Lecture 6

State observers
Output-feedback controllers

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State observers

Motivations: in several applications

- not all scalar states are accessible
- sensors are costly $\rightarrow$ not convenient to measure all scalar states

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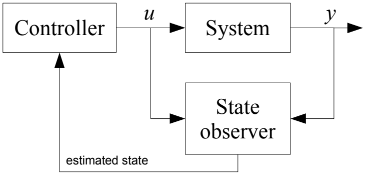
Problems

Knowing just the output, but not the state, prevents from using state feedback controllers

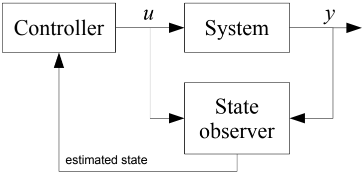
Solutions

- Build an observer, that is a dynamical system with
  - inputs: the inputs and outputs of the system Σ under observation
  - outputs: the estimated state of Σ
- Use the estimated state in the controller

State observers



State observers

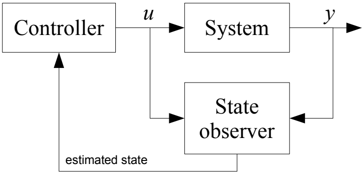


Terminology: state observers or state estimators

Challenges: stability of the closed-loop system? Performance?

Other uses: estimates of the internal state are also very useful for detecting malfunctioning and faults on components

State observers



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Outline of the lecture

- Full-order observers and filters
  - Output-feedback controllers: the separation principle
- Reduced-order observers
- Choice of the closed-loop eigenvalues

Later: Kalman filtering (stochastic framework)

Full-order observer

$$\Sigma : \begin{cases} x^+ = Ax + Bu \\ y = Cx \\ x(0) = x_0 \end{cases}$$

- **Full-order:** reconstruct the whole state $x(k)$
- Define $\hat{x}(k|k-1)$ the estimate available at time k of $x(k)$ using data (inputs and outputs) known up to $k-1$
  $\hookrightarrow$ Notation for these sequences : u^{k-1}, y^{k-1}

Full-order observer

Luenberger observer

$$\hat{\Sigma} = \begin{cases} \hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) - L\left[y(k) - C\hat{x}(k|k-1)\right] \\ \hat{y}(k) = C\hat{x}(k|k-1) \\ \hat{x}(0) = \hat{x}_0 \end{cases}$$

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Remarks

- $L \in \mathbb{R}^{n \times p}$: observer gain. Multiplies the output estimation error
- Recursive update at $\hat{x}$: the observer is an LTI system
- The dynamic of Σ and $\hat{\Sigma}$ are identical, up to the output-error term

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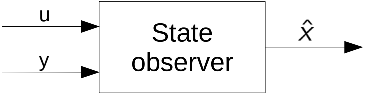
- Recursive update at $\hat{x}$: the observer is an LTI system
- The dynamic of Σ and $\hat{\Sigma}$ are identical, up to the output-error term
- If, at $\bar{k}$, $\hat{x}(\bar{k}|\bar{k}-1) = x(\bar{k})$, then, $\hat{x} = x$, $\forall k \geq \bar{k}$ (perfect reconstruction)
- x_0 is not known, $\hat{x}_0$ is chosen using "common sense" and, unavoidably, $\hat{x}_0 \neq x_0$

Observer stability

Observer stability



Goal

Guarantee that the estimation error $e(k|k-1) = x(k) - \hat{x}(k|k-1)$ goes to zero as $k \to \infty$.
$\hookrightarrow$ recovers from the mismatch $x_0 \neq \hat{x}_0$.

Key point for observers: analyse the error dynamics (not $\hat{x}$)!

