Problem 1.

/25

When there are multiple choices in the following, select all statements that are true.

1. Consider the system $x^+ = 2x + 1 + u$ subject to the constraints $u \leq 0$. Which of the following sets are control invariant?

   ◯ $\{0, 1, 2\}$
   ◯ $\{x \mid \|x\| \leq 1\}$
   ◯ $\{x \mid x \leq 0\}$
   ◯ $\{x \mid x \geq 0\}$
   ◯ $\mathbb{R}$

2. Consider the system $x^+ = 4x + 5 + u$ subject to the constraints $|u| \leq 5$ and $|x| \leq 1$.

i) What is the pre-set of the set $\Omega = \{x \mid \alpha \leq x \leq \beta\}$?

◯ $[\alpha - 10, \beta]$ ◯ $[\alpha, \beta + 5]$ ◯ $\frac{1}{4}[\alpha - 10, \beta]$ ◯ $\frac{1}{2}[-\alpha, -\beta]$ ◯ $\mathbb{R}$

$\frac{1}{4}[\alpha - 10, \beta]$

ii) What is the maximum control invariant set?

◯ $[-1, 1]$ ◯ $\frac{1}{4}[-1, 1]$ ◯ $\emptyset$ ◯ $[-1, 0]$ ◯ $\mathbb{R}$

3. Consider the following standard MPC problem which generates the control law $\pi_N(x)$

$$
\begin{aligned}
J_N^\star(x) = \min \ & \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& (x_i, u_i) \in X \times U \\
& x_N = 0
\end{aligned}
$$

i) Assume for all of the following cases that the state x is feasible and non-zero and mark the correct statements.

   ◯ $J_N^\star(x) < J_{N-1}^\star(x)$
   ◯ $J_{N-1}^\star(x) < J_N^\star(Ax + B\pi_N(x))$
   ◯ $J_{N-1}^\star(x) > J_N^\star(Ax + B\pi_{N-1}(x))$
   ◯ $J_N^\star(Ax + B\pi_{N-1}(x)) = J_{N-1}^\star(x)$
   ◯ $J_{N-1}^\star(Ax + B\pi_N(x)) = J_N^\star(x) - l(x, \pi_N(x))$
   ◯ $J_{N-1}^\star(Ax + B\pi_N(x)) < J_N^\star(x) - l(x, \pi_N(x))$

ii) Let $\{z_i^N(x)\}$ be the closed-loop sequence generated by the dynamic system $z^+ = Az + B\pi_N(z)$ starting at state x, and the function $\bar{J}_N(x) = \sum_{i=0}^{\infty} l(z_i^N(x), \pi_N(z_i^N(x)))$ be the resulting closed-loop cost. Mark all statements that are correct for all feasible, non-zero x

- ◯ $\bar{J}_N(x) \leq J^\star(x)$
- ◯ $\bar{J}_N(x) \leq J_{N-1}(x)$
- ◯ $\bar{J}_N(x) \geq J_{N-1}(x)$
- ◯ $\lim_{N\to\infty} \bar{J}_N(x) = \lim_{N\to\infty} J_{N-1}(x)$
- ◯ $\lim_{N\to\infty} \bar{J}_N(x) > \lim_{N\to\infty} J_{N-1}(x)$
- ◯ $\lim_{N\to\infty} \bar{J}_N(x) < \lim_{N\to\infty} J_{N-1}(x)$

iii) If the state x is feasible for the MPC problem above and the controller $\pi_N(x)$ is applied, then the closed-loop system will arrive at the origin in exactly N steps.

◯ True ◯ False

4. Consider the following parametric optimization problem:

$$J^\star(x,y) = \min_u \; -4x^2 + xu_1 + u_1^2 + yu_1u_2 + 3 + xu_1^2$$
$$\text{s.t.} \; \left\| \begin{bmatrix} 1 & 2 \\ 3y & 4 \end{bmatrix} u + (7-x)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_\infty \leq x + \begin{bmatrix} 3 & 7 \end{bmatrix} u$$

(Mark all correct statements)

i) The feasible set of this problem is convex for

- ◯ any x or y
- ◯ no x or y
- ◯ $x \geq 0$ and any y
- ◯ $x = 0$ and $y = 2/3$
- ◯ $x \geq -1$ and $y = 0$

ii) The problem is convex for

- ◯ any x or y
- ◯ no x or y
- ◯ $x \geq 0$ and any y
- ◯ $x = 0$ and $y = 2/3$
- ◯ $x \geq -1$ and $y = 0$

Hint: A 2x2 symmetric matrix A is positive semidefinite if and only if $\det A \geq 0$ and $\operatorname{trace} A \geq 0$

5. Mark all invariant sets of the system $x^+ = x^2$ contained within the constraint set $X = \{x \,|\, |x| \leq 2\}$

- ◯ $\{x \,|\, -1 \leq x \leq 1\}$
- ◯ X
- ◯ $\{x \,|\, 0 \leq x \leq 1\}$
- ◯ $\{0\}$
- ◯ $\{1\}$
- ◯ $\{-1, 0, 1\}$
- ◯ $\{-1\}$

6. If X is an invariant set for the system $x^+ = f(x)$, then αX is too. Mark all conditions under which this statement is true.

