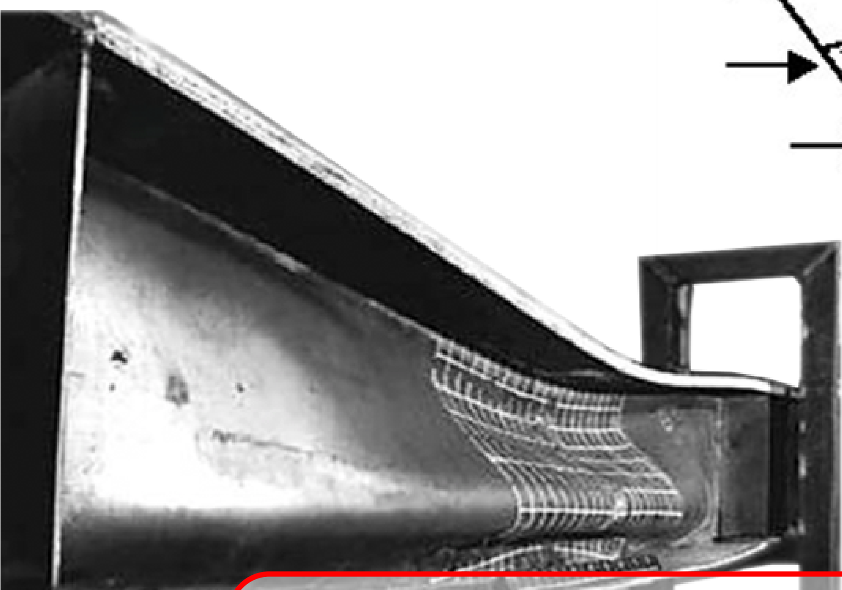
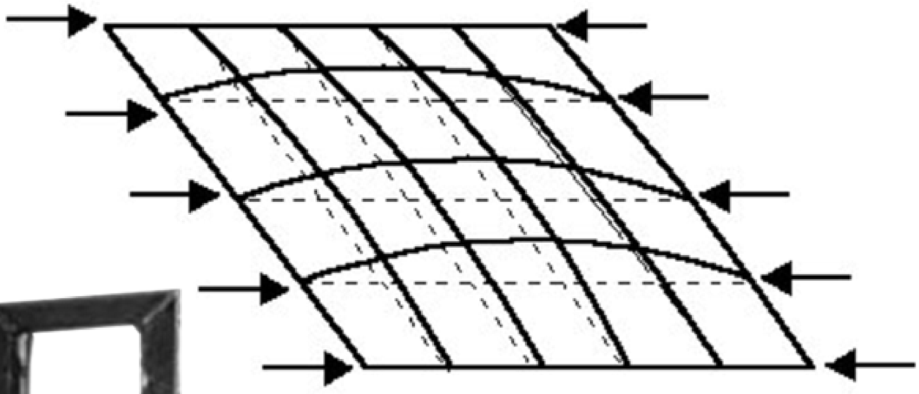
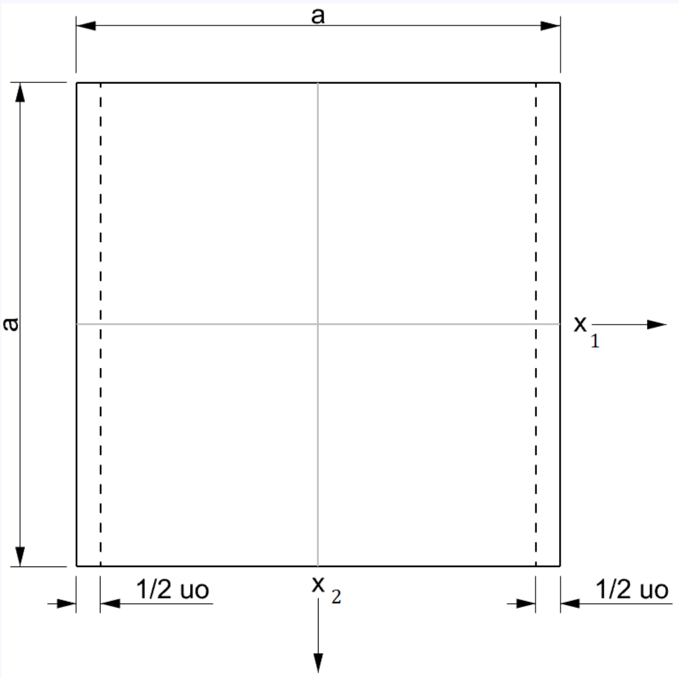




Studio 9: Buckling of plates

Exercise 9.1



S 9.0.1 **Problem 1**

In lectures, we derived the critical force P_{cr} required to buckle a rectangular plate by assuming the deformed shape is sinusoidal with a certain wavenumber. However, we did not study the amplitude of the buckled solution. In this studio, we investigate the post-buckling deformation by considering the in-plane stretching of the plate. The plate under consideration is square with side length a. We assume all edges are simply supported and a total in-plane displacement u_0 is applied symmetrically along the x_1-direction, as sketched above. The origin of the coordinate system is taken to be the centre of the square.

