Exercise 1

Discrete-time systems and Lyapunov Theory

Giancarlo Ferrari Trecate[1]

[1]Automatic Control Laboratory
École Polytechnique Fédérale de Lausanne (EPFL), Switzerland
giancarlo.ferraritrecate@epfl.ch

Outline

- Linear Time Invariant (LTI) system in Discrete Time (DT)
  - Equilibria
  - Stability: definitions and test through eigenvalues
  - Stability test through Lyapunov functions
- DT Linear Time Varying (LTV) systems
  - Definitions of stability
  - DT linear switched systems: stability test through Lyapunov functions

Discrete-time (DT) linear systems

- $k \in \mathbb{N}$: discrete time

  LTV models:

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

  LTI models if A, B, C and D do not depend on k.

  - alternative notation:
    - $x_{k+1} = x(k+1)$
    - drop k and define $x^+ = x_{k+1}$

$x(k+1) = x(k)K$

$$x^+ = Ax + Bu \qquad\qquad x_{k_0} = x_0$$
$$y = Cx + Du$$

- Transition map $x_k = \phi(k, k_0, x_0, u)$
- For LTV models, the initial time k_0 of the experiment is important
- Superposition principle, Lagrange formula, free and forced states are given in the Appendix (very similar to the CT case)

Stability of equilibria of LTI systems

$$x^+ = Ax + Bu \qquad\qquad x(0) = x_0$$

- $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium if $\bar{x} = A\bar{x} + B\bar{u}$

$$(I - A)\bar{x} - B\bar{u} = 0$$

- $(\bar{x}, \bar{u}) = (0, 0)$ is always an equilibrium.

Definitions

$\forall \varepsilon \geq 0, \exists \delta > 0 : \|x(0) - \bar{x}\| < \delta \Rightarrow \|x(k) - \bar{x}\| < \varepsilon \quad \forall k \geq 0$

- Stability, AS, instability: same definitions in the CT case replacing t with k
- $\bar{x}$ is (globally) exponentially stable (ES) if there are $\alpha > 0, \rho \in [0, 1)$ such that

$$\|x(k) - \bar{x}\| \leq \alpha\rho^k \|x(0) - \bar{x}\|, \ \forall x(0) \in \mathbb{R}^n,$$

$\rho^k = e^{\beta k}$

and the constant β such that $\rho = e^\beta$ is the decay rate.

Stability - relevant properties

$$x^+ = Ax + Bu \qquad\qquad x(0) = x_0$$

For a linear systems, all equilibria have the same stability properties

- Focus on the stability of $(\bar{x}, \bar{u}) = (0, 0)$
- The whole system can be termed stable/AS/unstable/ES

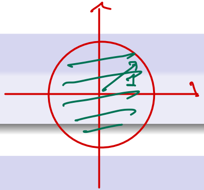
Theorem (stability and free states)

The above system is

- stable $\Leftrightarrow$ free states $x(k) = \phi(k, k_0, x_0, 0)$ are bounded $\forall x_0 \in \mathbb{R}^n$
- AS $\Leftrightarrow$ ES $\Leftrightarrow$ all the free states converge to zero $\forall x_0 \in \mathbb{R}^n$

$x(k+1) = A\,x(k)$, $x(0) = \tilde{x}$ → $x(1) = A\tilde{x}$, $x(2) = A^2\tilde{x}$, …, $x(k) = A^k\tilde{x}$

Stability test through the eigenvalues of A



Definition

A is Schur if all eigenvalues $\lambda \in \text{Spec}(A)$ verify $|\lambda| < 1$

Theorem (stability test)

An LTI system is

- AS $\Leftrightarrow$ if A is Schur
- unstable if there is $\lambda \in \text{Spec}(A)$ with $|\lambda| > 1$
- stable if all $\lambda \in \text{Spec}(A)$ verify $|\lambda| \leq 1$ and those verifying $|\lambda| = 1$ are simple.

Remark

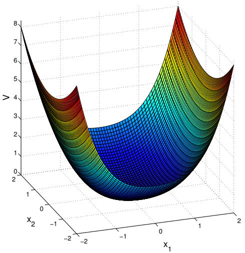
Similar to the continuous time case, multiple eigenvalues with $|\lambda| = 1$ can lead either to stability or instability.

Lyapunov stability theory

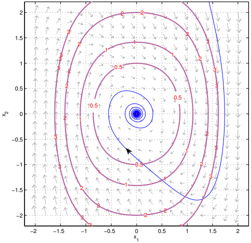
- We focus on **stability of the origin** for the LTI system $x^+ = Ax$
- Idea: if an energy-like function of the state decreases to zero, the origin is stable.
  - what is an energy function?