- ◯ f is linear, $\alpha > 0$
- ◯ f is linear, $\alpha < 0$
- ◯ f is smooth
- ◯ Always false
- ◯ $\alpha > 0$
- ◯ $\alpha = 0$

7. T and Q are invariant sets for the system $x^+ = f(x)$. Mark the correct statements.

- ◯ $T \cap Q$ is invariant
- ◯ $T \cup Q$ is invariant

8. Give an example of a system that benefits from using soft state constraints.

____

9. Give an example of a system that benefits from using soft input constraints.

____

Any system where we're limiting the input to prevent wear on the actuators, but they are physically capable of more.

10. Consider the following optimization problem, which has a local optimizer at $x = x^\star$

$$\min c^T x$$
$$\text{s.t. } x^T Q x + q^T x \geq 1$$

Which conditions guarantee that $x^\star$ is also a global minimum?

- ◯ Q positive definite
- ◯ Q positive semi-definite
- ◯ Q negative semi-definite
- ◯ $c > 0$
- ◯ $q > 0$

11. Suppose that Y is a robust control invariant set for the system $x^+ = Ax + Bu + w$ subject to the state and input constraints X and U and the noise $w \in W$. For which of the following scenarios is Y still a robust invariant set for the resulting system?

- ◯ Disturbance set changes to $2W$
- ◯ Disturbance set changes to $0.5W$
- ◯ Input constraint set changes to $2U$
- ◯ Input constraint set changes to $U + 1$
- ◯ State constraint set changes to $-X$

12. Let x_0 be a state in the maximum control invariant set $\mathcal{C}_\infty$ for the system $x^+ = f(x, u)$ under the constraints $(x, u) \in \mathbb{X} \times \mathbb{U}$. Is it possible that there exists a controller $\kappa(x)$ such that the 3-step sequence $\{x_0, x_1, x_2\}$ is in $\mathbb{X}$ and $\{\kappa(x_0), \kappa(x_1)\}$ is in $\mathbb{U}$, where $x_1 = f(x_0, \kappa(x_0))$ and $x_2 = f(x_1, \kappa(x_1))$?

◯ Yes ◯ No

13. A controller is called static if its output depends only on its input, and not on an internal state. Which of the controllers studied in class are static?

- ◯ Nominal MPC regulation
- ◯ Offset-free MPC
- ◯ Tube-MPC
- ◯ Open-loop robust MPC

14. Consider the system $x^+ = Ax + u + w$ subject to the constraints $\|x\| \leq 1$ and $\|u\| \leq 1$ and a bounded disturbance w, $\|w\| \leq 0.1$. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1.2 \end{bmatrix}$, what is the set of feasible states of an open-loop robust MPC problem as the horizon goes to infinity?

   ◯ $\emptyset$
   ◯ It won't converge
   ◯ $\mathbb{R}^n$
   ◯ A circle

15. If Y is a robust control invariant set for the system $x^+ = f(x, u) + w$ subject to the constraints $(x, u) \in X \times U$ and $w \in W$, is Y also a control invariant set for the system $x^+ = f(x, u)$?

   ◯ Yes ◯ No

Problem 2.

/15

Consider the finite-horizon optimal control problem

$$V_N^\star(x_0) = \min \sum_{i=0}^{N} l(x_i, u_i) \quad \text{s.t.} \quad x_{i+1} = 2x_i + u_i$$

where the stage cost is

$$l(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

and let $\kappa_N(x)$ be the resulting MPC controller.

1. Give the bellman recursion for this problem.
2. Compute the smallest horizon N° such that the closed-loop system $x^+ = Ax + B\kappa_{N^\circ}(x)$ is asymptotically stable.

   $N^\circ = 2$

3. Suppose now that the system is subject to input constraints $u \in U$ and consider the MPC controller below

$$\begin{aligned} \min \; & \sum_{i=0}^{N} l(x_i, u_i) + V_{N^\circ}(x_N) \\ \text{s.t.} \; & x_{i+1} = 2x_i + u_i \\ & u_i \in U \end{aligned}$$

   Will the closed-loop system $x^+ = 2x + u^\star(x)$ be asymptotically stable, where $u^\star(x)$ is the optimal solution of the above MPC problem? Why, or why not?

   No. The system is unstable and there is no terminal set. With bounded inputs there will be a sufficiently large feasible initial state x such that there is insufficient control authority to stabilize the system.

