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Problem 1.

When a question has more than one correct answer, mark all statements that are true.

1) Suppose that Y is a robust control invariant set for the system $x^+ = Ax + Bu + w$ subject to the state and input constraints X and U and the noise $w \in W$. For which of the following scenarios is Y still a robust invariant set for the resulting system?

○ Disturbance set changes to $2W$
● Disturbance set changes to $0.5W$
● Input constraint set changes to $2U$
○ Input constraint set changes to $U + 1$
○ State constraint set changes to $-X$

2) Consider the system $x^+ = Ax + u + w$ subject to the constraints $\|x\| \leq 1$ and $\|u\| \leq 1$ and a bounded disturbance w, $\|w\| \leq 0.1$. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1.2 \end{bmatrix}$, what is the set of feasible states of an open-loop robust MPC problem as the horizon goes to infinity?

● $\emptyset$
○ It won't converge
○ $\mathbb{R}^n$
○ A circle

3) Let $\pi(x) = K(x - z_0^\star(x)) + v_0^\star(x)$ be a tube-MPC control law, designed for the system $x^+ = Ax + Bu + w$ which is subject to the constraints $x \in X$, $u \in U$ and the disturbance $w \in W$. Let the set $\mathcal{E}$ be the minimum robust invariant set for the system $x^+ = (A + BK)x + w$ for $w \in W$. Suppose now that this controller is applied to the system but that the observed noise only lies within the set $\bar{W} = 0.5W$.

What is the set to which the closed-loop system will converge to in the limit?

○ Not enough information
○ The system does not converge
○ $\mathcal{E}$
○ The terminal set of the MPC control law
● $0.5\mathcal{E}$

4) Consider the system $x^+ = Ax + Bu$, which is subject to the constraints $x \in X$ and $u \in U$. Let $u^\star(x) = \text{argmin}\{\|u\| \mid Ax + Bu \in C_\infty,\ u \in U\}$, where C_∞ is the maximum control invariant set for this system. Let the state input sequence $\{x_i, u_i\}$ be generated by this system with $x_0 \in C_\infty$. Mark the correct statements.

● $x_i \in X$ and $u_i \in U$ for all i
○ There may exist an $x_0 \in C_\infty$ such that $x_i \notin X$ for some i
○ The system is asymptotically stable
○ $\lim_{i\to\infty} \|x_i\|_\infty = 0$
○ $\lim_{i\to\infty} \|u_i\|_1 = 0$

5) Consider the following optimization problem, which has a local optimizer at $x = x^\star$

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } x^T Q x + q^T x \geq 1 \end{aligned}$$

Which conditions guarantee that $x^\star$ is also a global minimum?

○ Q positive definite
○ Q positive semi-definite
● Q negative semi-definite
○ $c > 0$
○ $q > 0$

6) Consider the quadratic programming problem

$$\begin{aligned} p^\star = \min\ & \frac{1}{2} z^T H z + q^T z \\ \text{s.t.}\ & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} z \leq d \\ & \begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} z = b \end{aligned}$$

i) Is it possible to choose d and b so that the feasible set is empty?

● Yes
○ No
○ Depends on H and q

ii) Is it possible to choose d and b so that the feasible set is nonconvex?

○ Yes
● No
○ Depends on whether H is symmetric

iii) Let $z \in \mathbb{R}^2$. Which of the following H matrices results in a nonconvex optimization problem?

○ $\begin{bmatrix} 0.9 & 0 \\ 0 & 3 \end{bmatrix}$
○ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
● $\begin{bmatrix} 0.8 & 0 \\ 0 & -0.8 \end{bmatrix}$
○ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
○ None of the above

iv) Does there exist a bounded matrix H, a vector q and vectors d and b such that

$p^* = -\infty$ ● Yes ○ No ○ Not enough information

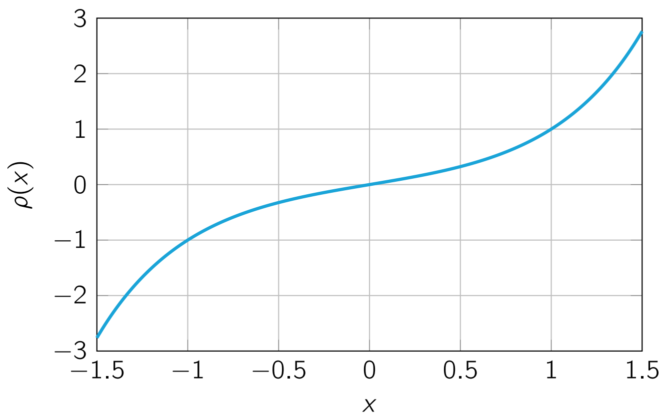
$p^* = \infty$ ○ Yes ● No ○ Not enough information

7) For which of the following cases is the optimization problem convex?

$$\min \; f(x) \quad \text{s.t. } g(x) \geq 5$$

- ○ $f(x) = x^T x - \|x\|_1$ $\qquad g(x) = x^T x$
- ○ $f(x) = -x^T x + \|x\|_1$ $\qquad g(x) = -x^T x$
- ○ $f(x) = \begin{bmatrix}1 & 2\end{bmatrix} x$ $\qquad g(x) = \|-4x\|_\infty$
- ○ $f(x) = \begin{bmatrix}1 & 2\end{bmatrix} x$ $\qquad g(x) = e^{-x_1} + e^{-x_2}$
- ● None of the above

8) What is the maximum invariant set of the scalar system $x^+ = \rho(x)$ contained in the set $X = \{x \,|\, \|x\|_\infty \leq 1.5\}$ where the function $\rho(x)$ is plotted below



- ● $\{x \,|\, -1 \leq x \leq 1\}$
- ○ X
- ○ $\{x \,|\, x \geq 0\}$
- ○ $\emptyset$
- ○ $\mathbb{R}$
- ○ None of the above

9) Consider the system $x^+ = 4x + 5 + u$ subject to the constraints $|u| \leq 5$ and $|x| \leq 1$.

i) What is the pre-set of the set $\Omega = \{x \,|\, \alpha \leq x \leq \beta\}$?