Lyapunov stability theory

Energy $V(x)$



(x_1, x_2)-plane



- $V(x)$ is a measure of the distance of x from the origin
  - If $V(x)$ can only decrease over time, then $\bar{x} = 0$ should be stable
- Next: make statements more rigorous!

Review: positive-definite matrices and quadratic functions

Definition

A symmetric matrix $M \in \mathbb{R}^{n \times n}$ is

(a) positive definite (pd) if $x \neq 0 \Rightarrow x^{\mathrm{T}} M x > 0$. Notation: $M > 0$

(b) positive semidefinite (psd) if $x^{\mathrm{T}} M x \geq 0$, $\forall x \in \mathbb{R}^n$. Notation: $M \geq 0$

(c) negative definite/semidefinite (nd/nsd) if $-M$ is pd/psd. Notation: $M < 0/M \leq 0$

Properties of the quadratic function $x^{\mathrm{T}} M x$

- A symmetric matrix M has real eigenvalues
- If $M > 0$, defining $\lambda_{min}(M)$ and $\lambda_{max}(M)$ as the minimum and maximum eigenvalue of M, respectively, one has

$$\lambda_{min}(M)\|x\|^2 \leq x^{\mathrm{T}} M x \leq \lambda_{max}(M)\|x\|^2$$

Energy forward difference

$$x^+ = Ax$$

Consider a quadratic energy-like function: $V(x) = x^T P x$, where $P \in \mathbb{R}^{n \times n}$ is symmetric and positive definite

$x(k+1)^T P x(k+1) = x^T(k) A^T P A x(k)$

- Compute $\Delta V(x) = V(x(k+1)) - V(x(k))$

$$\Delta V(x) = x^T A^T P A x - x^T P x = x^T (A^T P A - P) x$$

- We are sure that $\Delta V(x) \leq 0$ if

$$A^T P A - P \leq 0$$

Lyapunov theorems

Theorem 1: stability

The LTI system $x^+ = Ax$ is stable, if and only if there is $P > 0$ such that $A^T PA - P \leq 0$

Theorem 2 (AS/ES)

For the LTI system $x^+ = Ax$, the following statements are equivalent

(a) the system is ES

(b) for *an arbitrary symmetric matrix* $Q > 0$, there is a matrix $P^T = P > 0$ solving the Lyapunov equation

$$A^T PA - P = -Q$$

(c) there is $P = P^T > 0$ verifying $A^T PA - P < 0$.

Lyapunov theorems

Terminology

- $V(x) = x^T P x$ is a candidate Lyapunov function
- If $V(x)$ verifies one of the two theorems, it is a Lyapunov function

Remark

- $A^T P A - P = -Q$ is a system of linear equations in the elements of P, for a given Q
- $A^T P A - P \leq 0$ is a Linear Matrix inequality (LMI) in the elements of P - see next lecture!

Proof that (b) ⇒ (a)

The positive definiteness of Q implies that $\exists \gamma > 0$ verifying $-x^T Qx \leq -\gamma \|x\|^2$. For instance, one can choose $\gamma \in (0, \lambda_{min}(Q)]$. Similarly, $P > 0$ implies that

$$\lambda_{min}(P)\|x\|^2 \leq x^T Px \leq \lambda_{max}(P)\|x\|^2 \quad (1)$$

Step 1: using the forward difference, deduce how much V decreases.

From $\Delta V(x) = -x^T Qx$

$$\Delta V(x) \leq -\gamma \|x\|^2 \leq \frac{-\gamma}{\lambda_{max}(P)} x^T Px \leq \frac{-\gamma}{\lambda_{max}(P)} V(x) \quad (2)$$

which implies

$$V(x(k+1)) \leq \left(1 - \frac{\gamma}{\lambda_{max}(P)}\right) V(x(k)) \quad (3)$$

Since γ can be chosen arbitrarily small, select it such that $\rho^2 = 1 - \frac{\gamma}{\lambda_{max}(P)}$ verifies $\rho \in [0, 1)$.

Proof that (b) $\Rightarrow$ (a) (ctd.)

Step 2: iterate backwards to relate $V(x(k))$ to $V(x(0))$.
From

$$V(x(k+1)) \leq \rho^2 V(x(k)) \tag{4}$$

one has

$$V(x(k)) \leq \rho^{2k} V(x(0))$$

Step 3: use bounds on V to make states appear.
Using (1) and defining $m^2 = \frac{\lambda_{max}(P)}{\lambda_{min}(P)}$, one obtains

$$\|x(k)\| \leq m\rho^k \|x(0)\|$$

Proof that (a) $\Rightarrow$ (b) @ Home

$\Rightarrow$ For a given $Q > 0$, if A is Schur (which is guaranteed by ES), it can be shown that the Lyapunov equation has a solution $P = P^T$ given by

$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k = Q + A^T Q A + \dots$$

Show at home that this P fulfills the Lyapunov equation !
Since $Q > 0$ and $(A^T)^k Q A^k \geq 0,\ k \geq 1$, one has $P > 0$.

