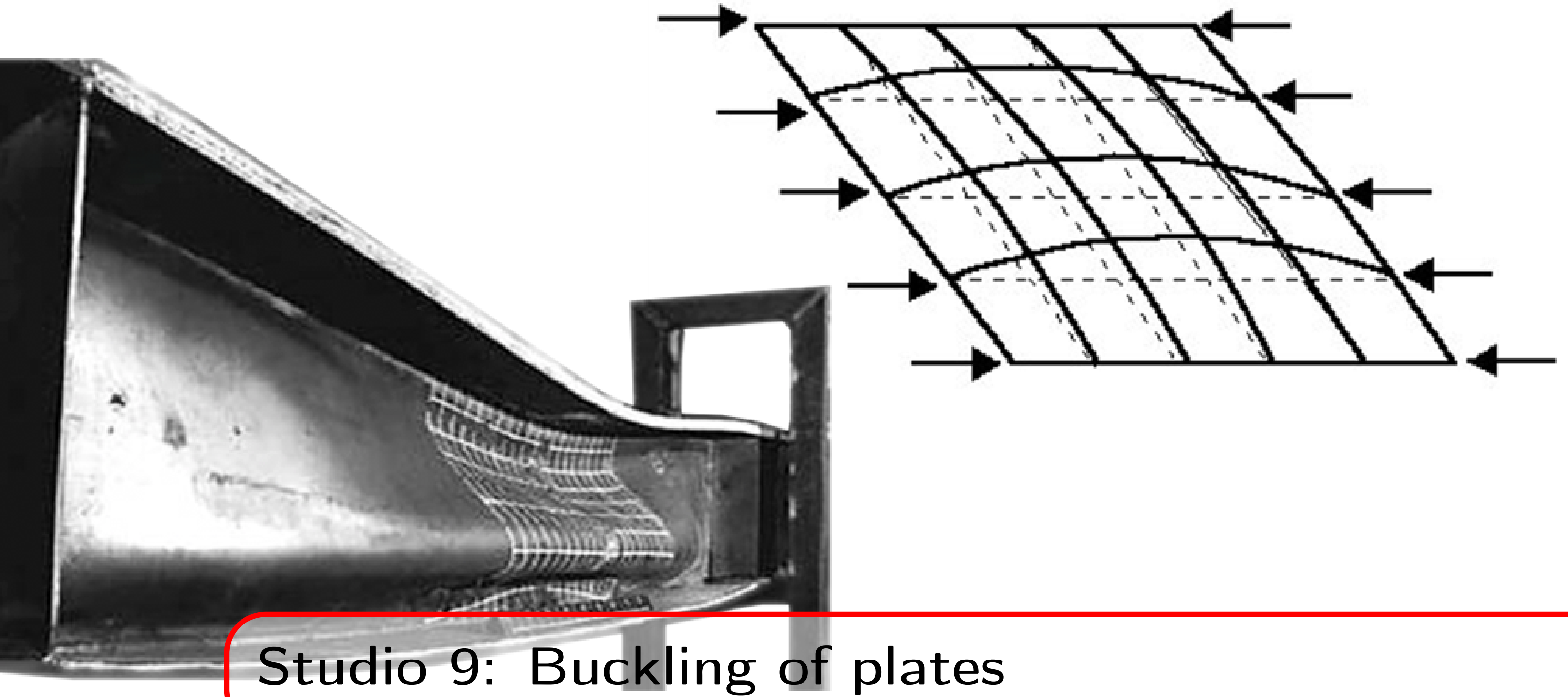
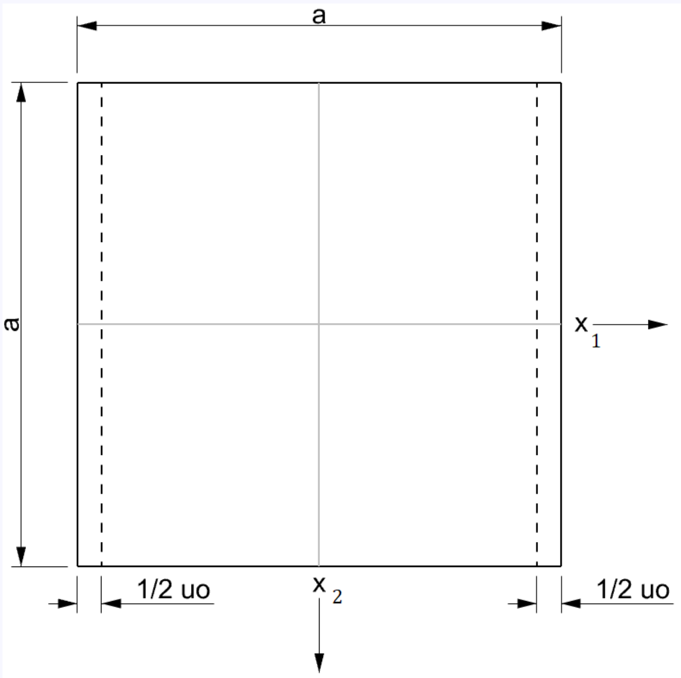