Questions:

1. What is the function that describes the out-of-plane deformation $w(x_1, x_2)$ assuming a sinusoidal buckled shape with amplitude $\tilde{w}$, i.e., how many half-waves are there?

   Solution:
   The simply supported conditions imply that the out-of-plane deformation is zero at the plate edges:

   $$w(\pm a/2, x_2) = w(x_1, \pm a/2) = 0, \qquad x_1, x_2 \in (-a/2, a/2). \tag{9.1}$$

   Assuming a sinusoidal shape in each direction x_i $(i = 1, 2)$, the displacement is then either symmetric or anti-symmetric about the centre of the square $x_i = 0$; the symmetric modes have the form

   $$\sin\left(\frac{n\pi x_i}{a}\right) \qquad n \text{ even}, \tag{9.2}$$

   while the anti-symmetric modes are

   $$\cos\left(\frac{n\pi x_i}{a}\right) \qquad n \text{ odd}. \tag{9.3}$$

   Hence, there are four types of buckling modes, depending on whether the displacement is symmetric or anti-symmetric in each direction:

   $$w_{nm} = \begin{cases} \tilde{w}\sin\left(\frac{n\pi x_1}{a}\right)\sin\left(\frac{m\pi x_2}{a}\right) & n, m \text{ even}, \\ \tilde{w}\sin\left(\frac{n\pi x_1}{a}\right)\cos\left(\frac{m\pi x_2}{a}\right) & n \text{ even}, \; m \text{ odd}, \\ \tilde{w}\cos\left(\frac{n\pi x_1}{a}\right)\sin\left(\frac{m\pi x_2}{a}\right) & n \text{ odd}, \; m \text{ even}, \\ \tilde{w}\cos\left(\frac{n\pi x_1}{a}\right)\cos\left(\frac{m\pi x_2}{a}\right) & n, m \text{ odd}. \end{cases}$$

   (These modes can also be derived from the expression given in lectures, i.e. equation 6.127, when $a = b$ and we shift the coordinates so that the origin is at the square centre.) The fundamental buckling mode, obtained by setting $n = m = 1$, is composed of one half-wave in each direction:

   $$w_{11} = \tilde{w}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right). \tag{9.4}$$

   We begin by deriving the bending energy U_b in the following questions.

2. Calculate the components of the curvature tensor $\mathcal{K}_{\alpha\beta}$, assuming the out-of-plane deformation is given by the fundamental $(n = 1)$ buckling mode:

   $$w = \tilde{w}\,\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right). \tag{9.5}$$

Solution:
The curvature tensor is given by

$$\mathcal{K}_{\alpha\beta} = -\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}. \tag{9.6}$$

Substituting the above expression for the fundamental buckling mode w gives

$$\mathcal{K}_{11} = -\frac{\partial^2 w}{\partial x_1^2} = \frac{\tilde{w}\pi^2}{a^2}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right), \tag{9.7}$$

$$\mathcal{K}_{22} = -\frac{\partial^2 w}{\partial x_2^2} = \frac{\tilde{w}\pi^2}{a^2}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right), \tag{9.8}$$

$$\mathcal{K}_{12} = -\frac{\partial^2 w}{\partial x_1 \partial x_2} = -\frac{\tilde{w}\pi^2}{a^2}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right). \tag{9.9}$$

3. State the bending energy density in terms of the components of the curvature tensor.

Solution:
From lectures (e.g. section 6.2, page 88), the bending energy density, $u_{\rm b}$, is given by

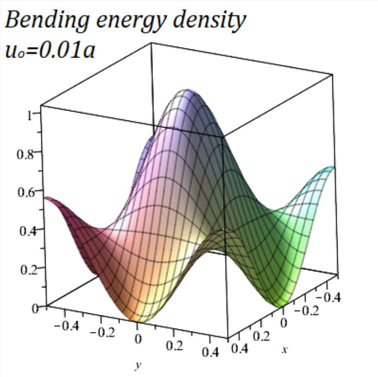
$$u_{\rm b} = D\left[4\mathcal{H}^2 - 2(1-\nu)\mathcal{K}_{\rm G}\right], \tag{9.10}$$

where D is the bending stiffness, $\mathcal{H}$ is the mean curvature and $\mathcal{K}_{\rm G}$ is the Gaussian curvature of the midsurface. These are given in terms of the curvature tensor $\mathcal{K}_{\alpha\beta}$:

$$\mathcal{H} = \frac{\mathcal{K}_{11} + \mathcal{K}_{22}}{2}, \quad \mathcal{K}_{\rm G} = \mathcal{K}_{11}\mathcal{K}_{22} - \mathcal{K}_{12}^2. \tag{9.11}$$