Observer stability

System

$$x^+ = Ax + Bu$$
$$y = Cx$$

Observer ($\hat{x} = \hat{x}(k|k-1)$)

$$\hat{x}^+ = A\hat{x} + Bu - L(y - C\hat{x})$$
$$\hat{y} = C\hat{x}$$

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Error dynamics ($e = e(k|k-1) = x(k) - \hat{x}(k|k-1)$)

$$e^+ = Ae + \cancel{Bu} - \cancel{Bu} + L(Cx - C\hat{x}) = (A + LC)e$$

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Definition

The observer is asymptotically stable (AS) if the error dynamics has this property

- The error dynamics is an autonomous system
  - AS $\Rightarrow \hat{x} \to x$ irrespectively of x_0 and $\hat{x}_0$

Definition

The eigenvalues of $A + LC$ are called the eigenvalues of the observer

Observer design

Problem

Find L such that $A + LC$ is Schur

- $\text{Spec}(A + LC) = \text{Spec}(A^T + C^T L^T)$
  The problem is identical to the design of K such that $(A + BK)$ has prescribed eigenvalues, up to the following replacements

$$A \to A^T$$
$$B \to C^T$$
$$K \to L^T$$

Observer design

- For single-output systems, $y \in \mathbb{R}$, use Ackermann's formula: if

$$M_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T \quad \text{is full rank, set}$$

$$L^T = -\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} (M_o)^{-1} P^D \left(A^T\right)$$

where P^D is the desired characteristic polynomial of $A + LC$

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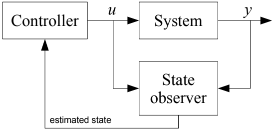
  where P^D is the desired characteristic polynomial of $A + LC$
- For multi-output systems all methods seen for MIMO control design can be applied to observer design

Output-feedback controllers: the separation principle

Observer & state feedback design: the separation principle



System

$$x^+ = Ax + Bu$$
$$y = Cx$$

Observer

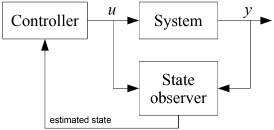
$$\hat{x}^+ = A\hat{x} + Bu - L(y - C\hat{x})$$

Recall: $\hat{x} = \hat{x}(k|k-1)$

Controller

$$u = K\hat{x}$$

Observer & state feedback design: the separation principle



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Closed-loop system

$$\begin{bmatrix} x^+ \\ \hat{x}^+ \end{bmatrix} = \underbrace{\begin{bmatrix} A & BK \\ -LC & A+BK+LC \end{bmatrix}}_{F} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Separation principle

$$\text{Spec}(F) = \text{Spec}(A+BK) \cup \text{Spec}(A+LC)$$

- If L stabilizes $A+LC$ and K stabilizes $A+BK$, the closed-loop system is AS
- Each gain is designed independently of the other one

Proof of the separation principle

Make the error $e = x - \hat{x}$ appear through the change of variables

$$\begin{bmatrix} x \\ e \end{bmatrix} = T \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \quad T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \Rightarrow T = T^{-1}$$

Proof of the separation principle

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- Dynamics of $\begin{bmatrix} x \\ e \end{bmatrix}$: $\begin{bmatrix} x^+ \\ e^+ \end{bmatrix} = \hat{F} \begin{bmatrix} x \\ e \end{bmatrix}$ where

$$\hat{F} = T \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} T^{-1} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix}$$

Proof of the separation principle

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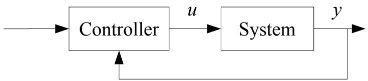
- Block-diagonal structure $\Rightarrow$ Spec$(\hat{F})$ =Spec$(A + BK)\cup$Spec$(A + LC)$

Output feedback controllers



State observer + state feedback provides a method for designing output feedback controllers

$$
\begin{aligned}
\hat{x}^+ &= A\hat{x} + Bu + L(y - C\hat{x}) \\
u &= K\hat{x}
\end{aligned}
$$

- The controller is a *dynamical* system of order n

Problem

Is it possible to reduce the order of the controller? Yes, by using reduced-order observers (see later).

