Ex 6: Nonlinear Model Predictive Control

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Problem 1

Consider a slot car racing track for which the curve of the track in a 2D plane is parameterized by $(x(\lambda), y(\lambda))$, for $\lambda \in \mathbb{R}$.

For any given λ we will say that the slot car is located at the point $(x(\lambda), y(\lambda)) \in \mathbb{R}^2$ on the curve and thus the position of the car on the track is entirely determined by λ, its velocity by $v = \dot{\lambda}$, and the state vector of the car is (λ, v).

The curvature of the track is given by the function $\kappa(\lambda)$ and the car is known to flip out from the track if at any $\lambda \in \mathbb{R}$, the speed v exceeds $\frac{1}{1+\kappa(\lambda)}$.

(x, y)

$\begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda \\ v \end{bmatrix}$

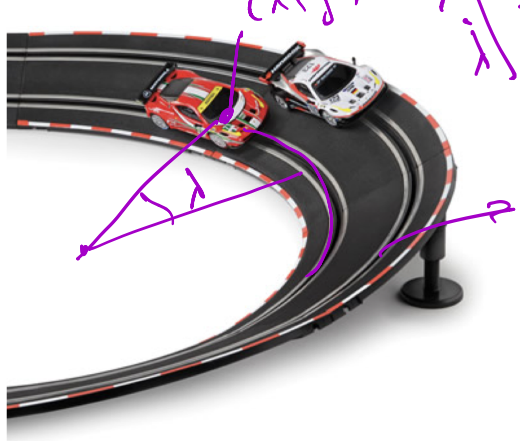


Figure 1: A slot car race.

Let the car have a forward acceleration input u_1, a brake u_2 and a viscous frictional force acting on it. The dynamics of the car can be written as

$$\begin{aligned} \dot{v} &= \gamma u_1 - \alpha u_2 v - \beta v^3 \\ \dot{\lambda} &= v \end{aligned} \tag{1}$$

Problem 1

Prob 1 | Integration / Discretization

a) Implement the RK4 and Forward Euler integrators and use them to define a corresponding $f_{discrete}$. Test your integrators from the initial condition $X(t_0) = (0, 0.5)$ and $U(t_0) = (0, -0.01)$ by computing $X(t_0 + h) = f_{discrete}(X_0(t_0), U(t_0)))$

b) For $X(0) = (0, 0.5)$ and a given input $U(t)$, simulate the movement of the car for 10 seconds using $f_{discrete}$ corresponding to your RK4 and Euler implementations for two cases $h = 0.1$ and $h = 0.5$. Plot and compare the integration errors for your trajectories with the ODE45 simulation given in the code template.

Problem 1(a)

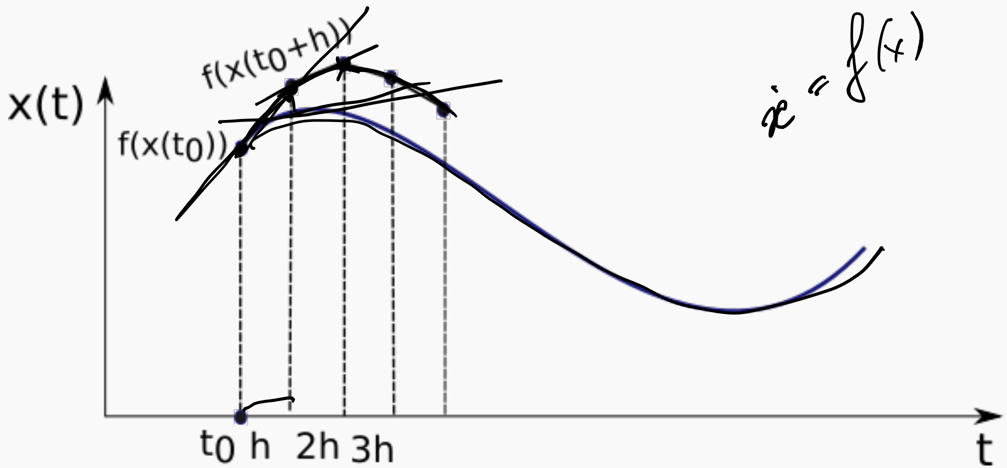
Task: Implement the explicit Euler and Runge-Kutta 4^{th} order integrators for ODE of the form: $\dot{x}(t) = f(x(t), u(t))$