Studio 9: Buckling of plates

Exercise 9.1



S 9.0.1 **Problem 1**

In lectures, we derived the critical force P_{cr} required to buckle a rectangular plate by assuming the deformed shape is sinusoidal with a certain wavenumber. However, we did not study the amplitude of the buckled solution. In this studio, we investigate the post-buckling deformation by considering the in-plane stretching of the plate. The plate under consideration is square with side length a. We assume all edges are simply supported and an in-plane displacement $\tilde{u}$ is applied along the x_1-direction, as sketched above. The origin of the coordinate system is taken to be the centre of the square.

Questions:

1. What is the function that describes the out-of-plane deformation $w(x_1, x_2)$ assuming a sinusoidal buckled shape with amplitude $\tilde{w}$, i.e., how many half-waves are there?

   We begin by deriving the bending energy U_b.

2. Calculate the components of the curvature tensor $\mathcal{K}_{\alpha\beta}$, assuming out-of-plane deformation is given by the fundamental ($n = 1$) buckling mode derived in lectures:

$$w = \tilde{w} \cos\left(\frac{\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right). \tag{9.1}$$

3. State the bending energy density in terms of the components of the curvature tensor.

4. Using the curvature components $\mathcal{K}_{\alpha\beta}$ computed in question 2, calculate the total bending energy (assuming D is the bending stiffness of the plate). Your answer should be proportional to $\tilde{w}^2/a^2$.

   We now focus on the stretching energy U_s.

5. What conditions do the in-plane displacements u_1 and u_2 satisfy?
   Hint: recall that the plate is simply supported, the origin is at the plate center (i.e. there is symmetry about x_1 and x_2), and a displacement $\tilde{u}$ is applied in the x_1-direction.

6. Derive the components of the in-plane strain tensor $E_{\alpha\beta}$ assuming the following in-plane deformation:

$$u_1 = q_1 \sin\left(\frac{2\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right) - \frac{1}{2}\tilde{u}\, x_1, \tag{9.2}$$

$$u_2 = q_1 \sin\left(\frac{2\pi x_2}{a}\right) \cos\left(\frac{\pi x_1}{a}\right). \tag{9.3}$$

   In view of your answers to question 5, why is this a reasonable ansatz for u_1 and u_2?

7. State the stretching energy in terms of the components of the in-plane strain tensor, assuming the in-plane stiffness is C. (Note that the derivation is tedious to do by hand.)