Problem 3.

/15

1. Consider the dynamic system $x^+ = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} x$

i) Show that the set $E = \{x \,|\, x'x \leq 1\}$ is invariant
ii) Give a Lyapunov function for this system

2. Consider the MPC controller $u^\star(x)$ defined by the following optimization problem

$$\begin{aligned} \min\ & \sum_{i=0}^{N-1} \|x_i\|^2 + u_i^2 + V_f(x_N) \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & -1 \leq u_i \leq 1 \\ & x_N \in X_f \end{aligned}$$

i) State conditions on V_f and X_f such that the MPC controller is recursively feasible and invariant and the closed-loop system $x^+ = Ax + Bu^\star(x)$ is asymptotically stable.
ii) Give a non-trivial set X_f (i.e., not $\{0\}$) and non-zero function V_f that satisfy your conditions for the following system

$$A = \begin{bmatrix} 2.5 & 3.5 \\ 4.5 & 5.5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hint : Helpful relation

$$13x'x \geq x' \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} x \quad \forall x$$

Problem 4.

/15

Consider the dynamic system

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

subject to input constraint U, which is non-empty and contains the origin, and the following MPC formulation

$$\begin{aligned} J(x, r) = \min_{\mathrm{x,u,y},x_s,u_s} \quad & \sum_{i=0}^{N-1} l(x_i - x_s, u_i - u_s) + V_f(x_N - x_s) \\ \text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \quad i = 0, \dots, N-1 \\ & u_i \in U \quad\quad\quad\quad i = 0, \dots, N-1 \\ & x_N - x_s \in X_f \\ & x_s = Ax_s + Bu_s \\ & r = Cx_s \\ & u_s \in U \\ & x_0 = x \end{aligned} \tag{2}$$

Denote the MPC control law resulting from the above optimization problem as $\pi(x, r)$.

1. Give conditions on the reference r, the stage cost l, the terminal cost V_f, the terminal set X_f and the system dynamics A, B, C such that the output of the closed-loop system $x^+ = Ax + B\pi(x, r)$, asymptotically converges to the reference r without violating the system constraints; $\lim_{i\to\infty} y_i = r$ and $\pi(x_i, r) \in U, \forall i$.

2. Sketch a proof showing that the system converges for a given constant r under the conditions given in the previous question.

3. Propose a modification to (2), so that the following conditions are met:

   - If a feasible solution exists for $(x, 0)$ in your controller, then for all r a feasible solution exists for (x, r)
   - If the closed-loop system converges to the reference without violating constraints for controller (2), then it does for your controller too.

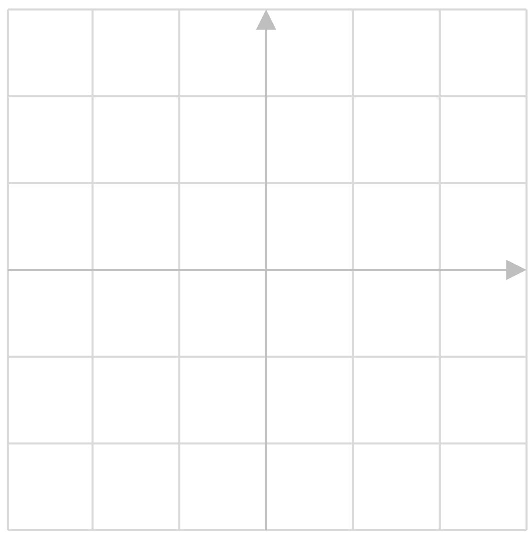
Problem 5.

/15

Consider the following parametric LCP

$$
\begin{aligned}
w - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} z &= Qx + q \\
w, z &\geq 0 \\
w^T z &= 0
\end{aligned}
$$

1. Give the complementarity cones for this problem and sketch them.



Enumerate all complementarity bases

$$z_1 = z_2 = 0 \qquad \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} = q$$

$$\begin{pmatrix}w_1\\w_2\end{pmatrix} = q$$

$$w_1 = w_2 = 0 \qquad \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} = q$$

$$\begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} q$$

$$z_1 = w_2 = 0 \qquad \begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix}\begin{pmatrix}z_2\\w_1\end{pmatrix} = q$$

$$\begin{pmatrix}z_2\\w_1\end{pmatrix} = \begin{bmatrix}0 & -1\\1 & 1\end{bmatrix} q$$

$$w_1 = z_2 = 0 \qquad \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\begin{pmatrix}z_1\\w_2\end{pmatrix} = q$$

Not invertable - empty set

So the complementarity cones are:

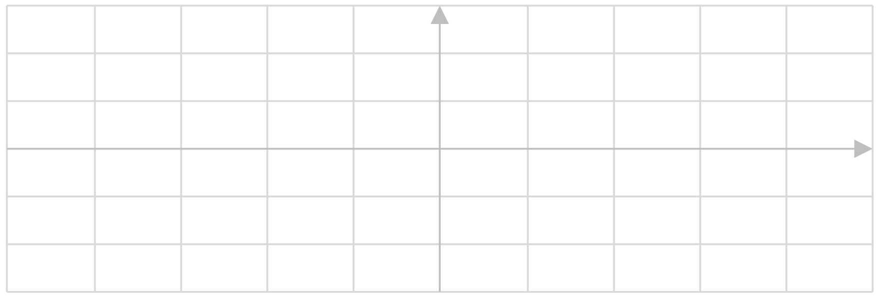
$$K_1 = \{q \mid q \geq 0\}$$

$$K_2 = \left\{q \mid \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} q \geq 0\right\}$$

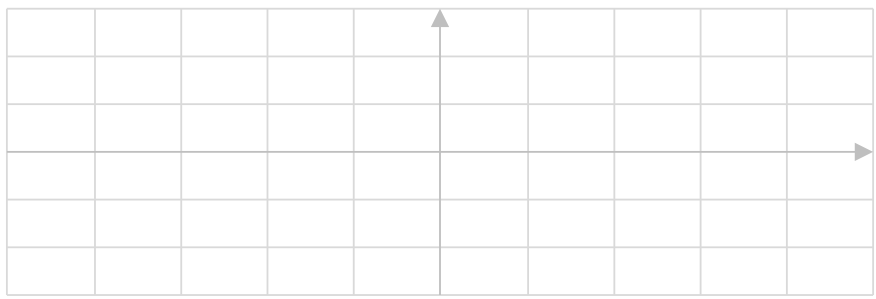
$$K_3 = \left\{q \mid \begin{bmatrix}0 & -1\\1 & 1\end{bmatrix} q \geq 0\right\}$$

2. Sketch the solution $w(x)$ and $z(x)$ for $Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$w_1(x)$



$w_2(x)$



$z_1(x)$

$z_2(x)$

Problem 6.

/15

Consider the following nonlinear discrete-time dynamic system

$$x^+ = \frac{1}{6}(x-1)^3 + \frac{1}{6} + u := f(x,u)$$

subject to the state and input constraints $X = \{x \,|\, |x| \leq 1/2\}$ and $U = \{u \,|\, |u| \leq 1\}$.

Your goal is to design a robust *linear* MPC controller for this system to regulate it to the origin.

1. Find values a and b by linearizing the system around the origin and the smallest $\bar{w}$ such that the evolution of the following uncertain linear dynamic system contains that of the nonlinear system above within the set X.

$$x^+ = ax + bu + w \qquad w \in W = \bar{w} \cdot [-1, 1] \tag{3}$$

$a =$ ________ $b =$ ________ $\bar{w} =$ ________

Hint: Ensure that $\forall x \in X \; \exists w \in W$ such that $ax + bu + w = f(x,u)$

$a = 0.5$ $b = 1$ $\bar{w} = \frac{7}{48}$

2. Compute the maximum robust invariant set X_f for the system (3) within the constraint set X for the control law $u = 0$. Use the value $\bar{w} = 0.2$ for this question.

$X_f =$ ____________________

$X_f = X$

3. Fill in the blanks in the following set so that any controller $u(x) \in C(x)$ will ensure robust constraint satisfaction for system (3). Use the value $\bar{w} = 0.2$ for this question.

$$C(x) := \left\{ u_0 \;\middle|\; \exists u_1 \begin{pmatrix} \; \\ \; \\ \; \\ \; \end{pmatrix} \leq \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ x \end{pmatrix} \leq \begin{pmatrix} \; \\ \; \\ \; \\ \; \end{pmatrix} \right\}$$

Hint: $a[-1,1] \ominus b[-1,1] = (a-b)[-1,1]$ for $a \geq b \geq 0$

This is just two-stage open-loop robust MPC in dense formulation.

$$\left\{ u_0 \;\middle|\; \exists u_1 \; - \begin{pmatrix} \frac{17}{48} \\ \frac{27}{96} \\ 1 \\ 1 \end{pmatrix} \leq \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ 1 & & \\ & 1 & \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ x \end{pmatrix} \leq \begin{pmatrix} \frac{17}{48} \\ \frac{27}{96} \\ 1 \\ 1 \end{pmatrix} \right\}$$

4. Will the dynamic system $x^+ = f(x, u^\star(x))$ satisfy the constraints $x \in X$ and $u \in U$ where $u^\star(x)$ be the robust MPC controller defined above? *Explain your answer.*

◯ Yes
◯ No
◯ Not enough information

Yes. The controller robustly satisfies the constraints of system (3) and we designed this uncertain system to contain the dynamics of the nonlinear system. As a result, the constraints of the nonlinear system are also satisfied.