○ $[\alpha - 10, \beta]$ ○ $[\alpha, \beta + 5]$ ○ $\frac{1}{4}[\alpha - 10, \beta]$ ○ $\frac{1}{2}[-\alpha, -\beta]$ ○ $\mathbb{R}$

$\frac{1}{4}[\alpha - 10, \beta]$

ii) What is the maximum control invariant set?

○ $[-1, 1]$ ○ $\frac{1}{4}[-1, 1]$ ○ $\emptyset$ ○ $[-1, 0]$ ○ $\mathbb{R}$

$[-1, 0]$

10) Consider the system $x^+ = 2x + 1 + u$ subject to the constraints $u \leq 0$. Which of the following sets are control invariant?

- ● $\{0, 1, 2\}$
- ● $\{x \,|\, \|x\| \leq 1\}$
- ● $\{x \,|\, x \leq 0\}$
- ● $\{x \,|\, x \geq 0\}$
- ● $\mathbb{R}$

11) Let x_0 be a state in the maximum control invariant set $\mathcal{C}_\infty$ for the system $x^+ = f(x, u)$ under the constraints $(x, u) \in \mathbb{X} \times \mathbb{U}$. Is it possible that there exists a controller $\kappa(x)$ such that the 3-step sequence $\{x_0, x_1, x_2\}$ is in $\mathbb{X}$ and $\{\kappa(x_0), \kappa(x_1)\}$ is in $\mathbb{U}$, where $x_1 = f(x_0, \kappa(x_0))$ and $x_2 = f(x_1, \kappa(x_1))$?

○ Yes ○ No

yes

12) Consider the following standard MPC problem which generates the control law $\pi_N(x)$

$$
\begin{aligned}
J_N^\star(x) = \min \; & \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& (x_i, u_i) \in X \times U \\
& x_N = 0
\end{aligned}
$$

i) Assume for all of the following cases that the state x is feasible and non-zero and mark the correct statements.

- ● $J_N^\star(x) < J_{N-1}^\star(x)$
- ○ $J_{N-1}^\star(x) < J_N^\star(Ax + B\pi_N(x))$
- ● $J_{N-1}^\star(x) > J_N^\star(Ax + B\pi_{N-1}(x))$
- ○ $J_N^\star(Ax + B\pi_{N-1}(x)) = J_{N-1}^\star(x)$
- ● $J_{N-1}^\star(Ax + B\pi_N(x)) = J_N^\star(x) - l(x, \pi_N(x))$
- ○ $J_{N-1}^\star(Ax + B\pi_N(x)) < J_N^\star(x) - l(x, \pi_N(x))$

ii) Let $\{z_i^N(x)\}$ be the closed-loop sequence generated by the dynamic system $z^+ = Az + B\pi_N(z)$ starting at state x, and the function $\bar{J}_N(x) = \sum_{i=0}^{\infty} l(z_i^N(x), \pi_N(z_i^N(x)))$ be the resulting closed-loop cost. Mark all statements that are correct for all feasible, non-zero x

- ● $\bar{J}_N(x) \leq J_N^\star(x)$
- ○ $\bar{J}_N(x) \leq \bar{J}_{N-1}(x)$
- ○ $\bar{J}_N(x) \geq \bar{J}_{N-1}(x)$
- ● $\lim_{N\to\infty} \bar{J}_N(x) = \lim_{N\to\infty} J_{N-1}^\star(x)$
- ○ $\lim_{N\to\infty} \bar{J}_N(x) > \lim_{N\to\infty} J_{N-1}^\star(x)$
- ○ $\lim_{N\to\infty} \bar{J}_N(x) < \lim_{N\to\infty} J_{N-1}^\star(x)$

iii) If the state x is feasible for the MPC problem above and the controller $\pi_N(x)$ is applied, then the closed-loop system will arrive at the origin in exactly N steps.

○ True ○ False

false

13) A controller is called static if its output depends only on its input, and not on an internal state. Which of the controllers studied in class are static?

- ● Nominal MPC regulation
- ○ Offset-free MPC
- ● Tube-MPC
- ● Open-loop robust MPC

14) The function V_f is a Lyapunov function for the system $x^+ = Ax$, the stage cost l is positive definite and the MPC controller $\pi(x)$ defined by the following optimization problem is recursively feasible and stabilizes the closed-loop system $x^+ = Ax + B\pi(x)$ for a prediction horizon of $N = N_0$. Mark all statements that are correct.

$$\begin{aligned} \min \; & \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N) \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \quad \forall i = 0, \ldots, N-1 \\ & u_i \in U \quad \forall i = 0, \ldots, N-1 \end{aligned}$$

The closed-loop system...

