Multivariable Control (ME-422) - Exercise session 12 SOLUTIONS

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1. Consider the system

$$\begin{aligned} x_{k+1} &= 0.5x_k + w_k \qquad w_k \sim N(0,1) \\ y_k &= x_k + v_k \qquad v_k \sim N(0,1) \\ x_0 &\sim N(0,1) \end{aligned}$$

and let the usual statistical assumptions for KF hold.
Derive a <u>nonrecursive</u> expression of $\mathrm{E}[x_1|y_0, y_1]$.

Solution: We start by computing the distribution of the vector

$$X = \begin{bmatrix} x_1 \\ Y \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

Using the system dynamics, we have

$$\begin{aligned} y_0 &= x_0 + v_0 \\ x_1 &= \frac{1}{2}x_0 + w_0 \\ y_1 &= x_1 + v_1 = \frac{1}{2}x_0 + w_0 + v_1 \end{aligned}$$

which is

$$\begin{bmatrix} x_1 \\ y_0 \\ y_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 1 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_0 \\ v_0 \\ w_0 \\ v_1 \end{bmatrix}.$$

Setting $\xi = \begin{bmatrix} x_0 & v_0 & w_0 & v_1 \end{bmatrix}^T$ we have that it is a Gaussian random vector with $\mathrm{E}[\xi] = 0$ and $var[\xi] = I$.
Hence, X is a Gaussian random vector with

$$\begin{aligned} \mathrm{E}[X] &= \bar{A}\mathrm{E}[\xi] = 0 \\ var[X] &= \bar{A}var[\xi]\bar{A}^T = \bar{A}\bar{A}^T = \begin{bmatrix} 1.25 & 0.5 & 1.25 \\ 0.5 & 2 & 0.5 \\ 1.25 & 0.5 & 2.25 \end{bmatrix}. \end{aligned}$$

Partition $var[X]$ as $\begin{bmatrix} \Sigma_{x_1,x_1} & \Sigma_{x_1,Y} \\ \Sigma_{Y,x_1} & \Sigma_{Y,Y} \end{bmatrix}$.
Using the results seen in the lectures, $x_1|Y$ is Gaussian with

$$\mathrm{E}[x_1|Y] = \underbrace{\mathrm{E}[x_1]}_{0} + \Sigma_{x_1,Y}\Sigma_{Y,Y}^{-1}\left(Y - \underbrace{\mathrm{E}[Y]}_{0}\right) = \begin{bmatrix} 0.1176 & 0.5294 \end{bmatrix} Y$$

2. Consider the first-order system

$$
\begin{aligned}
x_{k+1} &= \alpha x_k + w_k \\
y_k &= \gamma x_k + v_k \\
x_0 &\sim N\left(\bar{x}_0, \bar{\Sigma}_0\right)
\end{aligned} \tag{1}
$$

where α and γ are parameters, $w_k \sim N(0, \beta^2)$, $v_k \sim N(0, 1)$, and the usual statistical assumptions for Kalman Filtering hold (see the lectures).

We will study the Difference Riccati Equation (DRE) associated to the Kalman Predictor.

(a) Assume the system is unstable, that is $\alpha > 1$, and that $\beta \neq 0$, $\gamma \neq 0$. We want to check if the Kalman Predictor can "track" the state even when it diverges.

i. Show that, using Σ_k for $\Sigma_{k|k-1}$, the DRE is

$$
\Sigma_{k+1} = \beta^2 + \frac{\alpha^2 \Sigma_k}{1 + \gamma^2 \Sigma_k}. \tag{2}
$$

Solution: As seen in the lectures, the DRE is

$$
\Sigma_{k+1} = A\Sigma_k A^T + W - A\Sigma_k C^T \left[C\Sigma_k C^T + V\right]^{-1} C\Sigma_k A^T.
$$

Setting $A = \alpha$, $C = \gamma$, $W = \beta^2$, and $V = 1$ gives the formula.

ii. The values of Σ_k can be computed from (2) using the following graphical procedure (valid also for a generic nonlinear system $x_{k+1} = f(x_k)$).

A. Plot $f(\Sigma_k) = \beta^2 + \frac{\alpha^2 \Sigma_k}{1+\gamma^2 \Sigma_k}$ and the line $l(\Sigma_k) = \Sigma_k$. The intersections of f and l gives the solutions to the ARE

$$
\bar{\Sigma} = \beta^2 + \frac{\alpha^2 \bar{\Sigma}}{1 + \gamma^2 \bar{\Sigma}}. \tag{3}
$$

This plot is given in Figure 1. Consider only the positive solution.

