Correction Series 4.

Exercice 1

a) Let T be one facet of the considered polyedron approximation. The plane of this triangle cut the sphere along a circle $\mathcal{C}$ of radius ρ.

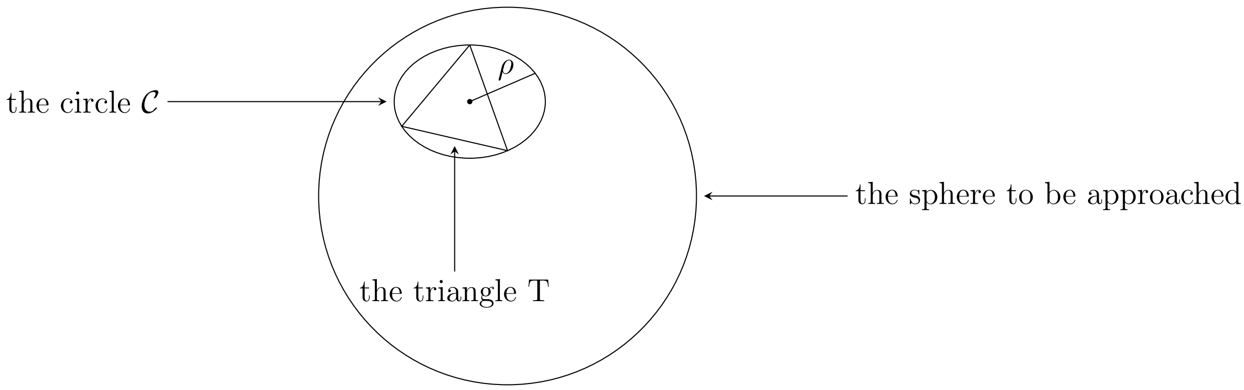

Figure 1: The sphere and the triangle T

The radius ρ is linked to the deviation height by the relation (see Fig. 2)

$$h = R - \sqrt{R^2 - \rho^2} = R(1 - \sqrt{1 - \frac{\rho^2}{R^2}})$$

i.e

$$h \simeq \frac{\rho^2}{2R}, \tag{1}$$

if the ratio $\frac{\rho}{R}$ is sufficently small to justify the Taylor expansion:

$$\sqrt{1 - \frac{\rho^2}{R^2}} \simeq 1 - \frac{\rho^2}{2R^2}.$$

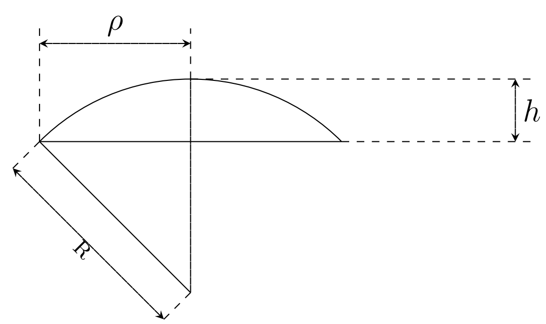

Figure 2: The triangle T and the sphere (cross section)

The exercise essentially amounts to link ρ and the area A of triangle T.
It can be observed that $\mathcal{C}$ is the circumcircle to triangle T. The radius ρ of the circumcircle to an equilateral triangle with side L is (see Fig. 3):

$$\rho = \frac{L}{2}\frac{1}{\cos 30^o} = \frac{L}{\sqrt{3}}. \tag{2}$$

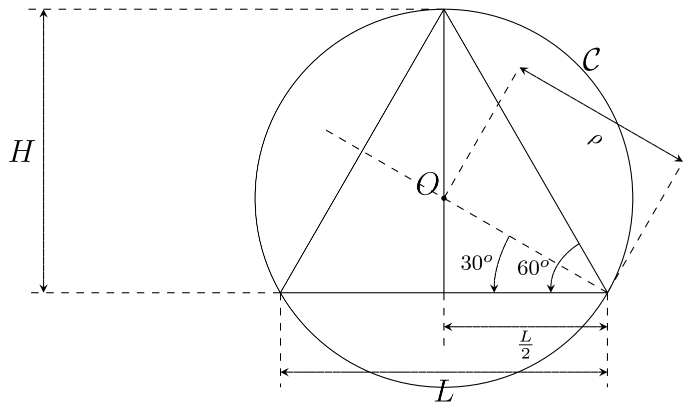

Figure 3: The elements of an equilateral triangle

On the other hand, the area of an equilateral triangle equilateral with side L is (see Fig. 3):

$$A = \frac{1}{2}L \cdot H = \frac{1}{2}L \cdot \frac{L}{2}\tan 30^o = \frac{\sqrt{3}}{4}L^2. \tag{3}$$

Using (3) to replace L in (2), we find

$$\rho^2 = \frac{4}{3\sqrt{3}}A.$$

and the relationship to be proved is a direct consequence of (1).

b) If T is a common triangle, the relationship (1) is still valid but the formula for the deviation height is replaced by a more general expression:

$$h \simeq \frac{C(T)}{2\pi}\frac{A}{R} \tag{4}$$

where $C(T)$ is the proportionality constant between the area of the triangle and the area of its circumscircle:

$$C(T) \equiv \frac{\pi\rho^2}{A}. \tag{5}$$

We saw that this constant was

$$C(T) = \frac{4\pi}{3\sqrt{2}} \simeq 2.4184$$

in the case of an equilateral triangle T. In any other cases, the value of $C(T)$ is bigger:

$$C(T) > 2.4184 \text{ if } T \text{ is not equilateral.}$$

In particular, it is very large if T has an angle which tends to 0

$$C(T) \to \infty \text{ if } \text{angle}_{min}(T) \to 0.$$

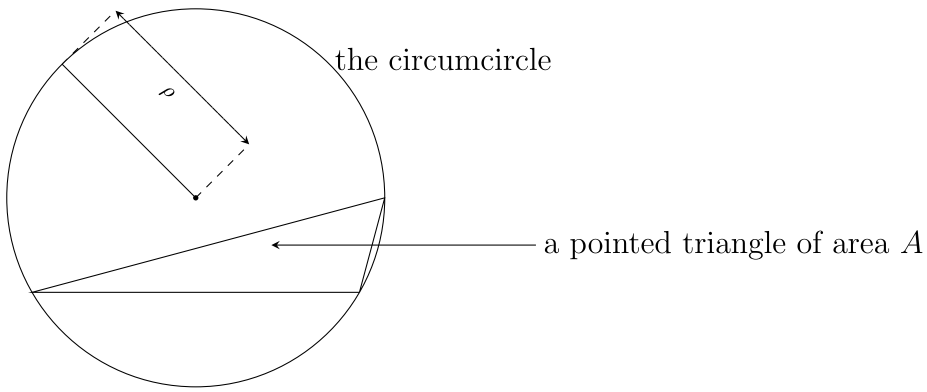

Figure 4: Situation where the ration $\frac{\pi\rho^2}{A}$ is very large

In that case, the estimate (4) of the deviation height is quite pessimistic.