- ◯ ... is stable for a prediction horizon $N > N_0$, but it may not be recursively feasible
- ● ... is stable and recursively feasible for any prediction horizon $N \geq N_0$
- ◯ ... may be stable for a prediction horizon $N < N_0$, but it may not be recursively feasible
- ● ... is stable and recursively feasible for any prediction horizon $N \leq N_0$
- ◯ ... is recursively feasible for a prediction horizon $N < N_0$, but may not be stable

15) Consider the following MPC problems

$$\begin{aligned} J^\star_{LQR}(x_0) = \min \; & \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N) \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in X, \; u_i \in U \\ & x_N \in X_{LQR} \end{aligned}$$

$$\begin{aligned} J^\star_{C_\infty}(x_0) = \min \; & \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N) \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in X, \; u_i \in U \\ & x_N \in C_\infty \end{aligned}$$

where $l(x, u) = x^T Qx + u^T Ru$ and $V_f(x)$ is the optimal value function of the corresponding LQR controller. X_{LQR} is the maximum invariant set for the system $x^+ = (A+BK)x$ for the LQR control law K subject to the constraints $x \in X$ and $Kx \in U$, and C_∞ is the maximum control invariant set for the system $x^+ = Ax + Bu$ subject to the constraints $x \in X$ and $u \in U$. If the problem is infeasible, we define the optimal value to be $+\infty$. Mark all the correct statements.

- ◯ $J^\star_{LQR}(x) \leq J^\star_{C_\infty}(x)$ for all x
- ● $J^\star_{LQR}(x) \geq J^\star_{C_\infty}(x)$ for all x
- ● $J^\star_{C_\infty}(x)$ is recursively feasible
- ◯ $J^\star_{C_\infty}(x)$ is a Lyapunov function for the system $x^+ = Ax + Bu^\star_{LQR}(x)$
- ◯ The domain of $J^\star_{LQR}$ contains the domain of $J^\star_{C_\infty}$
- ● The domain of $J^\star_{C_\infty}$ contains the domain of $J^\star_{LQR}$

16) Give an example of a system for which you cannot use soft state constraints. Explain.

17) Give an example of a system for which you cannot use soft input constraints. Explain.

18) Let $\pi(x)$ be the MPC control law defined by the following optimization problem

$$\begin{aligned} J^\star(x) = \min \ & \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N) \\ \text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\ & (x_i, u_i) \in X \times U \\ & x_N \in X_f \\ & x_0 = x \end{aligned}$$

where X, X_f and U are polyhedral sets, and l and V_f are positive definite quadratic functions that are zero at zero.

i) The control law $\pi(x)$ ...

- [ ] ... is a convex function
- [x] ... is a piecewise affine function
- [ ] ... is an affine function
- [ ] ... is a piecewise quadratic function
- [ ] ... is a quadratic function
- [x] ... has a polyhedral domain

ii) The value function $J^\star(x)$ ...

- [x] ... is a convex function
- [ ] ... is a piecewise affine function
- [ ] ... is an affine function
- [x] ... is a piecewise quadratic function
- [ ] ... is a quadratic function
- [x] ... has a polyhedral domain

19) Two infinite horizon LQR controllers are designed with weighting matrices Q_1, R_1 used for the first one and Q_2, R_2 used for the second one. It holds that $Q_1 = 10Q_2$ and $R_1 = 0.1R_2$. Which of the following statements is true?

- [ ] The first LQR controller will tend to use less input than the second one and will thus have slower convergence to the origin.
- [x] The second LQR controller will tend to use less input than the first one and will thus have slower convergence to the origin.
- [ ] The speed of response depends only on the choice of the Q matrix.
- [ ] None of the above

20) Consider the following finite-horizon LQR controller $u_0^\star$ applied to the system $x^+ = Ax + Bu$ in a receding horizon fashion.

$$\min \sum_{i=0}^{N-1} x_i^T x_i + u_i^T u_i$$
$$\text{s.t. } x_{i+1} = Ax_i + Bu_i$$

Mark the correct statements:

- ○ If the system is stable for $N = 5$, then it will also be stable for $N = 6$
- ○ The system will be stable if N is at least five times larger than the largest eigenvalue of A
- ○ The performance of the system will be better for $N = 6$ than it will be for $N = 5$
- ● None of the above

21) $V(x) = x^T x$ is a Lyapunov function for which of the following systems?

- ○ $x^+ = x + u$
- ○ $x^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$
- ○ $x^+ = \begin{bmatrix} 1.4 & 0 \\ 1 & 0.1 \end{bmatrix} x$
- ● $x^+ = 0.6x$
- ○ None of the above

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Problem 2.

Consider the finite-horizon optimal control problem

$$V_N^\star(x_0) = \min \sum_{i=0}^{N} l(x_i, u_i) \quad \text{s.t.} \quad x_{i+1} = 3x_i + u_i$$

where the stage cost is

$$l(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

and let $\kappa_N(x)$ be the resulting MPC controller.

1) If the cost-to-go is $V_{i+1}(x) = x^T H x$, give an expression for $\kappa_i(x)$ and $V_i(x)$ as functions of H.

