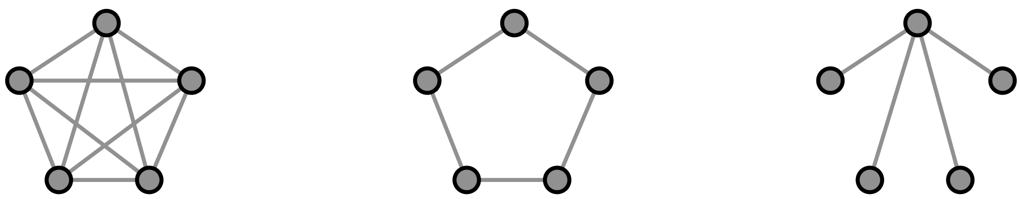
(a) the complete graph,

(b) the ring graph, and

(c) the star graph with node 1 as center.

i. In MatLab, compute the consensus values using the results seen in the lecture.

ii. Verify the results through simulations (use the code developed in the previous exercise sessions).



(a) Complete graph (b) Ring graph (c) Star graph

Solution: Download the MatLab code from Moodle.

2. **Discrete-time control of mobile robots.** Consider $n = 3$ robots moving on the line with positions $z_1, z_2, z_3 \in \mathbb{R}$. In order to gather at a common location (i.e., reach rendezvous), each robot heads for the centroid of its neighbors, that is,

$$\dot{z}_i = \frac{1}{n-1}\left(\sum_{j=1, j\neq i} z_j\right) - z_i.$$

Consider the Euler discretization of the above closed-loop dynamics with sampling rate $T > 0$:

$$z_i(k+1) = z_i(k) + T\left(\frac{1}{n-1}\left(\sum_{j=1, j\neq i} z_j(k)\right) - z_i(k)\right).$$

If $T \in [0, \frac{1}{2}]$, will rendezvous be guaranteed? If yes, to which position will the robots meet?

Solution:

$$z_i^+ = (1-T)z_i + \frac{T}{2}\sum_{j=1, j\neq 1} z_j.$$

Setting $z = [z_1\ z_2\ z_3]$, one has that $z_i^+ = Az$ where

$$A = \begin{bmatrix} 1-T & \frac{T}{2} & \frac{T}{2} \\ \frac{T}{2} & 1-T & \frac{T}{2} \\ \frac{T}{2} & \frac{T}{2} & 1-T \end{bmatrix}.$$

From the results seen in the lectures, consensus is guaranteed if A is primitive and doubly-stochastic. If $T \in (0,1)$, A is positive and hence primitive. Moreover, A is doubly stochastic by construction. Hence, from the results seen in the lectures, we have average consensus, i.e.,

$$\lim_{k\to\infty} z(k) = \left(\frac{1}{3}\sum_{i=1}^{3} z_i(0)\right).$$

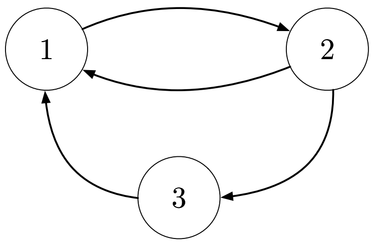
3. **The exponent of a primitive matrix.**

   (a) Let G be the digraph with nodes $\{1,\dots,3\}$ and edges $(1,2),(2,1),(2,3),(3,1)$. Explain if and why G is strongly connected and aperiodic.

   (b) Recall a non-negative matrix A is primitive if there exists a number k such that $A^k > 0$; the smallest such number k is called the *exponent* of the primitive matrix A. Using G in point (a), show that k can be larger than the number of nodes.

   (c) Provide an intuitive motivation for the result in point (b), by reasoning on the meaning of elements $(A^k)_{ij}$.

   Solution: Regarding statement (a), the digraph G contains the directed cycle $(1,2,3,1)$ of length 3, hence it is strongly connected. It also contains a cycle of length 2, that is, $(1,2,1)$, hence it is aperiodic.



   Regarding statement (b), the binary adjacency matrix of the digraph in statement (a) is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

   By statement (a), we know that A is primitive. It is easy to see that A^3 is not positive and that A^5 is the first positive power of A. Hence, the exponent is 5, which is strictly greater than $n = 3$.

   As for (c), the intuition is that, for reaching all other nodes from one node in exactly k steps, it might take longer than n hops. In the above digraph, with 3 hops one can not reach 3 from 1 then $(A^3)_{13} = 0$.

4. Let $A \in \mathbb{R}^{n\times n}$ be a non-negative matrix. Prove that if A is irreducible, then $\tilde{A} = (A + I)$ is primitive.