In terms of the curvature components, we then have

$$u_{\rm b} = D\left[\mathcal{K}_{11}^2 + \mathcal{K}_{22}^2 + 2\nu\mathcal{K}_{11}\mathcal{K}_{22} + 2(1-\nu)\mathcal{K}_{12}^2\right]. \tag{9.12}$$



4. Using the curvature components $\mathcal{K}_{\alpha\beta}$ computed in question 2, calculate the total bending energy (assuming D is the bending stiffness of the plate). Your answer should be proportional to $\tilde{w}^2/a^2$.

Solution:
The total bending energy is obtained by integrating u_b over the plate:

$$\begin{aligned} U_{\mathrm{b}} &= \frac{1}{2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} u_{\mathrm{b}}\,\mathrm{d}x_1\mathrm{d}x_2 \\ &= \frac{D}{2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\left[\mathcal{K}_{11}^2+\mathcal{K}_{22}^2+2\nu\mathcal{K}_{11}\mathcal{K}_{22}+2(1-\nu)\mathcal{K}_{12}^2\right]\,\mathrm{d}x_1\mathrm{d}x_2. \end{aligned} \tag{9.13}$$

We substitute the expressions for the components in 9.7–9.9 and simplify using the integrals:

$$\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\cos^2\left(\frac{\pi x_1}{a}\right)\cos^2\left(\frac{\pi x_2}{a}\right)\,\mathrm{d}x_1\mathrm{d}x_2 = \frac{a^2}{4}, \tag{9.14}$$

$$\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\sin^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right)\,\mathrm{d}x_1\mathrm{d}x_2 = \frac{a^2}{4}. \tag{9.15}$$

We obtain

$$U_{\mathrm{b}} = \frac{\pi^4 D\,\tilde{w}^2}{2a^2}. \tag{9.16}$$

We now focus on the stretching energy U_{s}.

5. What conditions do the in-plane displacements u_1 and u_2 satisfy?
Hint: recall that the plate is simply supported, the origin is at the plate center (i.e. the domain is symmetric with respect to x_1 and x_2), and a total displacement u_0 is applied in the x_1-direction.

Solution:
The applied displacement u_0 implies that

$$u_1(\pm a/2, x_2) = \mp u_0/2, \quad u_2(x_1, \pm a/2) = 0, \qquad x_1, x_2 \in (-a/2, a/2). \tag{9.17}$$

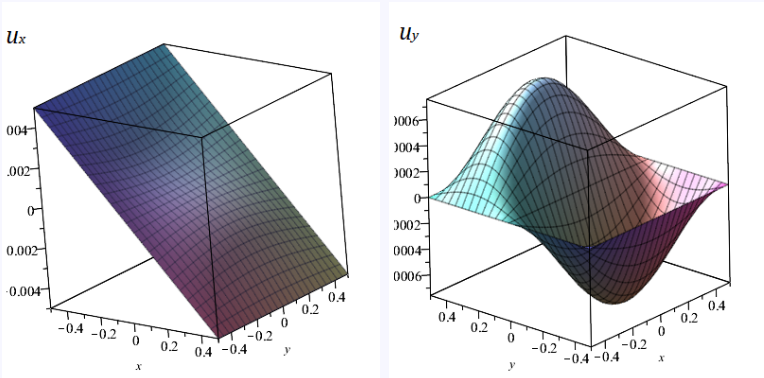
(We note that u_1 is of magnitude $u_0/2$ at $x_1 = \pm a/2$ since the loading is applied symmetrically.) The symmetry about the origin implies that the displacement u_1 is anti-symmetric about $x_1 = 0$ (to be consistent with the boundary conditions in 9.17) while u_2 is symmetric about $x_2 = 0$, i.e.

$$u_1(-x_1, x_2) = -u_1(x_1, x_2), \quad u_2(x_1, -x_2) = u_2(x_1, x_2), \qquad x_1, x_2 \in (-a/2, a/2). \tag{9.18}$$

Since the plate cannot tear or intersect itself at the origin, this symmetry also implies that the in-plane deformation is zero there:

$$u_1(0,0) = u_2(0,0) = 0. \tag{9.19}$$

The surface plot below illustrates the antisymmetry and symmetry of $u_1 = u_x$ and $u_2 = u_y$ with respect to x_1 and x_2 about the plate centre.