$\kappa_i(x) = -\frac{1+3H}{2+H}x$

$V_i(x) = \frac{1+13H}{2+H}$

$$\begin{aligned} J(x,u;H) &= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + (3x+u)^2 H \\ &= x^2 + 2u^2 + 2xu + H9x^2 + Hu^2 + 6xuH \\ &= x^2(1+9H) + u^2(2+H) + u(2x + 6Hx) \end{aligned}$$

Take the derivative and set to zero

$$\begin{aligned} \frac{\partial J}{\partial u} &= 2u(2+H) + 2x + 6Hx = 0 \\ u &= -\frac{1+3H}{2+H}x \end{aligned}$$

Plug the control law into the cost to get the optimal value function

$$\begin{aligned} J(x, -\frac{1+3H}{2+H}x; H)/x^2 &= (1+9H) + (-\frac{1+3H}{2+H})^2(2+H) + (-\frac{1+3H}{2+H}x)(2x+6Hx) \\ &= (1+9H) + \frac{(1+3H)^2}{(2+H)^2}(2+H) - \frac{1+3H}{2+H}(2+6H) \\ &= \frac{(1+9H)(2+H)}{2+H} + \frac{(1+3H)^2}{2+H} - 2\frac{(1+3H)^2}{2+H} \\ &= \frac{(1+9H)(2+H)}{2+H} - \frac{(1+3H)^2}{2+H} \\ &= \frac{(1+9H)(2+H) - (1+3H)^2}{2+H} \\ &= \frac{2 + H + 18H + 9H^2 - 1 - 9H^2 - 6H}{2+H} \\ &= \frac{1+13H}{2+H} \end{aligned}$$

2) Compute the smallest horizon N° and the controller $\kappa_{N^\circ}(x)$ such that the closed-loop system $x^+ = Ax + B\kappa_{N^\circ}(x)$ is asymptotically stable.

$N^\circ =$ 2 $\qquad\qquad$ $\kappa_{N^\circ}(x) =$ $-2.5x$

Solve the first step:

$$V_0^\star(x_0) = \min_{u_0} \; x_0^2 + 2u_0^2 + 2x_0u_0$$

Take derivative and set to zero:

$$4u_0 + 2x_0 = 0 \qquad \Rightarrow \qquad u_0 = -\frac{1}{2}x_0$$

$$\begin{aligned} V_0^\star(x_0) &= x_0^2 + 2(-\frac{1}{2}x_0)^2 + 2x_0(-\frac{1}{2}x_0) \\ &= x_0^2(1 + \frac{1}{2} - 1) = 1/2 \end{aligned}$$

Now we just iterate using the result from the previous question:

$$\begin{aligned} H_1(x) &= \frac{1+13H}{2+H} = \frac{1+13/2}{2+1/2} = \frac{15}{5} = 3 \\ H_2(x) &= \frac{1+13H}{2+H} = \frac{1+13\cdot 3}{2+3} = \frac{40}{5} = 8 \end{aligned}$$

And the controllers and closed-loop systems are

$$\begin{aligned} \kappa_0(x) &= -\frac{1}{2}x_0 & (A+BK) &= 3 - 0.5 = 2.5 \\ \kappa_1(x) &= -\frac{1+3H}{2+H}x = -\frac{1+3\cdot 3}{2+3}x = -\frac{10}{5}x = -2x & (A+BK) &= 3 - 2 = 1 \\ \kappa_2(x) &= -\frac{1+3H}{2+H}x = -\frac{1+3\cdot 8}{2+8}x = -\frac{25}{10}x = -2.5x & (A+BK) &= 3 - 2.5 = 0.5 \end{aligned}$$

3) Suppose now that the system is subject to input constraints $|u| \leq 1$ and consider the MPC controller below

$$\begin{aligned} \min \quad & \sum_{i=0}^{10} l(x_i, u_i) + V_{N^\circ}(x_{10}) \\ \text{s.t.} \quad & x_{i+1} = 3x_i + u_i \quad \forall i = 0, \dots, 9 \\ & |u_i| \leq 1 \qquad\qquad \forall i = 0, \dots, 9 \end{aligned}$$

What is the set of states $x_0 \in X_0$ for which this MPC controller has a solution?

$X_0 =$ $\mathbb{R}$

Will the closed-loop system $x^+ = 3x + u^\star(x)$ be asymptotically stable for all $x \in X_0$, where $u^\star(x)$ is the optimal answer of the above MPC problem?

◯ Yes ◯ No

If yes, prove it. If no, explain why.

No. The system is unstable with a bounded input. There is no such controller.

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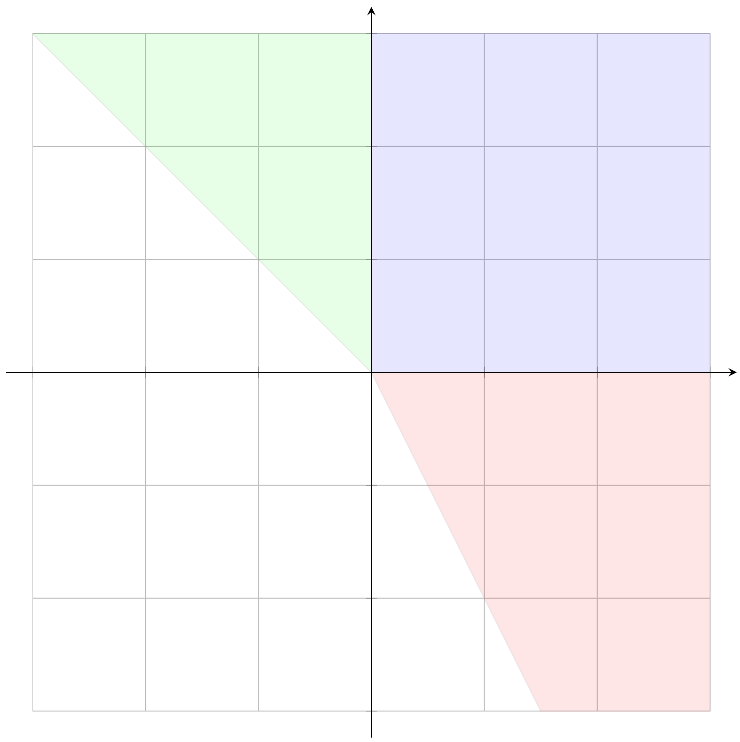
Problem 3.