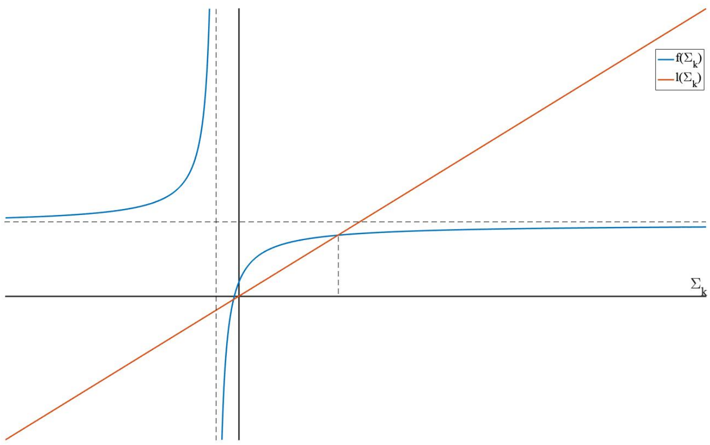

Figure 1: $f(\Sigma_k)$ and $l(\Sigma_k)$

B. Set $k = 0$ and fix $\Sigma_0 > 0$ on the horizontal axis. One has $\Sigma_{k+1} = f(\Sigma_k)$ and

- If $\Sigma_k < \bar{\Sigma}$, then $\Sigma_{k+1} > \Sigma_k$ (because $f(\Sigma_k) > l(\Sigma_k)$) and Σ_k increases monotonically towards $\bar{\Sigma}$
- If $\Sigma_k > \bar{\Sigma}$, then $\Sigma_{k+1} < \Sigma_k$ (because $f(\Sigma_k) < l(\Sigma_k)$) and Σ_k decreases monotonically towards $\bar{\Sigma}$

Plot on the figure, the (qualitative) sequences Σ_0, Σ_1, Σ_2, ... when starting from $\Sigma_0 > \bar{\Sigma}$ and $\Sigma_0 < \bar{\Sigma}$.

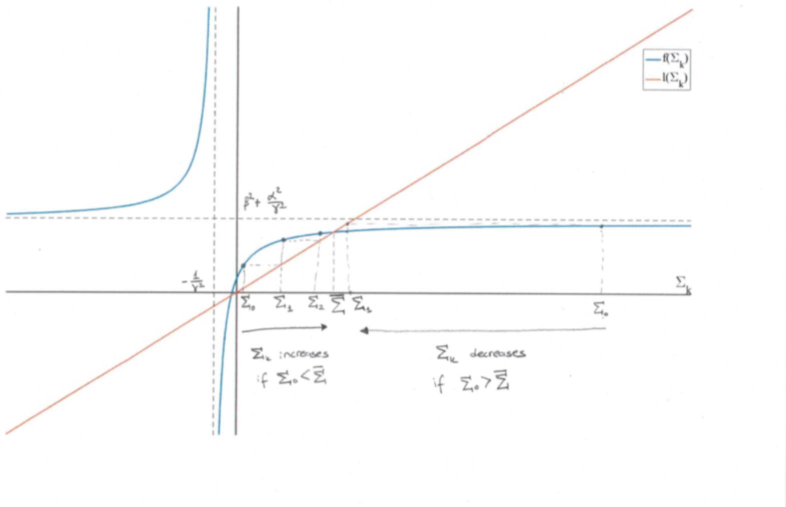

Figure 2: Convergence of Σ_k sequences

Solution: See Figure 2 for the sequences Σ_0, Σ_1, Σ_2, $\dots$.

(b) Assume now that $|\alpha| > 1$, $\beta \neq 0$ but $\gamma = 0$.

i. Derive the DRE.

Solution: From (2), we have $\Sigma_{k+1} = \beta^2 + \alpha^2 \Sigma_k$ by replacing $\gamma = 0$.

ii. Adapt the graphical method in point (2.a.ii.) for assessing if the DRE is converging or not.

Solution: For the graphical method, we need to first plot $f(\Sigma_k) = \alpha^2 \Sigma_k + \beta^2$ and $l(\Sigma_k) = \Sigma_k$. After doing so, the graphical procedure can be still applied, but the conditions in the point (2.a.ii.B.) become:

- If $\Sigma_k < \bar{\Sigma}$, then $\Sigma_{k+1} < \Sigma_k$ (because $f(\Sigma_k) < l(\Sigma_k)$) and Σ_k decreases monotonically
- If $\Sigma_k > \bar{\Sigma}$, then $\Sigma_{k+1} > \Sigma_k$ (because $f(\Sigma_k) > l(\Sigma_k)$) and Σ_k increases monotonically

Therefore, the DRE is diverging. This can also be seen from $\Sigma_{k+1} = \alpha^2 \Sigma_k + \beta^2$, which is an unstable LTI system fed by the constant input β^2. The graphical method is shown in Figure 3.