Full-order observers with no delay (filters)

Observers with no delay

Previously

Compute $\hat{x}(k|k-1)$ using u^{k-1} (inputs up to $k-1$) and y^{k-1} (outputs up to $k-1$)

Next

Compute $\hat{x}(k|k)$ using u^{k-1} and y^{k}

- Controller $u(k) = K\hat{x}(k|k)$

Observers with no delay

Pros

- $\hat{x}(k|k)$ is "better" than $\hat{x}(k|k-1)$ as it uses more information.
  - Example: a disturbance at time k can be captured by y^k but not by y^{k-1}

Observers with no delay

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- $\hat{x}(k|k)$ is "better" than $\hat{x}(k|k-1)$ as it uses more information.
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Cons

Timing of the computations (uniform sampling period T)

- Measure $y(k) = y(kT)$ at time kT
- $\hat{x}(k|k)$ is available at time $kT + \epsilon$
- $u(k)$ is available at time $kT + \epsilon$, at earliest

Applicable if ϵ is "small" compared to T

Observers with no delay

Terminology

- $\hat{x}(k|k)$: filtered estimate
- $\hat{x}(k+1|k)$: predicted estimate
- $\hat{x}(k-1|k)$: smoothed estimate

Luenberger filter

Filter dynamics ($\hat{x}(k) = \hat{x}(k|k)$)

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) - L(y(k+1) - \underbrace{C\left(A\hat{x}(k) + Bu(k)\right)}_{\text{estimate of } y(k+1)})$$

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Error dynamics ($e(k) = x(k) - \hat{x}(k)$)

$$\begin{aligned} e(k+1) &= Ax(k) + \cancel{Bu(k)} - A\hat{x}(k) - \cancel{Bu(k)} + LCx(k+1) - LCA\hat{x}(k) \\ &\quad - LCBu(k) \\ &= Ae(k) + LC\left(Ax(k) + Bu(k)\right) - LCA\hat{x}(k) - LCBu(k) \\ &= (A + LCA)\, e(k) \end{aligned}$$

Filter design : find L such that $A + LCA$ is Schur

- Eigenvalue assignment for the pair (A, CA), if it is observable

Definition

The eigenvalues of $A + LCA$ are termed the filter eigenvalues

Luenberger filter

Remarks

- (A, C) observable $\not\Rightarrow$ (A, CA) observable
- The observability matrix for (A, CA) is

$$\tilde{M}_o = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}^T = A^T M_o \qquad M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T$$

If $\det(A) = 0$, $\tilde{M}_o$ is not full rank, even if M_o is

Luenberger filter

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  If $\det(A) = 0$, $\tilde{M}_o$ is not full rank, even if M_o is
- Possible to prove that if $\lambda = 0$ is an eigenvalue of A, then it is an unobservable eigenvalue of (A, CA)
  - Not critical, however, because the mode associated to $\lambda = 0$ is AS (vanishes in n steps)

Example

$$x^+ = Ax + Bu \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$y = Cx$$

Observability analysis: $\det(M_o) \neq 0$ but $\det(\tilde{M}_o) = 0$

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Assignment of the observer eigenvalues

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \quad A + LCA = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & l_1 + 1 \\ 0 & l_2 + 1 \end{bmatrix}$$

- $\text{Spec}(A + LCA) = \{0, l_2 + 1\}$. The eigenvalue $\lambda = 0$ cannot be modified. However, this is not a problem for the observer stability

Separation principle

Proceeding as in the case of the observer with state $\hat{x}(k|k-1)$, one can prove that

$$\{\text{closed-loop eigenvalues}\} = \text{Spec}(A+BK) \cup \text{Spec}(A+LCA)$$

- If (A, B) is controllable and (A, CA) is observable, all eigenvalues of the closed-loop system can be assigned by choosing K and L independently

Reduced-order observers

Reduced-order observers

Goal: Design observers with order strictly less than n

Idea

$y = Cx \in \mathbb{R}^p$ carries (partial) information on x.
Find a state transformation $\bar{x} = Tx$ such that p states coincide with y and reconstruct only the remaining $n - p$ states through an observer of order $n - p$.