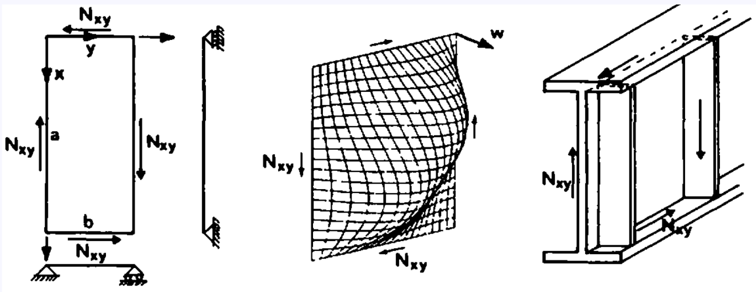
8. Assuming that the total energy is $U = U_\mathrm{b} + U_\mathrm{s}$, derive an expression for q_1.
   Hint: use a variational argument, i.e. $\partial U/\partial q_1 = 0$.

9. Hence obtain an equation for the buckled amplitude $\tilde{w}$.

10. Noting the different solutions that are possible, sketch how $\tilde{w}$ evolves with $\tilde{u}$.

S 9.0.2 **Problem 2: shear buckling**

In this problem we consider the following question: given a rectangular plate of side lengths a and b, what is the shear force N_{12} that causes the plate to buckle? Note that the solution ansatz for w used in the previous problem (i.e. equation 9.1) is no longer applicable, since the deformation due to shear is generally not symmetric about the plate centre. However, we can still approximate the solution in terms of trigonometric functions. In this problem, for simplicity, we will limit ourselves to the pre-buckling behavior. The origin of the coordinate system is now at the plate corner, as sketched below.



Questions:

1. Compared to the first problem, since we consider only the pre-buckling behavior, what simplifications can we say about the in-plane displacements (u_1, u_2) and the in-plane stress resultants (N_{11}, N_{22})?

2. Calculate the total bending energy, assuming the following out-of-plane behavior:

$$w = q_1 \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{b}\right) + q_2 \sin\left(\frac{2\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{b}\right). \tag{9.4}$$

3. What is the total stretching energy in terms of the applied shear N_{12}?

4. What is the critical shear force N_{cr} required to buckle the plate? (Hint: use the variational arguments for q_1 and q_2.)

5. At what aspect ratio does the plate require the least amount of shear to buckle?

6. Is it possible to solve for q_1 and q_2 given the assumptions we have made? Why/why not?

Further hints for Problem 1

Questions 1 to 6 can be answered by following Lecture 9 directly.

7. The total stretching energy is

$$U_s = -\frac{C}{8a^2}\Bigg\{ - a^4\tilde{u}^2 + \left[(\nu - 9)\,\pi^2 {q_1}^2 + \frac{\pi^2\,(\nu+1)\,\tilde{u}\,\tilde{w}^2}{2} - \frac{64\,(\nu+1)\,{q_1}^2}{9} \right] a^2 \\ - 4\,(\nu - 5/3)\,\pi^2\, q_1 \tilde{w}^2 a - \frac{5\,\pi^4 \tilde{w}^4}{16} \Bigg\}. \tag{9.5}$$

8. The solution for q_1 is

$$q_1 = \frac{(18\,\nu - 30)\,\pi^2 \tilde{w}^2}{a\,[9\pi^2\,(\nu-9) - 64\,(\nu+1)]} \tag{9.6}$$

9. Use the variational argument, i.e. $\partial U/\partial \tilde{w} = 0$, and substitute in the solution for q_1.
10. See the lecture for a qualitatively similar curve.

Further hints for Problem 2

Question 1 can be answered by following Lecture 9 directly.

2. The total bending energy is

$$\begin{aligned} U_b &= \int_0^b \int_0^a u_b \, dx_1 dx_2 \\ &= D\,\frac{\pi^4\left({q_1}^2 + 16\,{q_2}^2\right)\left(a^2+b^2\right)^2}{8b^3a^3}. \end{aligned} \tag{9.7}$$

3. U_s can be derived as,

$$U_s = -\frac{32}{9}\,N_{12}\,q_1\,q_2. \tag{9.8}$$

4. The critical shear for buckling is

$$N_{cr} = \frac{9\pi^4}{32} Dab \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2. \tag{9.9}$$

5. Find the minima of N_{cr} as a function of a while keeping the rest constant.

■