Consider the following parametric LCP

$$
\begin{aligned}
w - \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} z &= Qx + q \\
w, z &\geq 0 \\
w^T z &= 0
\end{aligned}
$$

1) Give the complementarity cones for this problem and sketch them.



Enumerate all complementarity bases

$$z_1 = z_2 = 0 \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = q$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$CR_1 = \{\theta \,|\, \theta \geq 0\}$$

$$w_1 = w_2 = 0 \quad \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = q$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} q = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \theta + \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$CR_2 = \{\theta \,|\, \theta \leq -1\}$$

$$z_1 = w_2 = 0 \quad \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ z_2 \end{bmatrix} = q$$

$$\begin{bmatrix} z_2 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & -0.5 \end{bmatrix} q = \begin{bmatrix} 1 & 0.5 \\ 0 & -0.5 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

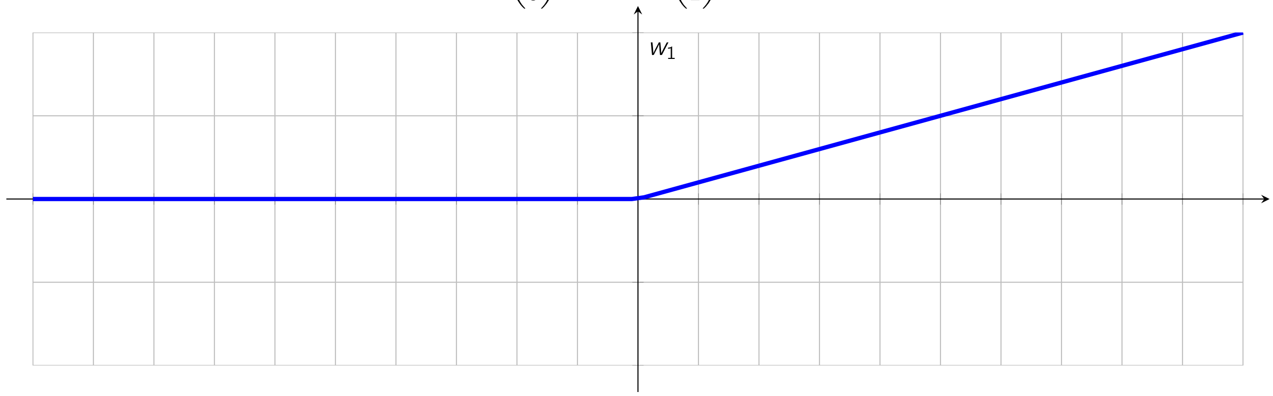
$$CR_3 = \emptyset$$

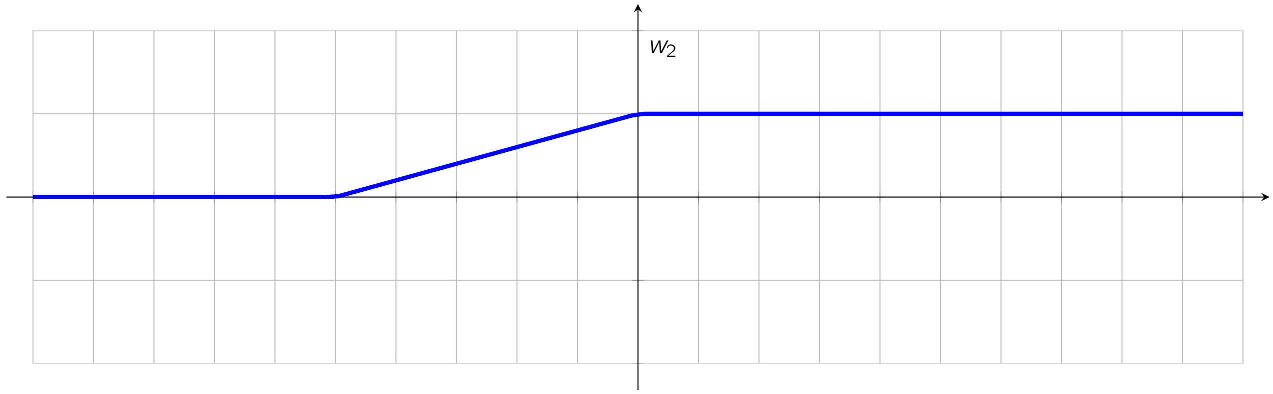
$$w_1 = z_2 = 0 \quad \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ w_2 \end{bmatrix} = q$$