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System dynamics

$$x^+ = Ax + Bu$$
$$y = Cx$$

Set

$$T = \begin{bmatrix} C \\ T_1 \end{bmatrix} \begin{matrix} \} \ p \text{ rows} \\ \} \ n-p \text{ rows} \end{matrix}$$

where T_1 is such that $\det(T) \neq 0$ (T_1 is not unique)

Change of coordinates

System with state $\bar{x} = Tx$

$$\begin{array}{rcl} \bar{x}^+ & = & \bar{A}\bar{x} + \bar{B}u \\ y & = & \bar{C}\bar{x} \end{array} \qquad (*)$$

where $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$ and, by construction,

$$\bar{x} = \begin{bmatrix} C \\ T_1 \end{bmatrix} x = \begin{bmatrix} y \\ w \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} I & 0 \end{bmatrix}$$

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- Partition $\bar{A}$, $\bar{B}$ as

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{A}_{11} \in \mathbb{R}^{p \times p}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}$$

- The system $(*)$ becomes

$$y(k+1) = \bar{A}_{11}y(k) + \bar{A}_{12}w(k) + \bar{B}_1u(k)$$
$$w(k+1) = \bar{A}_{21}y(k) + \bar{A}_{22}w(k) + \bar{B}_2u(k)$$

Observer design

Dynamics of $w(k)$

$$\Sigma_w = \begin{cases} w(k+1) = \bar{A}_{22} w(k) + \underbrace{\left[\bar{A}_{21} y(k) + \bar{B}_2 u(k)\right]}_{\text{Known input } \bar{u}(k)} \\ \underbrace{y(k+1) - \bar{A}_{11} y(k) - \bar{B}_1 u(k)}_{\text{Measured output } \bar{y}(k+1)} = \bar{A}_{12} w(k) \end{cases}$$

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Design of the estimator $\bar{y}(k) \to \hat{w}(k|k)$

Full order observer (without delay) for Σ_w.

Set $\hat{w}(k) = \hat{w}(k|k) \in \mathbb{R}^{n-p}$.

$$\hat{w}(k+1) = \bar{A}_{22}\hat{w}(k) + \bar{u}(k) - L\left(\bar{y}(k+1) - \bar{A}_{12}\hat{w}(k)\right)$$

Observer design

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Error dynamics (check at home)

$$\hat{e}(k+1) = w(k+1) - \hat{w}(k+1) = \left(\bar{A}_{22} + L\bar{A}_{12}\right) \hat{e}(k)$$

- $(\bar{A}_{22}, \bar{A}_{12})$ observable $\Rightarrow$ design $L \in \mathbb{R}^{(n-p)\times n}$ for assigning the eigenvalues of $\bar{A}_{22} + L\bar{A}_{12}$ with the usual procedures

Observer design

Lemma

If (A, C) is observable, then $(\bar{A}_{22}, \bar{A}_{12})$ is observable.
The (not obvious) proof is omitted...

Observer design

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Reconstruction of the full state

$$\hat{x}(k|k) = T^{-1} \begin{bmatrix} y(k) \\ \hat{w}(k|k) \end{bmatrix}$$

Remarks

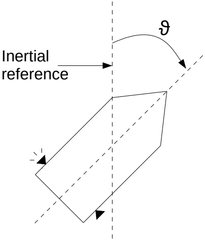
Using the control law

$$u(k) = K\hat{x}(k|k)$$

one can prove a separation principle: under the reachability and observability of suitable pairs, $2n - p$ eigenvalues of the closed-loop system can be assigned through K and L.

Example: satellite attitude control - output feedback

Attitude control: proper orientation of the satellite antenna with respect to earth



$$\ddot{\theta} = u + w, \quad u = \frac{M_C}{I}, \quad w = \frac{M_D}{I}$$

- I: moment of inertia of the satellite (about the mass center)
- M_C: control torque applied by thrusters
- M_D disturbance torque
- θ=angle of satellite

DT model ($x_1 = \theta, \quad x_2 = \dot{\theta}, \quad y = \theta$, exact discretisation)

As seen in lecture 5,

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}}_{B} (u + w)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

In the sequel :

$$w = 0, \quad T = 0.1$$

Problem

Design a state observer

First design: standard Luenberger observer

Goal

Synthetize a full-order observer with eigenvalues $z_{1,2} = 0.4 \pm j0.4$

$$\hat{x} = \hat{x}(k|k-1)$$

$$\begin{cases} \hat{x}^+ = A\hat{x} + Bu - L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases}$$