1h Ex1

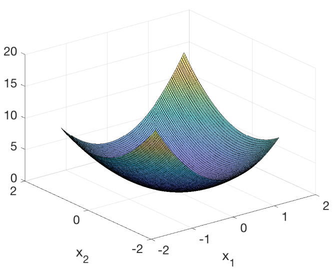
Example

$$x^+ = \begin{bmatrix} -0.81 & -0.09 \\ -0.45 & 0.63 \end{bmatrix} x \qquad \text{Spec}(A) = \{-0.8376, 0.6576\}$$

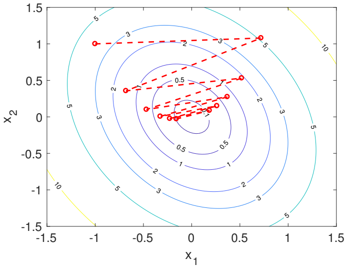
Set $Q = I$ and solve $A^T PA - P = -Q$ (`P=dlyap(A,Q)`)

$$P = \begin{bmatrix} 3.2661 & 0.7302 \\ 0.7302 & 2.0683 \end{bmatrix} \qquad \text{Spec}(P) = \{1.7728, 3.6116\}$$

Energy $V(x)$



Level sets



Why Lyapunov theory?

Much more flexible then the analysis of $\text{Spec}(A)$. Generalizes to

- nonlinear systems
- LTV systems - see next!

Moreover, Lyapunov theory allows to cast stability tests into optimization problems (see next lectures on LMIs)

Stability concepts for LTV systems

$$x(k+1) = A(k)x(k) + B(k)u(k) \qquad x(k_0) = x_0$$

How to define stability? Focus on $(\bar{x}, \bar{u}) = (0, 0)$

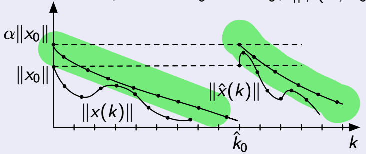
Definition:

The equilibrium $(\bar{x}, \bar{u}) = (0, 0)$ *is*

(1) stable if for all x_0 and k_0, $x(k) = \phi(k, k_0, x_0, 0)$ is bounded for $k \geq k_0$

(2) AS if $\forall x_0$ and k_0, $x(k) \to 0$ as $k \to +\infty$

(3) ES if $\exists \rho \in [0, 1)$ and $\alpha > 0$ such that, for all x_0 and k_0, $\|\phi(k, k_0, x_0, 0)\| \leq \alpha\rho^{k-k_0}\|x_0\|$



Abuse of language: "$x(k+1) = A(k)x(k)$ is AS", "A(k) is AS", etc..

Remarks

- α, ρ in (3) do not depend on k_0
- In (3), the constant $\beta \geq 0$ such that $\rho = e^{-\beta}$ is the decay rate
- AS $\not\Rightarrow$ ES (different from LTI system)

Example: for LTV systems AS $\not\Rightarrow$ ES

$$x(k+1) = A(k)x(k) \qquad A(k) = \left(\frac{k+1}{k+2}\right)^2$$

$$x(k_0) = x_0$$

Remark: $A(k) \to 1$ as $k \to +\infty$, implying slower and slower convergence rate

- Computations:

$$x(3) = \left(\frac{2+1}{2+2}\right)^2 x(2) = \frac{(2+1)^2}{(2+2)^2}\frac{(1+1)^2}{(1+2)^2}x(1) = \frac{(1+1)^2}{(2+2)^2}\frac{(0+1)^2}{(0+2)^2}x(0)$$

x_0:

$$x(k+1) = \left(\frac{k_0+1}{k+2}\right)^2 x_0 \Rightarrow \quad \text{AS since } x(k) \to 0, \forall x_0$$

- For studying ES, fix $k_0 = 0$, $x_0 = 1$. Assume that $\exists \alpha > 0, \rho \in [0,1)$ such that

$$\left(\frac{k_0+1}{k+2}\right)^2 x_0 = \left(\frac{1}{k+2}\right)^2 \leq \alpha\rho^k, \forall k \geq 0$$