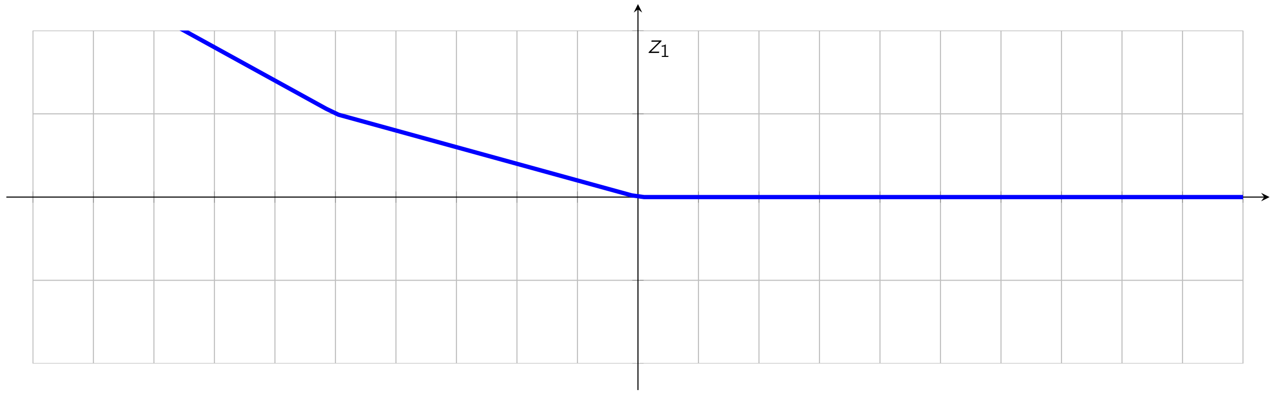
$$\begin{bmatrix} z_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} q = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \theta + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

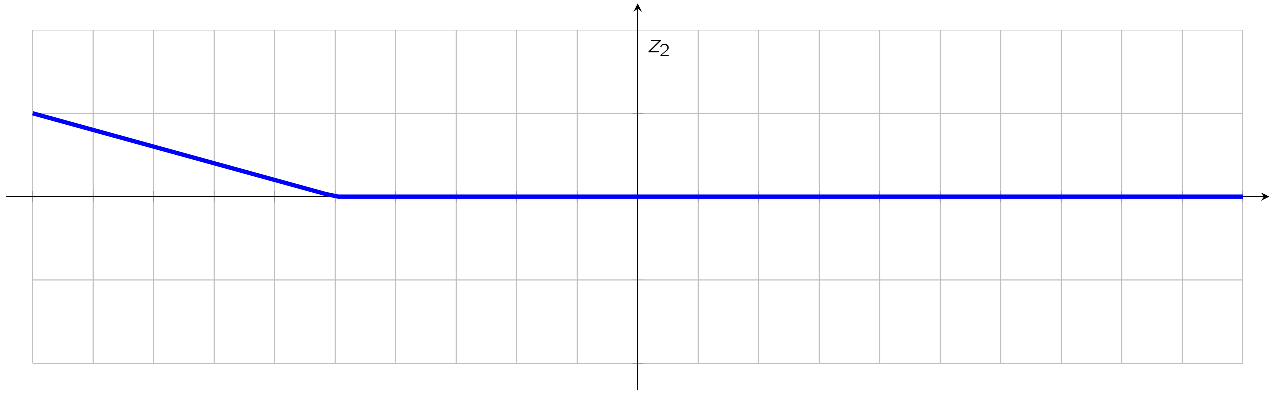
$$CR_4 = \{\theta \,|\, -1 \leq \theta \leq 0\}$$

2) Sketch the solution $w(x)$ and $z(x)$ for $Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.









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Problem 4.

Recall the following definitions

$X \oplus Y := \{x + y \mid x \in X,\ y \in Y\}$	Minkowski sum of X and Y
$\alpha X := \{\alpha x \mid x \in X\}$	Scaling of a set
If X convex: $x_1, x_2 \in X \Rightarrow \alpha x_1 + (1-\alpha)x_2 \in X\ \forall \alpha \in [0,1]$	Convex set

1) Let X_1 and X_2 be convex invariant sets for the system $x^+ = Ax$. Show that $\alpha X_1 \oplus (1-\alpha)X_2$ is also an invariant set for any $\alpha \in [0,1]$.

Want to show $x^+ \in \alpha X_1 \oplus (1-\alpha)X_2$ for all $x \in \alpha X_1 \oplus (1-\alpha)X_2$.
If $x \in \alpha X_1 \oplus (1-\alpha)X_2$, then there exists an $x_1 \in X_1$ and $x_2 \in X_2$ such that $x = \alpha x_1 + (1-\alpha)x_2$. From linearity, we have that $x^+ = \alpha A x_1 + (1-\alpha)Ax_2$. Invariance of X_1 and X_2 imply that $Ax_1 \in X_1$ and $Ax_2 \in X_2$ and convexity gives the result.

2) Let $X_1 \subseteq \mathbb{X}$ and $X_2 \subseteq \mathbb{X}$, where X_1, X_2 and $\mathbb{X}$ are convex sets. Show that $\alpha X_1 \oplus (1-\alpha)X_2 \subseteq \mathbb{X}$ for any $\alpha \in [0,1]$.

If $x \in \alpha X_1 \oplus (1-\alpha)X_2$, then there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that $x = \alpha x_1 + (1-\alpha)x_2$. Convexity then gives the result.