- Design of L in Matlab for assigning the eigenvalues of $A + LC$

```
T = 0.1
A = [1 T ; 0 1]
C = [1  0]
p = [0.4+i*0.4 ; 0.4-i*0.4]
L = -acker(A', C', p)'
```

$$L = \begin{bmatrix} 1.2 \\ 5.2 \end{bmatrix}$$

First design: Luenberger observer

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Synthetize a full-order observer with eigenvalues $z_{1,2} = 0.4 \pm j0.4$

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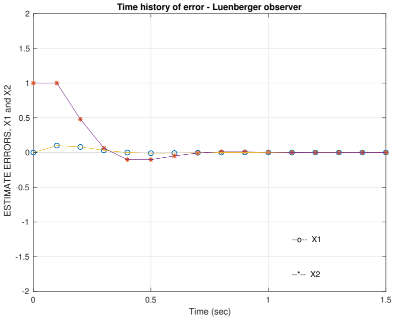

Second design: observer with no delays (filter)

$$\hat{x} = \hat{x}(k|k)$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu - \hat{L}\left(y(k+1) - C\left(A\hat{x}(k) + Bu(k)\right)\right)$$

- Design of $\hat{L}$ in Matlab for assigning the eigenvalues of $A + \hat{L}CA$

$\hat{L}$ = -acker(A^T, $A^T * C^T$, p)^T

$$\hat{L} = \begin{bmatrix} 0.68 \\ 5.2 \end{bmatrix}$$

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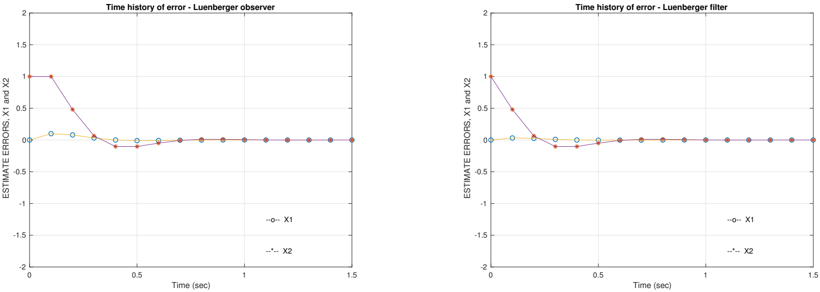

Third design: reduced-order observer

Goal

Synthetize a reduced-order observer with eigenvalue $z = 0.5$

First step: put the system dynamics in the reference form

$$\bar{x}^+ = \bar{A}\bar{x} + \bar{B}u$$
$$y = \bar{C}\bar{x}$$

with $\bar{x} = \begin{bmatrix} y \\ w \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

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- $(\bar{A}_{22}, \bar{A}_{12})$ is observable $\Rightarrow$ design $\bar{L}$ such that

$$\bar{A}_{22} + \bar{L}\bar{A}_{12} = 0.5 \quad \rightarrow \quad 1 + \bar{L}T = 0.5$$
$$T = 0.1 \quad \rightarrow \quad \bar{L} = -5$$

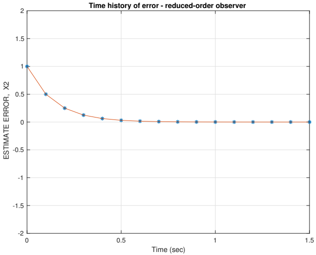
Estimator dynamics

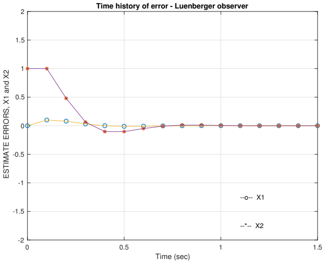
Since $w = x_2$, by using the notation $\hat{x}_2(k+1) = \hat{x}_2(k+1|k+1)$, we have