Method1: Explicit Euler method

h

$$x(t+h) = \underline{x(t)} + h * f(x(t), u(t)) \; or$$

$$x_{k+1} = x_k + h * f(x_k, u_k)$$



$\dot{x} = f(x)$

Problem 1(a)

Task: Implement the explicit Euler and Runge-Kutta 4^{th} order integrators for ODE of the form: $\dot{x}(t) = f(x(t), u(t))$

Method2: RK4 method

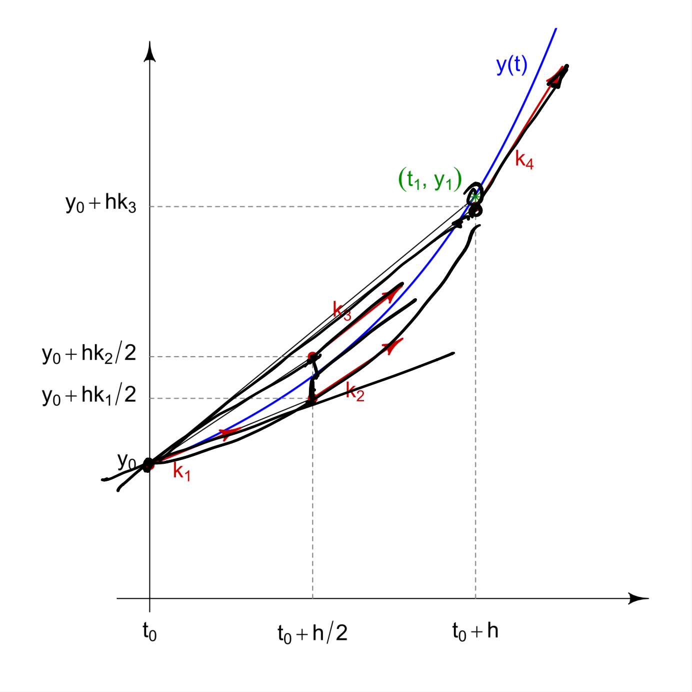
$$k_1 = f(t_k, x_k),$$

$$k_2 = f\left(t_k + \frac{h}{2}, x_k + h\frac{k_1}{2}\right),$$

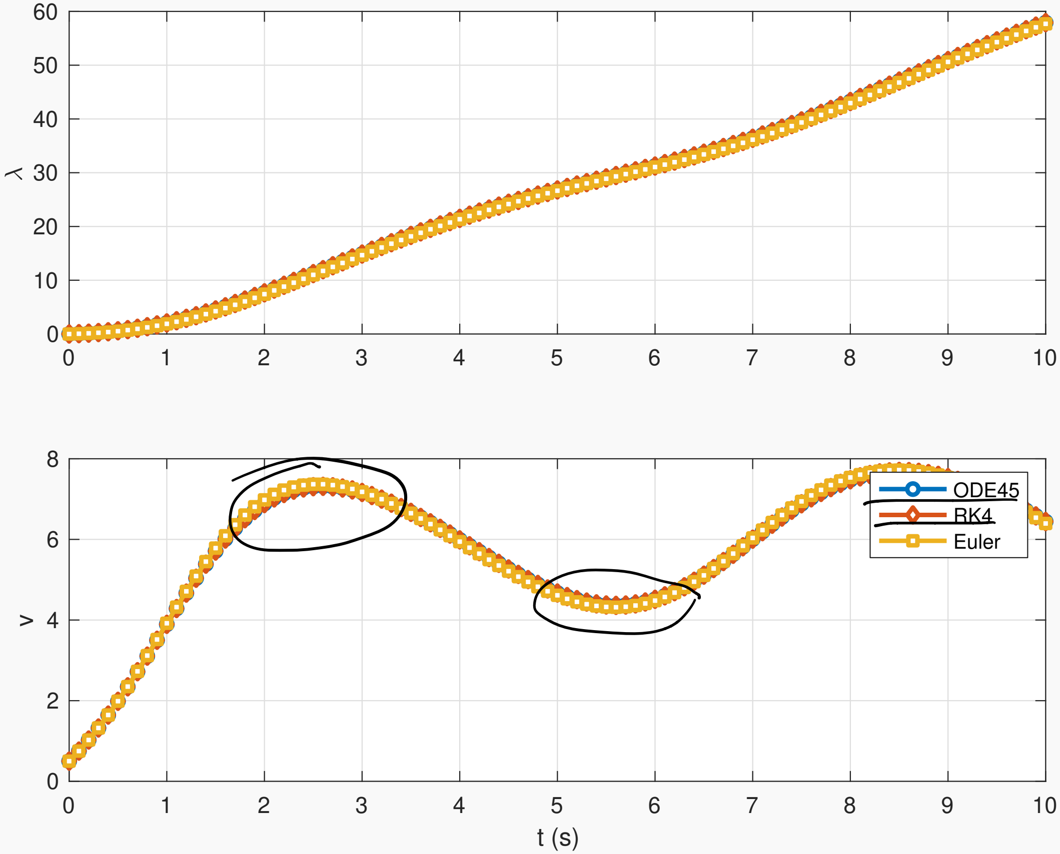
$$k_3 = f\left(t_k + \frac{h}{2}, x_k + h\frac{k_2}{2}\right),$$