3) Let $V_i(x) := x^T P_i x$ be a Lyapunov function for the system $x^+ = Ax$ for $i = 1,2$, with a rate of decrease of $x^T \Gamma x$, i.e.:

$$V_i(x^+) - V_i(x) \leq -x^T \Gamma x\ .$$

Show that $V(x) = \alpha V_1(x) + (1-\alpha)V_2(x)$ is also a Lyapunov function with a rate of decrease of $x^T\Gamma x$ for any $\alpha \in [0,1]$.

$$\begin{aligned} V(x^+) - V(x) &= \alpha x^T A^T P_1 A x + (1-\alpha)x^T A^T P_2 A x - \alpha x^T P_1 x - (1-\alpha)x^T P_2 x \\ &\leq -\alpha x^T \Gamma x - (1-\alpha) x^T \Gamma x \\ &= -x^T \Gamma x \end{aligned}$$

4) Let K be a stabilizing controller for the system $x^+ = Ax + Bu$, and $X_i \subset \mathbb{X}$ be a convex invariant set for the system $x^+ = (A + BK)x$, with $KX_i \subset \mathbb{U}$ for each $i = 1, 2$. $V_i(x) = x^T P_i x$ are Lyapunov functions for the system $x^+ = (A + BK)x$ with a rate of decrease of $Q + K^T RK$, for some $Q = Q^T \succ 0$ and $R = R^T \succ 0$.

$$
\begin{aligned}
J^\star(x(t)) = \min \sum_{i=0}^{N} & x_i^T Q x_i + u_i^T R u_i + \alpha V_1(x) + (1-\alpha) V_2(x) \\
\text{s.t. } & x_{i+1} = Ax_i + Bu_i,\ i = 0, \dots, N \\
& x_i \in \mathbb{X},\ i = 1, \dots, N \\
& u_i \in \mathbb{U},\ i = 0, \dots, N-1 \\
& x_N \in \alpha X_1 \oplus (1-\alpha) X_2 \\
& x_0 = x(t)
\end{aligned}
$$

Prove that this MPC controller is stabilizing and recursively feasible for any $\alpha \in [0, 1]$ by

i) listing sufficient conditions for stability and

ii) proving them

Hint: You can use the results of the previous three questions, even if you couldn't answer them.

- The stage cost is a positive definite function
  Q and R are positive definite.
- The terminal set is invariant under the local control law $\kappa_f(x) = Kx$
  Part a) proved this
- All state and input constraints are satisfied in X_f
  Part b) proved this
- The terminal cost is a Lyapunov function in the terminal set X_f
  Part c) proved this

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Problem 5.

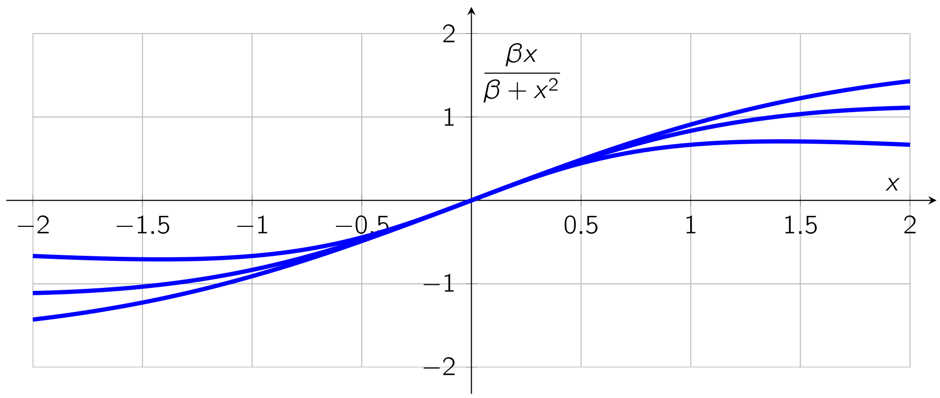
Consider the following nonlinear discrete-time dynamic system.

$$x^+ = \frac{\beta x}{\beta + x^2} + u := f(x,u)$$

subject to the state and input constraints $X = \{x \,|\, |x| \leq 1\}$ and $U = \{u \,|\, |u| \leq 0.2\}$. Treat β symbolically as a constant $\beta > 1$.

Your goal is to design a robust **linear** MPC controller for this nonlinear system to regulate it to the origin.

The function $\frac{\beta x}{\beta + x^2}$ is shown in the plot below for $\beta \in \{2, 5, 10\}$.



1) Find values a and b by linearizing the system around the origin and the smallest $\bar{w}$ such that the evolution of the following uncertain linear dynamic system contains that of the nonlinear system above for all $x \in X$ and all $u \in U$.

$$x^+ = ax + bu + w \qquad w \in W = \bar{w} \cdot [-1, 1] \tag{1}$$

$a =$ 1 $b =$ 1 $\bar{w} = \frac{1}{\beta + 1}$

Hint: Ensure that $\forall (x,u) \in X \times U \; \exists w \in W$ such that $ax + bu + w = f(x,u)$

We linearize the system:

$$a = 1$$
$$b = 1$$

We see that the maximum error will occur at the boundary $x = 1$, so we get that $\bar{w} = ax - f(x,0) = 1 - \frac{\beta}{\beta+1} = \frac{\beta+1-\beta}{\beta+1} = \frac{1}{\beta+1}$.

2) Compute the maximum robust invariant set X_f for the linear system in the previous part within the constraint set X for the control law $u = -0.5x$. Use the value $\bar{w} = 0.1$ for this question.