   Solution: Consider the associated graph G_A and $G_{\tilde{A}}$. $G_{\tilde{A}}$ is G_A with the addition of self-cycles to the nodes that have no self-cycle. (if the node v has a self-cycle in A, then $A_{vv} > 0$ and $A_{vv} + 1 > 0 \Rightarrow v$ has a self-cycle also in $\tilde{A}$). Since G is strongly connected, for arbitrary nodes i and j, $i \neq j$ one can reach i from j in no more than n hops. Thanks to self-cycles one can also reach i from j in exactly n hops $\Rightarrow A^n \succ 0$, i.e A is primitive.

5. **Leslie population model.** The Leslie model is used in population ecology to model the changes in a population of organisms over a period of time. In this model, the population is divided into n groups based on age classes; the indices i are ordered increasingly with the age, so that $i = 1$ is the class of the newborns. The variable $x_i(k), i \in \{1,\dots,n\}$, denotes the number of individuals in the age class i at time k; at every time step k the $x_i(k)$ individuals

- produce a number $\alpha_i x_i(k)$ of offsprings (i.e., individuals belonging to the first age class), where $\alpha_i \geq 0$ is a fecundity rate, and
- progress to the next age class with a survival rate $\beta_i \in [0, 1]$.

In the following, let $n = 3$. If $x(k)$ denotes the vector of individuals at time k, the Leslie population model reads

$$x(k+1) = Ax(k) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{bmatrix} \tag{2}$$

where A is referred to as the Leslie matrix. Assume $\alpha_i > 0$ for all $i \in \{1, \dots, 3\}$ and $0 < \beta_i \leq 1$ for all $i \in \{1, \dots, 2\}$.

(a) Prove that matrix A is primitive.

(b) Let $p_i(k) = \frac{x_i(k)}{\sum_{i=1}^{3} x_i(k)}$ denote the percentage of the total population in class i at time k. Call $p(k)$ the population distribution at time k. Compute by hand and in closed form $\lim_{k\to+\infty} p(k)$ as a function of the spectral radius $\rho(A)$ and the parameters $(\alpha_i, \beta_i), i \in \{1, \dots, 3\}$.

Solution:

(a) The graph associated with A is strongly connected and possesses a self loop in node 1. Therefore, the graph is aperiodic and, in turn, the matrix A is primitive.

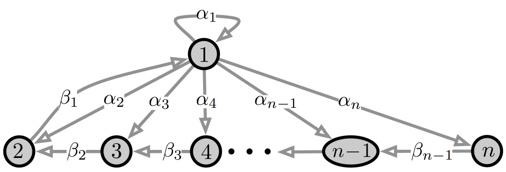


Figure 2: The associated graph for the general case of n groups. In our case, $n = 3$.

(b) The results shown in the lecture about convergence of $\left(\frac{A}{\rho(A)}\right)^k$ implies

$$\lim_{k\to+\infty} \frac{A^k}{\rho(A)^k} = vw^T,$$

where v is the right eigenvector associated with $\rho(A)$, w is the left eigenvector, and $w^T v = 1$. To compute the right eigenvector v, we write

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \rho(A) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \Rightarrow \quad \beta_i v_i = \rho(A) v_{i+1}, \ \forall i > 1,$$

so that we obtain

$$v = v_1 \begin{bmatrix} 1 \\ \frac{\beta_1}{\rho(A)} \\ \frac{\beta_1 \beta_2}{\rho(A)^2} \end{bmatrix}.$$

Note that

$$\begin{aligned} \lim_{k\to\infty} p(k) &:= \lim_{k\to\infty} \frac{A^k x(0)}{\mathbf{1}_n^T A^k x(0)} \\ &= \lim_{k\to\infty} \frac{\frac{A^k}{\rho(A)^k} x(0)}{\mathbf{1}_n^T \frac{A^k}{\rho(A)^k} x(0)} \\ &= \frac{v(w^T x(0))}{\mathbf{1}_n^T v(w^T x(0))} = \frac{v}{\mathbf{1}_n^T v}, \end{aligned}$$

where we used the fact that the limit of the ratio is the ratio of the limits, since the two limits are finite and strictly greater than 0. (Because the matrix A is primitive, v and w are positive for any $x(0) \neq 0$, we have $vw^T x(0) > 0$). Adopting the shorthand, $T := 1 + \frac{\beta_1}{\rho(A)} + \frac{\beta_1 \beta_2}{\rho(A)^2}$, we obtain

$$\lim_{k \to \infty} p(k) = \frac{1}{T} \begin{bmatrix} 1 \\ \frac{\beta_1}{\rho(A)} \\ \frac{\beta_1 \beta_2}{\rho(A)^2} \end{bmatrix}.$$