$$k_4 = f(t_k + h, x_k + hk_3)$$

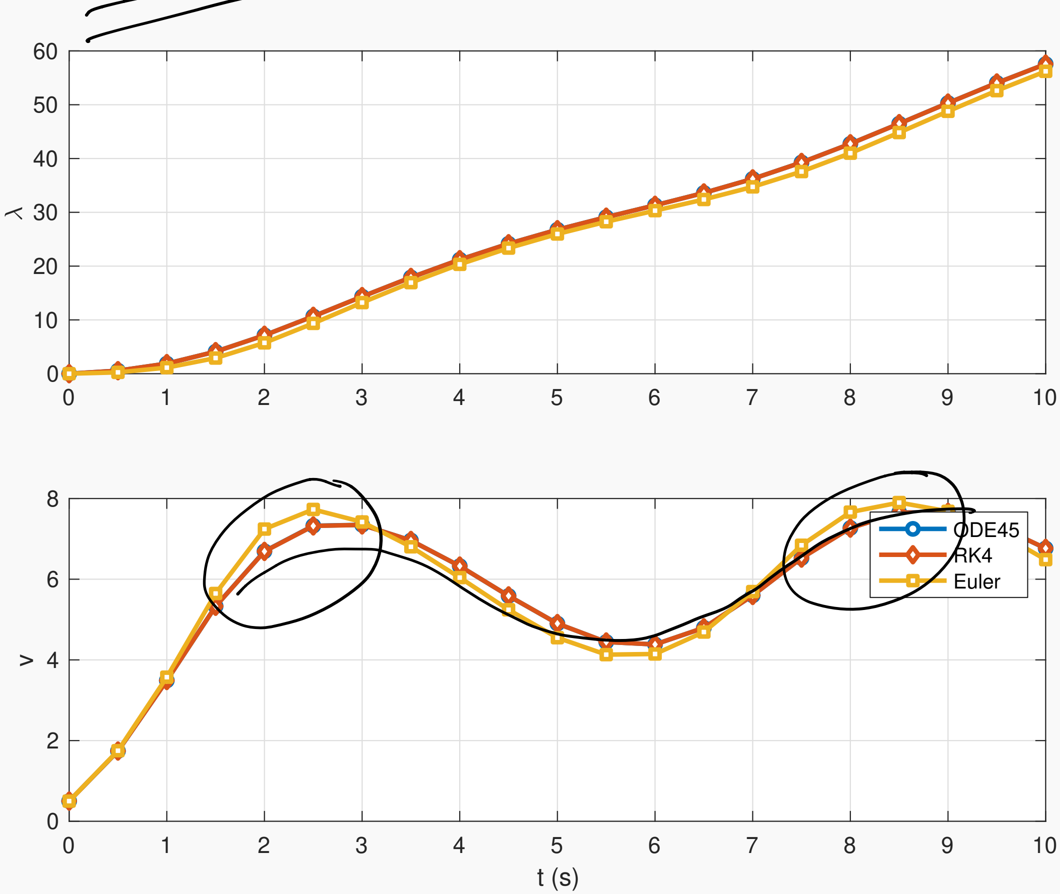
$$x_{k+1} = x_k + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$



Problem 1(b) : h = 0.1



Problem 1(b) : h = 0.5



Problem 2: Gradients

a) Write two functions `jac_x.m` and `jac_u.m` to compute the Jacobians $\nabla_X f_{discrete}$ and $\nabla_U f_{discrete}$ using a finite difference approximation.[1]