6. Derive the components of the in-plane strain tensor $E_{\alpha\beta}$ assuming the following in-plane deformation:

$$u_1 = q_1 \sin\left(\frac{2\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right) - \frac{\tilde{u}x_1}{2}, \tag{9.20}$$

$$u_2 = q_1 \cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right), \tag{9.21}$$

where we define the dimensionless end-displacement

$$\tilde{u} = \frac{2u_0}{a}. \tag{9.22}$$

In view of your answers to question 5, why is this a reasonable ansatz for u_1 and u_2?

Solution:
The components of the in-plane stress tensor are

$$E_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2, \tag{9.23}$$

$$E_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2, \tag{9.24}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right). \tag{9.25}$$

Substituting the above solution ansatz yields

$$E_{11} = \frac{2\pi q_1}{a}\cos\left(\frac{2\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right) - \frac{\tilde{u}}{2} + \frac{\pi^2\tilde{w}^2}{2a^2}\sin^2\left(\frac{\pi x_1}{a}\right)\cos^2\left(\frac{\pi x_2}{a}\right), \tag{9.26}$$

$$E_{22} = \frac{2\pi q_1}{a}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{2\pi x_2}{a}\right) + \frac{\pi^2\tilde{w}^2}{2a^2}\cos^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right), \tag{9.27}$$

$$\begin{aligned} E_{12} = &-\frac{\pi q_1}{2a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right] \\ &+ \frac{\pi^2\tilde{w}^2}{2a^2}\sin\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\cos\left(\frac{\pi x_2}{a}\right). \end{aligned} \tag{9.28}$$

The solution ansatz given in 9.20 and 9.21 for u_1 and u_2 are reasonable: they are anti-symmetric/symmetric about the origin according to 9.18, and we can verify that they satisfy the boundary conditions 9.17 and the condition 9.19 at the plate centre.

7. State the stretching energy in terms of the components of the in-plane strain tensor, assuming the in-plane stiffness is C. (Note that the derivation is tedious to do by hand.)

Solution:
We use the expression for the stretching energy density in terms of the strain components:

$$u_{\mathrm{s}} = C\left[E_{11}^2 + E_{22}^2 + 2\nu E_{11}E_{22} + 2(1-\nu)E_{12}^2\right]. \tag{9.29}$$

Substituting in the expressions for the components found in the last question, we obtain the somewhat lengthy result:

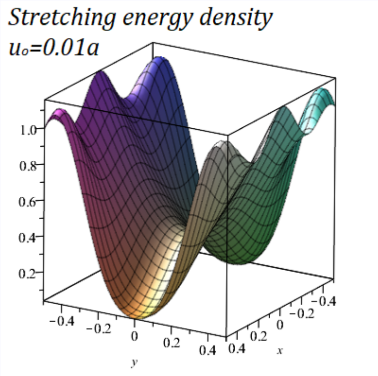
$$\begin{aligned} \frac{u_{\mathrm{s}}}{C} = &\left[\frac{2\pi q_1}{a}\cos\left(\frac{2\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right) - \frac{\tilde{u}}{2} + \frac{\pi^2\tilde{w}^2}{2a^2}\sin^2\left(\frac{\pi x_1}{a}\right)\cos^2\left(\frac{\pi x_2}{a}\right)\right]^2 \\ &+ \left[\frac{2\pi q_1}{a}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{2\pi x_2}{a}\right) + \frac{\pi^2\tilde{w}^2}{2a^2}\cos^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right)\right]^2 \\ &+ 2\nu\left[\frac{2\pi q_1}{a}\cos\left(\frac{2\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right) - \frac{\tilde{u}}{2} + \frac{\pi^2\tilde{w}^2}{2a^2}\sin^2\left(\frac{\pi x_1}{a}\right)\cos^2\left(\frac{\pi x_2}{a}\right)\right] \\ &\quad\times\left[\frac{2\pi q_1}{a}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{2\pi x_2}{a}\right) + \frac{\pi^2\tilde{w}^2}{2a^2}\cos^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right)\right] \\ &+ 2(1-\nu)\left\{-\frac{\pi q_1}{2a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]\right. \\ &\quad\left.+ \frac{\pi^2\tilde{w}^2}{2a^2}\sin\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\cos\left(\frac{\pi x_2}{a}\right)\right\}^2. \end{aligned} \tag{9.30}$$

We derive the total stretching energy by integrating over the plate dimensions:

$$U_{\rm s} = \frac{1}{2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} u_{\rm s}\,{\rm d}x_1{\rm d}x_2. \tag{9.31}$$