(c) Assume that $|\alpha| > 1$, $\beta^2 = 0$, $\gamma \neq 0$. Use MATLAB for plotting the new functions $f(\Sigma_k)$ and $l(\Sigma_k)$ and repeat the analysis about the convergence of the DRE.

Solution: The plot of $f(\Sigma_k) = \frac{\alpha^2 \Sigma_k}{1+\gamma^2 \Sigma_k}$ and $l(\Sigma_k) = \Sigma_k$ is given in Figure 4.

Proceeding as in point (a), one finds that

- There are two solutions to the ARE $\bar{\Sigma} = \frac{\alpha^2 \bar{\Sigma}}{1+\gamma^2 \bar{\Sigma}}$: one is $\bar{\Sigma}_1 = 0$ and the other $\bar{\Sigma}_2 > 0$.
- One obtains $\Sigma_k \to 0$ only if $\Sigma_0 = 0$. Otherwise, Σ_k converges to $\bar{\Sigma}_2$.

(d) The previous analysis shows that Σ_k diverges when $\gamma = 0$. Can you provide an interpretation of this result (**Hint:** Look at (1))?

Solution: The output does not carry information about the state of the system. The Kalman Predictor hence works in open-loop and tries to mimic the system behavior. So, $\hat{x}_{k+1|k}$ diverges

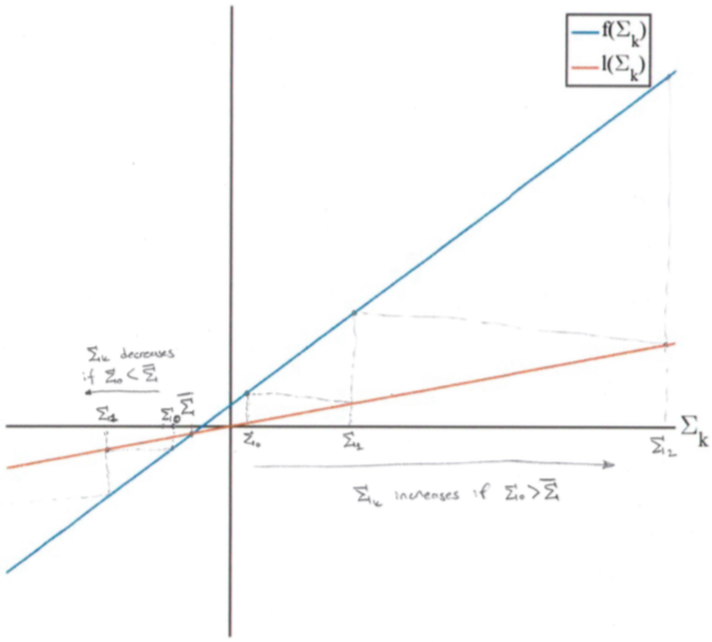

Figure 3: Divergence of Σ_k sequences

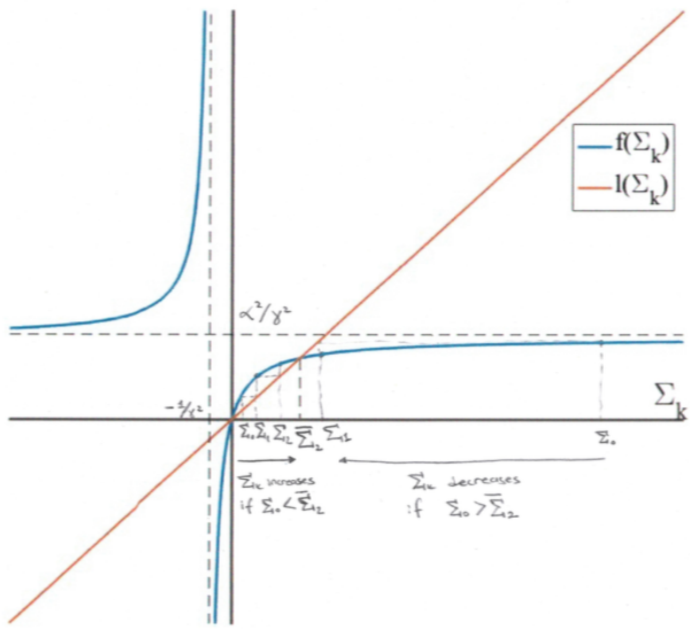

Figure 4: Convergence of Σ_k sequences

from x_k. Moreover, since $x_{0|-1} \sim N\left(\bar{x}_0, \bar{\Sigma}\right)$ and, in general, $\bar{x}_0 \neq x_0$, the error $e_k = x_k - \hat{x}_{k|k-1}$ diverges.

Therefore, one expects that the error variance Σ_k diverges as well.