Inverted pendulum

System model

Consider the system depicted in Figure 1 composed by a pendulum fixed on a cart and controlled through a DC motor.

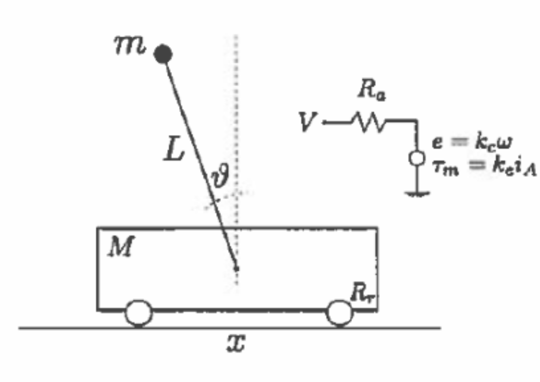

Figure 1: Cart-pendulum system.

The model parameters are given as *M = 10 kg, m = 2kg, L = 1m, Rr = 0.1m, Ra = 10Ω, Kc = 2V s/rad, Ke = 2Nm/A*. The goal of this work is to stabilize the pendulum in its vertical position (around *ϑ = 0*) acting on the motor voltage *v*.

To derive the dynamical model the system can be decomposed as in Figure2.

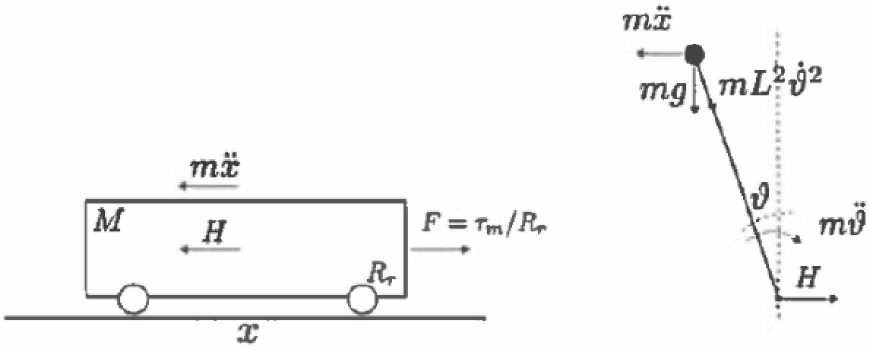

Figure 2: Cart-pendulum system.

- Cart:

$$M\,\ddot{x} + H = F$$

$$\tau_m = k_e\, i_a = k_e \frac{v - k_c\omega}{R_a} \quad , \quad \omega = \frac{\dot{x}}{R_r}$$

- Pendulum:

$$\begin{aligned} H &= m\,\ddot{x} + mL\ddot{\theta}\,\cos(\theta) - mL^2\dot{\theta}^2\sin(\theta) \\ m\,L^2\ddot{\theta} &= +mgLsin(\theta) - mL\ddot{x}\,\cos(\theta) \end{aligned}$$

Rearranging the terms to remove the force H we obtain

$$(M+m)\ddot{x} + mL\ddot{\theta}\cos(\theta) - mL^2\dot{\theta}^2\sin(\theta) = \frac{\tau_m}{R_r}$$

$$m\,L^2\ddot{\theta} - mgLsin(\theta) + mL\ddot{x}\cos(\theta) = 0$$

and substituting the terms $\ddot{\theta}$ and τ_m in the first equation we have the following model

$$\begin{cases} \left(M+m-m\,cos^2(\theta)\right)\ddot{x} = -mgLsin(\theta)\cos(\theta) + mL^2\dot{x}^2\sin(\theta) + \dfrac{k_e}{R_rR_a}v - \dfrac{k_ek_c}{R_r^2R_a}x_2 \\ \ddot{\theta} = \dfrac{g}{L}\sin(\theta) - \dfrac{\ddot{x}}{L}\cos(\theta) \end{cases}$$

To use a state-space representation define the variables $x_1 = x,\ x_2 = \dot{x},\ x_3 = \theta,\ x_4 = \dot{\theta}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{M+m-m\,cos^2(x_3)}\left(-mg\sin(x_3)\cos(x_3) + m\,L^2x_2^2\sin(x_3) + \dfrac{k_e}{R_rR_a}v - \dfrac{k_ek_c}{R_r^2R_a}x_2\right) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{g}{L}\sin(x_3) - \dfrac{\cos(x_3)}{L\left(M+m-m\,cos^2(x_3)\right)}\left(-mg\sin(x_3)\cos(x_3) + m\,L^2x_2^2\sin(x_3) + \dfrac{k_e}{R_rR_a}v - \dfrac{k_ek_c}{R_r^2R_a}x_2\right) \end{cases}$$

Equilibrium and linearization

We are now interested in linearizing the system model around the vertical pendulum position with zero input, i.e. $\bar{x} = (0,0,0,0)$ and $\bar{v} = 0$; the resulting linearized model is the following

$$\begin{cases} \delta\dot{x_1} = \delta x_2 \\ \delta\dot{x_2} = -\dfrac{k_ek_c}{MR_r^2R_a}\delta x_2 - \dfrac{gm}{M}\delta x_3 + \dfrac{k_e}{MR_aR_r}\delta v \\ \delta\dot{x_3} = \delta x_4 \\ \delta\dot{x_4} = -\dfrac{k_ek_c}{MR_r^2R_aL}\delta x_2 + \left(\dfrac{g}{L} + \dfrac{gm}{M\,L}\right)\delta x_3 - \dfrac{k_e}{M\,R_aR_rL}\delta v \end{cases}$$

Substituting the parameters values the linearized system model can be written as

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -4 & -1.9620 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 11.7720 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ -0.2 \end{bmatrix}\delta v$$

In matlab there are several ways of obtaining the linearized model without making all the computations by hand. **One possible solution is to construct a Simulink model of the system, put it at the equilibrium and use the 'time-based linearization' block.** You will do this in Exercise Session 6.