- This implies

$$\frac{1}{\alpha} \leq (k+2)^2\rho^k$$

which is a contradiction because $(k+2)^2\rho^k \to 0$ as $k \to +\infty$

Discrete-time Linear Switched system

System with a finite set $\mathcal{I} = \{1, \cdots, M\}$ of modes of operation and a switching signal indicating the active mode at each time instant

$$x_{k+1} = A_{\sigma(k)} x_k \quad x_k \in \mathbb{R}^n \quad \sigma(k) \in \mathcal{I} \qquad (5)$$

$\mathcal{I} = \{1, 2\}$, $A_1 = 0.5$, $A_2 = 2$

$k_0 = 0$, $x_0 = 1$; $\sigma(0) = 1$, $\sigma(1) = 2$, $\sigma(3) = 2$; $x(1) = A_1 x_0 = 0.5$, $x(2) = A_2 x(1)$, $x(3) = A_2 x(2)$

$\sigma(\cdot)$ is an exogenous input

- For any fixed sequence $\sigma(0), \sigma(1), \ldots$, system (5) is LTV: stability = stability of the zero solution.

Definition

The switched system (5) is exponentially stable if for any sequence $\sigma(k)$ the resulting LTV system is ES. Equivalently, for all x_0, k_0 and $\{\sigma(k)\}_{k=k_0}^{+\infty}$

$$\exists \rho \in [0, 1) \text{ and } \alpha > 0 \text{ such that } \|\phi(k, k_0, x_0, 0)\| \leq \alpha \rho^{k-k_0} \|x_0\|$$

Remark

- For stability, it is not sufficient that all matrices $A_i, i \in \mathcal{I}$ are Schur (examples in the exercise sessions!)

Discrete-time Linear Switched system

System with a finite set $\mathcal{I} = \{1, \cdots, M\}$ of modes of operation and a switching signal indicating the active mode at each time instant

$$x_{k+1} = A_{\sigma(k)} x_k \quad x_k \in \mathbb{R}^n \quad \sigma(k) \in \mathcal{I} \tag{5}$$

Theorem

If there is $P \in \mathbb{R}^{n \times n}$, $P = P^T > 0$ such that

$$A_i^T P A_i - P < 0, \quad \forall i \in \mathcal{I},$$

then (5) is exponentially stable

$\mathcal{I} = \{1, 2\}$

$A_1^T P A_1 - P < 0$ AND

$A_2^T P A_2 - P < 0$

- $V(x) = x^T P x$ is a *common* Lyapunov function for all the modes
- The condition is only sufficient and implies all modes of operation are exponentially stable
- How to find P? By solving a Linear Matrix Inequality (LMI) optimization problem (see next lecture)

Appendix

Superposition principle (LTV system)

- The *same* as for CT linear systems
- For $\alpha, \beta \in \mathbb{R}$, let
  - $x_a(k) = \phi(k, k_0, x_{0,a}, u_a)$ and $y_a(k)$ the corresponding output
  - $x_b(k) = \phi(k, k_0, x_{0,b}, u_b)$ and $y_b(k)$ the corresponding output
  - $x(k) = \phi(k, k_0, \alpha x_{0,a} + \beta x_{0,a}, \alpha u_0 + \beta u_0)$ and $y(k)$ the corresponding output
- Then, $\forall k \geq k_0$
  - $x(k) = \alpha x_a(k) + \beta x_a(k)$
  - $y(k) = \alpha y_a(k) + \beta y_b(k)$

LTI systems: Lagrange formula

How the transition map looks like for an LTI system?

$$x(k+1) = Ax(k) + Bu(k) \qquad\qquad x(0) = x_0$$
$$y(k) = Cx(k) + Du(k)$$

We assume, for simplicity, the experiment starts at time $k_0 = 0$. One has

$$x(1) = Ax_0 + Bu_0$$
$$x(2) = Ax(1) + Bu(1) = A^2 x(0) + ABu(0) + Bu(1)$$
$$x(3) = Ax(2) + Bu(2) = \cdots = A^3 x(0) + A^2 Bu(0) + ABu(1) + Bu(2)$$

- $$x(k) = \phi(k, 0, x_0, u) = \underbrace{A^(k - k_0)x_0}_{\phi(k,0,x_0,0)=free\ \ state} + \underbrace{\sum_{i=0}^{k-1} A^{(k-i-1)} Bu(i)}_{\phi(k,0,0,u)=forced\ \ response}$$
- $$y(k) = \phi(k, 0, x_0, u) = \underbrace{CA^k x_0}_{free\ \ output} + \underbrace{C\sum_{i=0}^{k-1} A^{(k-i-1)} Bu(i) + Du(k)}_{forced\ \ output}$$
- Easy to generalize for $k_0 \neq 0$ and for LTV systems - just more complex