While the expression for u_s is lengthy, we note that various terms integrate to zero due to periodicity of the trigonometric functions; the remaining integrals can be evaluated similarly to 9.14–9.15. The result is

$$U_{\rm s} = \frac{C}{8a^2}\left\{a^4\tilde{u}^2 + \left[\pi^2(9-\nu)\,{q_1}^2 - \frac{\pi^2(1+\nu)\,\tilde{u}\tilde{w}^2}{2} + \frac{64\,(1+\nu)\,{q_1}^2}{9}\right]a^2 + \frac{4\pi^2(3\nu-5)\,q_1\tilde{w}^2}{3}a + \frac{5\pi^4\tilde{w}^4}{16}\right\}. \tag{9.32}$$



8. Assuming that the total energy is $U = U_{\rm b} + U_{\rm s}$, derive an expression for q_1.
Hint: use a variational argument, i.e. $\partial U/\partial q_1 = 0$.

Solution:
We calculate the partial derivative

$$\frac{\partial U}{\partial q_1} = \frac{C}{4a}\left\{q_1\left[\frac{64\,(1+\nu)}{9} + \pi^2(9-\nu)\right]a - \frac{2\pi^2(5-3\nu)\,\tilde{w}^2}{3}\right\}. \tag{9.33}$$

The stationarity condition $\partial U/\partial q_1 = 0$ then gives

$$q_1 = \frac{6\pi^2(5-3\nu)\,\tilde{w}^2}{a\,[64\,(1+\nu) + 9\pi^2(9-\nu)]}. \tag{9.34}$$

We see that the amplitude of the in-plane deformation depends on the out-of-plane deformation $\tilde{w}$. This coupling between the in-plane strains and out-of-plane displacement is inherent to the Föppl–von Kármán equations, and will allow us to solve for the buckling amplitude as a function of the imposed end-displacement $\tilde{u}$.

9. Hence obtain an equation for the buckled amplitude $\tilde{w}$.

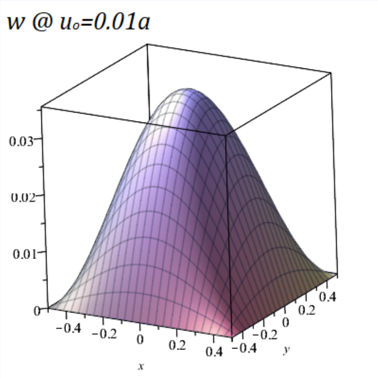
Solution:
We calculate the partial derivative

$$\frac{\partial U}{\partial \tilde{w}} = \frac{\pi^2 \tilde{w}}{a^2} \left\{ \left[\frac{5\pi^2 \tilde{w}^2}{32} - \frac{(1+\nu)\, a^2 \tilde{u}}{8} - \frac{(5-3\nu)\, a q_1}{3} \right] C + \pi^2 D \right\}. \quad (9.35)$$

Substituting the above solution for q_1 and simplifying,

$$\frac{\partial U}{\partial \tilde{w}} = \frac{\pi^2 \tilde{w}}{32a^2} \left\{ 32\pi^2 D - 4(1+\nu) a^2 C \tilde{u} + \pi^2 C \tilde{w}^2 \frac{45\pi^2(9-\nu) - 64(9\nu^2 - 35\nu + 20)}{9\pi^2(9-\nu) + 64(1+\nu)} \right\}. \quad (9.36)$$

The amplitude $\tilde{w}$ then satisfies the stationarity condition $\partial U / \partial \tilde{w} = 0$.



10. Noting the different solutions that are possible, sketch how $\tilde{w}$ evolves with $\tilde{u}$.

Solution:
The trivial solution $\tilde{w} = 0$, representing the unbuckled state, is always a solution of $\partial U / \partial \tilde{w} = 0$. The other solutions are found by considering when the term in braces in 9.36 is zero:

$$32\pi^2 D - 4(1+\nu) a^2 C \tilde{u} + \pi^2 C \tilde{w}^2 \frac{45\pi^2(9-\nu) - 64(9\nu^2 - 35\nu + 20)}{9\pi^2(9-\nu) + 64(1+\nu)} = 0. \quad (9.37)$$