Compare the errors of finite difference approximation to the algorithmic differentiated Jacobians of your RK4 integrator provided in the template (take note of the syntax to find Jacobian and defining functions using casadi for future use).

b) Linearize $f_{discrete}$ around a point (X_0, U_0) using finite differences and algorithmic differentiation and write the linearized discrete time dynamics

$$f_{lin}(X, U) = \nabla_X f(X_0, U_0)(X - X_0) + \nabla_U f(X_0, U_0)(U - U_0) + f(X_0, U_0)$$

Simulate the linearized system using your RK4 integrator with $h = 0.5$. Compare the result to simulation of the nonlinear f using RK4.

Problem 2(a)

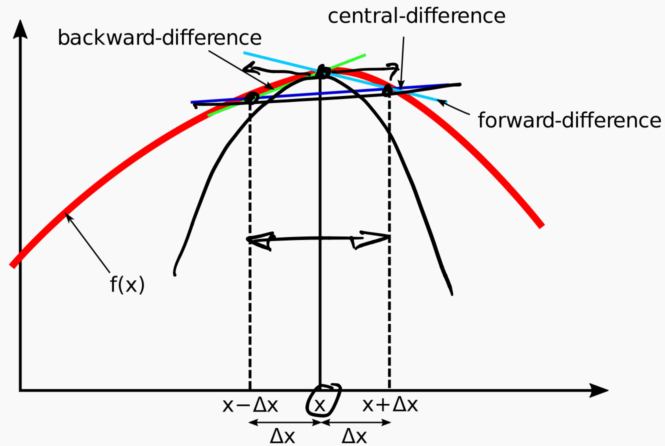
Task: Implement Jacobians $\nabla_x f_{discrete}$ and $\nabla_u f_{discrete}$

$$J = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Method: Central finite difference

$$f_x(x, u) \approx \frac{f(x + \delta, u) - f(x - \delta, u)}{2\delta}$$

$$f_u(x, u) \approx \frac{f(x, u + \delta) - f(x, u - \delta)}{2\delta}$$



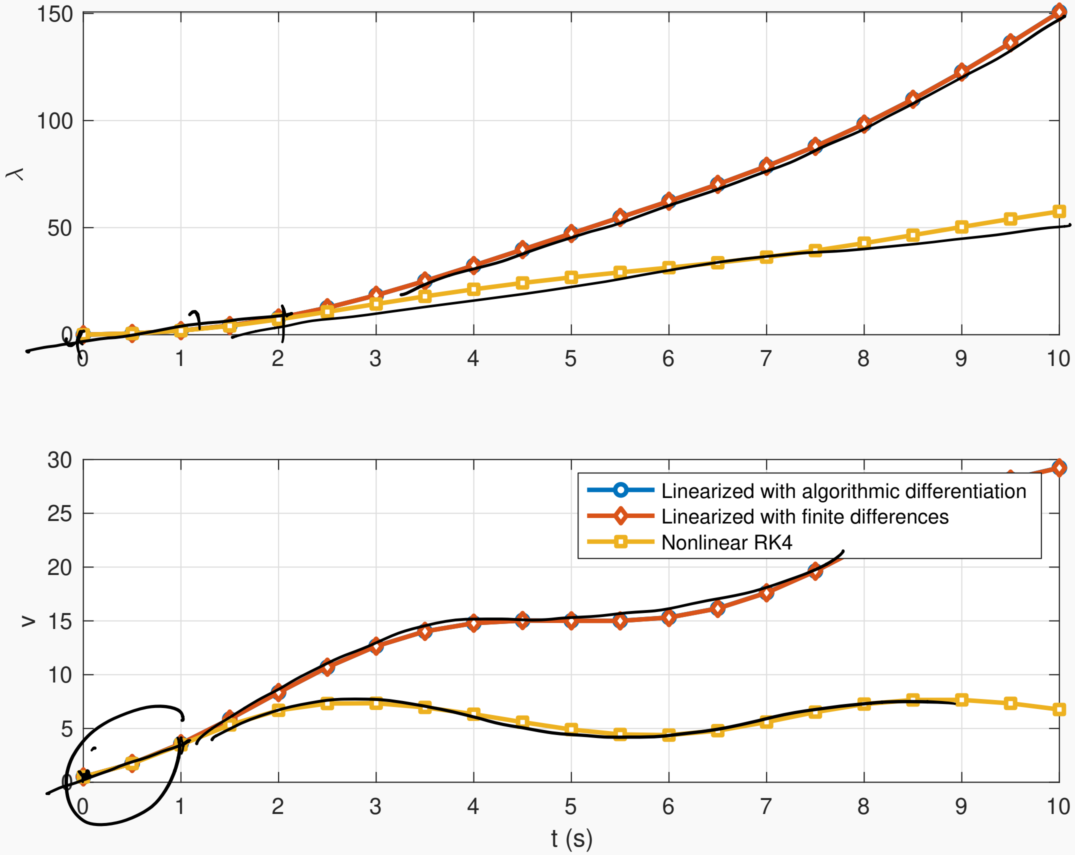
Problem 2(b)