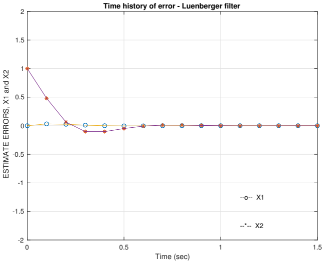
$$\hat{x}_2(k+1) = \bar{A}_{22}\hat{x}_2(k) + \bar{u}(k) - \bar{L}(\bar{y}(k+1) - \bar{A}_{12}\hat{x}_2(k))$$

$$\hat{x}(k) = \begin{bmatrix} y(k) \\ \hat{x}_2(k) \end{bmatrix}$$

Estimator dynamics







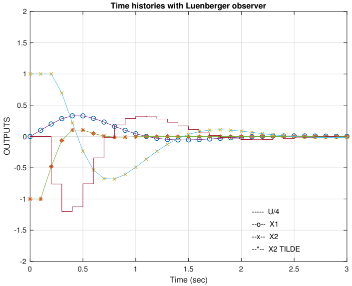
Output-feedback controller

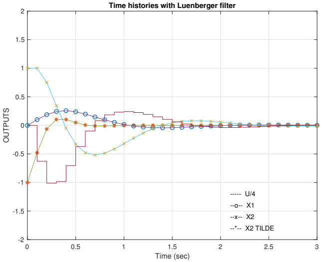
Goal

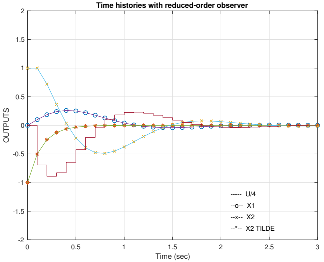
Design $u = K\hat{x}$ such that the remaining eigenvalues of the closed-loop system are $z = 0.8 \pm j0.25$

- Already done (see Lecture 5) $\rightarrow \quad K = \begin{bmatrix} -10 & -3.5 \end{bmatrix}$

Output-feedback controller







Output-feedback controller

- Luenberger filter: faster response than Luenberger observer (as expected)
- Reduced-order observer: first-order response of the estimator $\rightarrow$ slightly reduced control effort compared to Luenberger observer

Choice of the closed-loop eigenvalues

How to choose the closed-loop eigenvalues ?

- State feedback + full-order observer : assign $\nu = 2n$ eigenvalues
- State feedback + reduced-order observer : assign $\nu = n + (n - p)$ eigenvalues

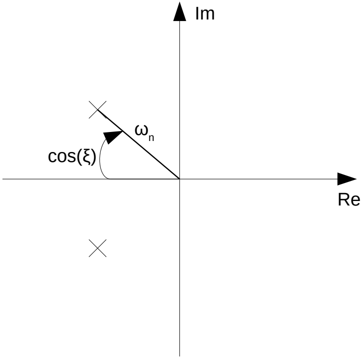
First heuristic approach

- Assign $\nu - 2$ eigenvalues to the origin
  - Deadbeat behaviour
- Set 2 "dominant" eigenvalues as desired

Assignment of dominant eigenvalues: first heuristic method

Desired closed-loop continuous-time transfer function

$$G_c(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \xi\text{: damping} \quad \omega_n\text{: natural frequency}$$



Assignment of dominant eigenvalues: first heuristic method

Exact discretisation of $G_c(s)$ with sampling time T_c results in a DT system with eigenvalues

$$p_1 = -2e^{-\xi\omega_n T_c}\cos(\omega_n T_c\sqrt{1-\xi^2})$$
$$p_2 = e^{-2\xi\omega_n T_c}$$

Algorithm

Choose desired ξ and $\omega_n \rightarrow$ compute p_1 and p_2

Recall the golden rules from basic control theory

- If the CT LTI system (A, B, C, D) is a low pass filter with pass-band $[0, \bar{\omega}]$, do not set $\omega_n >> \bar{\omega}$. Otherwise
  - the magnitude of control variables might be large and actuator limits might be reached
  - high-frequency disturbances might start playing a significant role

Second set of heuristic criteria

- Choose control eigenvalues no more than $2 \div 6$ times faster than open-loop eigenvalues
  - Good for limiting the actuator effort
- Choose observer eigenvalues faster than control eigenvalues
  - They do not impact on actuators
    - However, if estimation errors due to sensor noise are significant, one has to slow down the observer eigenvalues
  - Closed-loop performance will be dominated by control eigenvalues

Take home messages

- Observers are essential for systems where not all states can be measured
- Duality + separation principle: eigenvalue assignment is the key tool for designing output-feedback controllers

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Problem

How stochastic disturbances affect state estimation ?
See later (Kalman filtering)