Note the form of the three terms on the left-hand side: (i) a constant term; (ii) a term proportional to $\tilde{u}$ with negative coefficient; and (iii) a term that is proportional to $\tilde{w}^2$. It may also be shown that the coefficient of $\tilde{w}^2$ is positive, consistent with our expectation that $\partial U / \partial \tilde{w} \to +\infty$ as $\tilde{w} \to +\infty$. The equation $\partial U / \partial \tilde{w} = 0$ therefore admits real solutions (for which $\tilde{w}^2 > 0$) only when $\tilde{u}$ is above the critical value

$$\tilde{u}_{cr} = \frac{8\pi^2 D}{(1+\nu) a^2 C}, \quad (9.38)$$

which corresponds to the critical displacement needed for buckling. In dimensional terms, recalling from question 6 that $\tilde{u} = 2u_0/a$, the critical displacement is

$$u_{0cr} = \frac{4\pi^2 D}{(1+\nu) a C}. \quad (9.39)$$

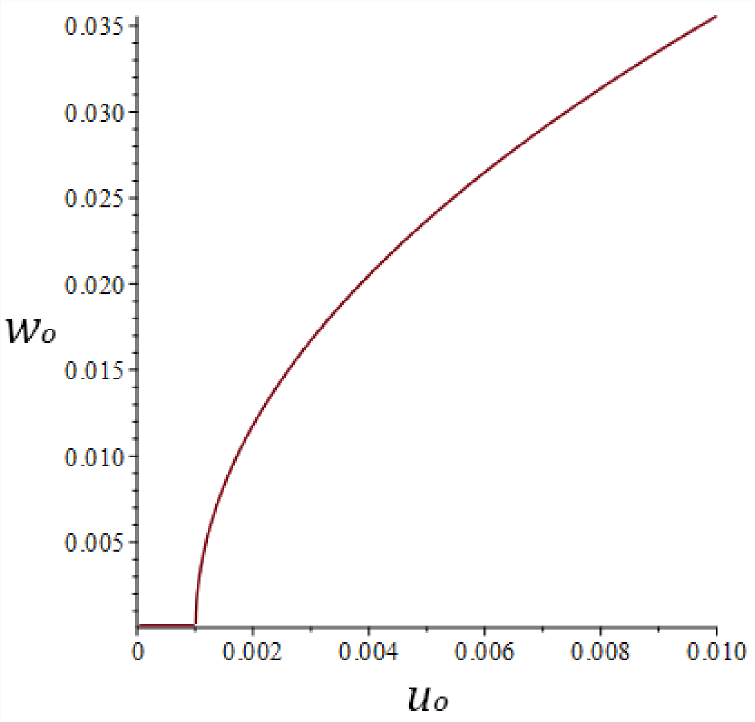
For example, for an isotropic material with Young's modulus E and plate thickness h, we have

$$C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{9.40}$$

which gives

$$u_{0cr} = \frac{\pi^2 h^2}{3(1+\nu)a}. \tag{9.41}$$

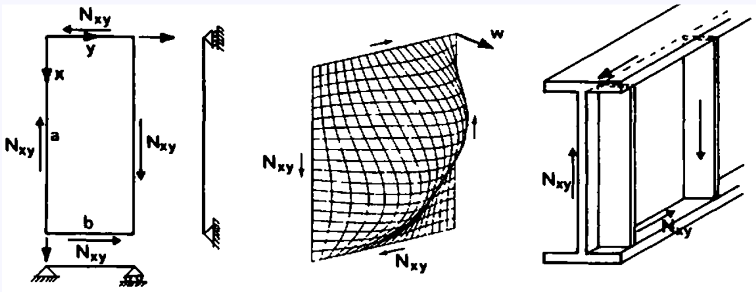
The typical behaviour of the solution branches, demonstrating the appearance of the buckled solution for $u_0 > u_{0cr}$, is shown in the plot below. Note that while $\tilde{w} = 0$ is always a solution, it can be shown that it is linearly unstable above u_{0cr} so that the buckled solution is always observed in practice.



S 9.0.2

Problem 2: shear buckling

In this problem we consider the following question: given a rectangular plate of side lengths a and b, what is the shear force N_{12} that causes the plate to buckle? Note that the solution ansatz for w used in the previous problem (i.e. equation 9.5) is no longer applicable, since the deformation due to shear is generally not symmetric (or anti-symmetric) about the plate centre. However, we can still approximate the solution in terms of trigonometric functions. In this problem, for simplicity, we will limit ourselves to the onset of buckling behavior (in which the buckling amplitude is infinitesimally small). Note that the origin of the coordinate system is now at the plate corner, as sketched below.



Questions:

1. Compared to the first problem, since we consider only the onset of buckling, what simplifications can we say about the in-plane displacements (u_1, u_2) and the in-plane stress resultants (N_{11}, N_{22})?

   Solution:
   Since the buckling amplitude is infinitesimally small, we can linearize the stress resultants about the stress field associated with the pre-buckled (planar) solution, for which the only non-zero stress resultant is the applied shear, N_{12}. Moreover, since the in-plane displacements u_1 and u_2 are zero prior to buckling (in contrast to Problem 1 where we imposed a non-zero displacement at the plate boundaries), we can approximate them as zero. This greatly simplifies the stretching energy calculation.