$X_f =$ $[-0.4, 0.4]$

Our linearized system is $x^+ = x - 0.5x + w$, for $|w| \leq 0.1$.
We compute the robust pre-set of an interval $[-\alpha, \alpha]$

$$\begin{aligned} \text{pre}([-\alpha,\alpha]) &= \{x \mid -\alpha \leq 0.5x + w \leq \alpha \ \forall |w| \leq 0.1\} \\ &= \{x \mid -\alpha + 0.1 \leq 0.5x \leq \alpha - 0.1\} \\ &= \{x \mid -2(\alpha - 0.1) \leq x \leq 2(\alpha - 0.1)\} \\ &= [-2(\alpha - 0.1), 2(\alpha - 0.1)] \end{aligned}$$

Our set is $\Omega_0 = X \cap U = [-1, 1] \cap \{x | |-0.5x| \leq 0.2\} = [-0.4, 0.4]$
The pre-set of Ω_0 is

$$\text{pre}(-0.4, 0.4) = [-2(0.4 - 0.1), 2(0.4 - 0.1)] = [-0.4, 0.4] = \Omega_0$$

3) Fill in the blanks in the following set so that any controller $u(x) \in C(x)$ will ensure robust constraint satisfaction for system (1). Use the value $\bar{w} = 0.1$ for this question.

$$C(x) := \left\{ u_0 \;\middle|\; \exists u_1 \begin{pmatrix} \; \\ \; \\ \; \\ \; \end{pmatrix} \leq \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ x \end{pmatrix} \leq \begin{pmatrix} \; \\ \; \\ \; \\ \; \end{pmatrix} \right\}$$

This is just two-stage open-loop robust MPC in dense formulation.

$$\begin{aligned} x_1 &= x_0 + u_0 + w_0 \in X \\ x_2 &= x_0 + u_0 + w_0 + u_1 + w_1 \in X_f \\ |u_0| &\leq 0.2 \\ |u_1| &\leq 0.2 \end{aligned}$$

$$\left\{ u_0 \;\middle|\; \exists u_1 - \begin{pmatrix} 0.9 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix} \leq \begin{bmatrix} 1 & & 1 \\ 1 & 1 & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ x \end{pmatrix} \leq \begin{pmatrix} 0.9 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix} \right\}$$

4) Define the controller $u^\star(x) = \text{argmin}_{u \in C(x)} u^T u$, where $C(x)$ is the function from the previous part. Will the dynamic system $x^+ = f(x, u^\star(x))$ satisfy the constraints $x \in X$ and $u \in U$? Explain your answer.

● Yes ○ No ○ Not enough information

Yes. The controller robustly satisfies the constraints of system (1) and we designed this uncertain system to contain the dynamics of the nonlinear system. As a result, the constraints of the nonlinear system are also satisfied.

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Problem 6.

Consider the following scalar system, which has the nonlinearity γ

$$\dot{x} = f(x,u) = x + u + \gamma(x) \tag{2}$$

subject to the constraints $|x| \le 25$ and $-1 \le u \le 15$.

1) Discretize the system $x^+ = \phi(x,u)$ with a sample period of 1 second using Runge-Kutta 2. Assume that during the sample period, the input is constant, i.e., $u(t) = u_i$ for all $t \in [hi, h(i+1)]$ for $i \in \{0, 1, 2, \dots\}$.

$$x^+ = \phi(x,u) \approx 2.5x + 1.5u + \gamma(x) + 0.5\gamma(2x + u + \gamma(x))$$

RK2 integration is:

$$k_1 = f(x,u) = x + u + \gamma(x)$$

$$\begin{aligned}
x + hk_1 &= x + x + u + \gamma(x) = 2x + u + \gamma(x) \\
k_2 &= f(x + hk_1) = 2x + u + \gamma(x) + u + \gamma(2x + u + \gamma(x)) \\
x^+ &\approx x + \frac{h}{2}k_1 + \frac{h}{2}k_2 \\
&= x + \frac{h}{2}(x + u + \gamma(x)) + \frac{h}{2}(2x + u + \gamma(x) + u + \gamma(2x + u + \gamma(x))) \\
&= 2.5x + 1.5u + \gamma(x) + 0.5\gamma(2x + u + \gamma(x))
\end{aligned}$$

2) Linearize the discrete-time system around the point $x_s = 0$ and $u_s = 0$ and for $\gamma(x) = \sin(x)$

$$x^+ \approx 5x + 2u$$

Compute Jacobian of ϕ with respect to x and u

$$\begin{aligned}
J_x &= 2.5 + \cos(0) + 0.5\cos(0)(2 + \cos(0)) = 5 \\
J_u &= 1.5 + 0.5\cos(0) = 2
\end{aligned}$$

3) Design a stabilizing and recursively feasible linear MPC controller for the linear discrete-time model computed in the last part.

Any standard MPC setup will do:

$$\begin{aligned}
\min \quad & \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{s.t.} \quad & x_{i+1} = 5x_i + 2u_i \\
& |x_i| \leq 25 \\
& -1 \leq u_i \leq 15 \\
& x_N = 0
\end{aligned}$$

4) If the input from your linear MPC controller is applied to the system (2), will the closed-loop system necessarily satisfy the constraints?

◯ Yes ◯ No

If yes, prove that this is the case.

If no, give suggestions for how you can modify your controller to ensure that it does.

No, it will not because the linearization and discretization are approximate.
One way to ensure that it does is to bound the difference between the linear discrete time system and the continuous time one, and then use robust MPC to compensate for this error.