Task: Compute linearised integrator: f_{discr_lin}

x_0, u_0

$$f_{discr\_lin}(x, u) = f_{discr}(x_0, u_0) + \nabla_x f_{discr}(x, u)\Big|_{x_0, u_0} * x + \nabla_u f_{discr}(x, u)\Big|_{x_0, u_0} * u$$

Problem 2: h = 0.5



Problem 3

$$\underset{\{x_k, u_k\}}{\text{minimize}} \; -\alpha v_N + \sum_{k=0}^{N-1} -\alpha v_k + \beta u_{1k}^2 + \gamma u_{2k}^2 + c\rho(\epsilon_k)$$

subject to: $\forall k = 0, ..., N-1$

$x_{k+1} = f_{discrete}(x_k, u_k), \quad x_k = [v_k, \lambda_k]^T, \quad u_k = [u_{1k}, u_{2k}]^T$

$v_k \leq \frac{1}{1+\kappa(\lambda_k)} + \epsilon_k$

$0 \leq u_{1k} \leq 1$

$0 \leq u_{2k} \leq 1$ ← braking

$\epsilon_k \geq 0$

$\rho(\epsilon_k) = \epsilon_k^2 + \epsilon_k$

RK4

acc

brake

Casadi

Bonus problem 1: exam 2015

1. $V(x) > 0, \quad x \in \mathbb{R} / \{0\}$
2. $V(0) = 0$
3. $\dot{V}(x) < 0$

$V(x^+) - V(x) < 0$

Given a linear system $x^+ = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ and a state-feedback control law $u = \begin{bmatrix} -3 & 1 \end{bmatrix} x$, which of the following is a Lyapunov function for the closed-loop system $x^+ = (A + BK)x$?

- [ ] $x^T x$
- [ ] $\cancel{\frac{\|x+3\|_\infty}{x^T x}}$
- [ ] $x^T \begin{bmatrix} 1 & 0.7 \\ -0.3 & 0.9 \end{bmatrix} x$
- [x] None of the above

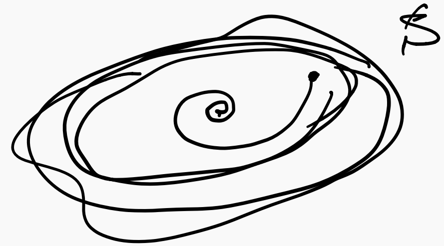
$x^+ = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} x$

$V(x^+) - V(x) \leq 0$

$V(x^T x) : x^T \underbrace{\left(\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}^T \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right)}_{< 0 \Rightarrow \det > 0} x \not< 0$

$x^T \underbrace{\left(\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}^T \begin{bmatrix} 1 & 0.7 \\ -0.3 & 0.9 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0.7 \\ -0.3 & 0.9 \end{bmatrix} \right)}_{\det \overset{?}{>} 0} x \not< 0$

Bonus problem 2: exam 2012



Which of the following statements implies that $S = \{x \mid x^T P x \leq 1\}$, $P \succeq 0$ is an invariant set for the system $x^+ = Ax$?

- [ ] $A^T P A \succeq P$
- [x] $A^T P A \preceq P$
- [ ] $A^T P A \succeq 0$
- [ ] $A^T P A \preceq 0$

$$x^+ \in S, \quad x \in S$$

$$x^{+T} P x^+ \leq x^T P x \leq 1$$

$$x^T A^T P A x \leq x^T P x$$

$$x^T \underbrace{\left[A^T P A - P\right]}_{\preceq 0} x \leq 0$$