2. Calculate the total bending energy, assuming the following out-of-plane behavior:

$$w = q_1 \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{b}\right) + q_2 \sin\left(\frac{2\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{b}\right). \tag{9.42}$$

   Solution:
   Following the same procedure as Problem 1, we first calculate the components

of the curvature tensor using the above ansatz for w:

$$\mathcal{K}_{11} = -\frac{\partial^2 w}{\partial x_1^2} = \frac{\pi^2}{a^2}\left[q_1 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{b}\right) + 4q_2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{b}\right)\right], \tag{9.43}$$

$$\mathcal{K}_{22} = -\frac{\partial^2 w}{\partial x_2^2} = \frac{\pi^2}{b^2}\left[q_1 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{b}\right) + 4q_2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{b}\right)\right], \tag{9.44}$$

$$\mathcal{K}_{12} = -\frac{\partial^2 w}{\partial x_1 \partial x_2} = -\frac{\pi^2}{ab}\left[q_1 \cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{b}\right) + 4q_2 \cos\left(\frac{2\pi x_1}{a}\right)\cos\left(\frac{2\pi x_2}{b}\right)\right]. \tag{9.45}$$

The total bending energy is then obtained by integrating the bending energy density u_b over the plate; noting that the origin of the coordinate system is now at the plate corner, this becomes

$$\begin{aligned} U_\mathrm{b} &= \frac{1}{2}\int_0^b \int_0^a u_\mathrm{b}\, \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \frac{D}{2}\int_0^b \int_0^a \left[\mathcal{K}_{11}^2 + \mathcal{K}_{22}^2 + 2\nu \mathcal{K}_{11}\mathcal{K}_{22} + 2(1-\nu)\mathcal{K}_{12}^2\right] \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \frac{\pi^4 \left(a^2+b^2\right)^2 \left({q_1}^2 + 16{q_2}^2\right) D}{8a^3 b^3}. \end{aligned} \tag{9.46}$$

3. What is the total stretching energy in terms of the applied shear N_{12}?

Solution:
The definition of the stretching energy density at the mid-surface is

$$u_\mathrm{s} = N_{\alpha\beta} E_{\alpha\beta}. \tag{9.47}$$

As discussed in question 1, since we consider a infinitesimal buckling amplitude, the only non-zero stress resultant is the constant applied shear $N_{12} = N_{21}$ (i.e. we ignore the correction of the stress and the non-zero in-plane displacements that arise from the out-of-plane displacement). Noting that we pick up both the $(\alpha, \beta) = (1, 2)$ and $(\alpha, \beta) = (2, 1)$ contributions, we therefore have, using symmetry of the stress and strain tensors,

$$u_\mathrm{s} = N_{12}E_{12} + N_{21}E_{21} = 2N_{12}E_{12} = N_{12}\frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}. \tag{9.48}$$

Substituting for w and simplifying the various integrals of trigonometric functions, the total stretching energy is

$$U_\mathrm{s} = \frac{1}{2}\int_0^b \int_0^a u_\mathrm{s}\, dx_1 dx_2 = -\frac{16}{9} N_{12}\, q_1\, q_2. \tag{9.49}$$

4. What is the critical shear force N_{cr} required to buckle the plate? (Hint: use the variational arguments for q_1 and q_2.)

Solution:
The total energy is,

$$U = U_{\mathrm{b}} + U_{\mathrm{s}} \tag{9.50}$$

We set the first variation of U with respect to q_1 and q_2 to zero to arrive at a system of two coupled equations:

$$\frac{\partial U}{\partial q_1} = \frac{\pi^4 D \left(a^2 + b^2\right)^2}{4a^3 b^3} q_1 - \frac{16}{9} N_{12}\, q_2 = 0, \tag{9.51}$$

$$\frac{\partial U}{\partial q_2} = -\frac{16}{9} N_{12}\, q_1 + \frac{4\pi^4 D \left(a^2 + b^2\right)^2}{a^3 b^3} q_2 = 0. \tag{9.52}$$

For a non-trivial buckled solution (with at least one of q_1 or q_2 being non-zero), we require that the determinant of the coefficient matrix is zero. This leads to

$$\left[\frac{\pi^4 D \left(a^2 + b^2\right)^2}{a^3 b^3}\right]^2 - \left(\frac{16 N_{12}}{9}\right)^2 = 0. \tag{9.53}$$

Solving for N_{12}, we obtain the critical value at which buckling occurs:

$$N_{cr} = \frac{9\pi^4 D \left(a^2 + b^2\right)^2}{16 a^3 b^3}. \tag{9.54}$$

5. At what aspect ratio does the plate require the least amount of shear to buckle?

Solution:
To explore the dependence of N_{cr} on the aspect ratio, we imagine fixing b and set

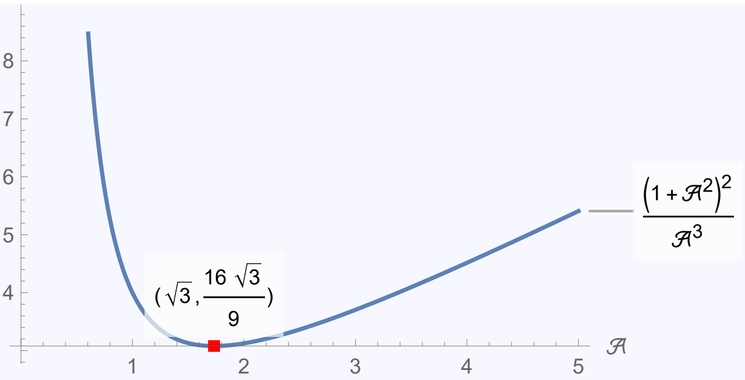
$$a = \mathcal{A} b. \tag{9.55}$$

The critical value 9.54 then becomes

$$N_{cr} = \frac{9\pi^4 D}{16 b^2} \frac{\left(1 + \mathcal{A}^2\right)^2}{\mathcal{A}^3}. \tag{9.56}$$

We then consider the graph of the function $A \mapsto (1 + \mathcal{A}^2)^2/\mathcal{A}^3$, as shown in the plot below. Differentiating shows that this has a single stationary point in the region $\mathcal{A} > 0$, corresponding to a global minimum, at

$$\mathcal{A} = \sqrt{3}, \quad \frac{\left(1 + \mathcal{A}^2\right)^2}{\mathcal{A}^3} = \frac{16\sqrt{3}}{9}. \tag{9.57}$$



Hence, the critical shear is minimized at an aspect ratio of $a = \sqrt{3}b$, with value

$$N_{cr} = \frac{\pi^4\sqrt{3}D}{b^2}. \tag{9.58}$$

6. Is it possible to solve for q_1 and q_2 given the assumptions we have made? Why/why not?

   Solution:
   To find the amplitude of the buckled solution as a function of applied shear, we need to consider the effect of the out-of-plane deformation on the in-plane stretching — this is exactly what we did in Problem 1 for the case of loading by end-displacement. Specifically, for Problem 2, this will lead to corrections to the pre-buckled stress field that will depend on q_1 and q_2.

Further hints for Problem 1

Questions 1 to 6 can be answered by following Lecture 9 directly.

7. The total stretching energy is

$$U_{\mathrm{s}} = -\frac{C}{8a^2}\left\{ -a^4\tilde{u}^2 + \left[(\nu-9)\,\pi^2{q_1}^2 + \frac{\pi^2\,(\nu+1)\,\tilde{u}\,\tilde{w}^2}{2} - \frac{64\,(\nu+1)\,{q_1}^2}{9}\right]a^2 \right.$$
$$\left. -\,4\,(\nu-5/3)\,\pi^2\,q_1\tilde{w}^2 a - \frac{5\,\pi^4\tilde{w}^4}{16}\right\}. \tag{9.59}$$

8. The solution for q_1 is

$$q_1 = \frac{6\pi^2\tilde{w}^2\,(5-3\,\nu)}{a\left[9\pi^2\,(9-\nu)+64\,(\nu+1)\right]} \tag{9.60}$$

9. Use the variational argument, i.e. $\partial U/\partial\tilde{w} = 0$, and substitute in the solution for q_1.
10. See the lecture for a qualitatively similar curve.

Further hints for Problem 2

Question 1 can be answered by following Lecture 9 directly.

2. The total bending energy is

$$\begin{aligned} U_{\mathrm{b}} &= \frac{1}{2}\int_0^b\int_0^a u_{\mathrm{b}}\,\mathrm{d}x_1\mathrm{d}x_2 \\ &= \frac{\pi^4 D\left({q_1}^2+16{q_2}^2\right)\left(a^2+b^2\right)^2}{8a^3b^3}. \end{aligned} \tag{9.61}$$

3. U_{s} can be derived as,

$$U_{\mathrm{s}} = -\frac{16}{9}\,N_{12}\,q_1\,q_2. \tag{9.62}$$

4. The critical shear for buckling is

$$N_{cr} = \frac{9\pi^4 D\left(a^2+b^2\right)^2}{16a^3b^3}. \tag{9.63}$$

5. Find the minima of N_{cr} as a function of a while keeping the rest constant.

■