# Studio 3: Elasto-gravity Bending

**Exercise 3.1** In class, we saw that in the linearized limit of small deformations, the elastic bending energy of a two dimensional sheet with shape $y(x)$ is

$$U[y(x)] = \frac{1}{2}\left[\frac{Ewh^3}{12(1-\nu^2)}\right]\int_0^l y''^2(x)\mathrm{d}x, \tag{3.1}$$

where $l$ is the projected length of the sheet, $h$ is the thickness, $w$ is the width, $\nu$ is the Poisson's ratio, and $E$ is the Young's modulus. The term $B = Ewh^3/[12(1-\nu^2)]$ is often referred to as the bending stiffness, which measures how stiff the sheet is under bending deformations.

For large deformations, the corresponding bending energy is

$$U[\theta(s)] = \frac{1}{2}B\int_0^L \theta'^2(s)\mathrm{d}s, \tag{3.2}$$

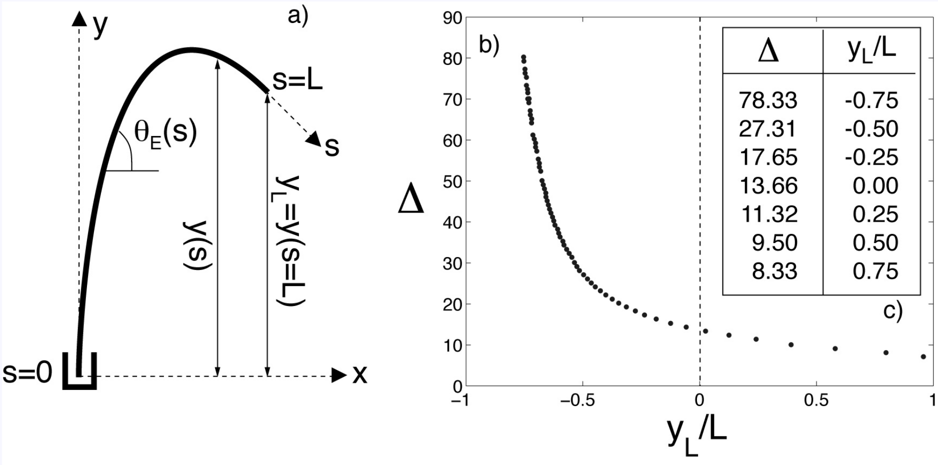
where $L$ is the total length of the sheet, $\theta$ is the angle that the sheet makes with the horizontal, and $s$ is the arc length along the neutral surface of the sheet. Eq. (3.2) can be read in *english* as: "*The bending energy of a thin sheet per unit width ($U/w$) equals one half of the bending modulus ($B/w$) times the curvature $\theta'(s)$ squared, integrated along its total length.*"

Consider the following configuration. A thin sheet of thickness $h$, width $w$, and total length $L$ is clamped vertically at $s = 0$ and is free to bend under gravity otherwise. The boundary conditions are that: 1) the sheet is vertical at the clamp, $\theta(s = 0) = \pi/2$ and 2) there is no curvature at the free end, $\theta'(s = L) = 0$. Assume that the sheet has a linear density (mass per unit length) given by $\rho_l = \rho h w$, where $\rho$ is the volumetric density. A schematic diagram of this configuration is given in the figure below.

**Questions:**

1. Write the total energy functional, $\mathcal{E}[\theta(s)]$, for this thin strip bent under gravity. Make sure that your expression for the energy only depends on $s$, $\theta(s)$ and $\theta'(s)$. (*hint: the fact that* $\mathrm{d}y = \sin\theta \mathrm{d}s$, *i.e.*, $y(s) = \int_0^s \sin\theta \mathrm{d}s$ *will help*).
2. Assuming small functional perturbations, $\delta\theta(s)$, on the equilibrium shape of the beam $\theta_E(s)$, use calculus of variations ($\delta\mathcal{E} \sim \mathcal{E}[\theta_E(s) + \delta\theta(s)] - \mathcal{E}[\theta_E(s)] \to 0$) to show that the equilibrium shape satisfies the following differential equation

$$B\theta_E''(s) = \rho_l g(L - s)\cos\theta_E(s). \tag{3.3}$$




a) Schematic diagram of a thin sheet clamped vertically and bending under gravity. The sheet has dimensions: length $L$, thickness $h$, and width/span $w$. Note that $y_L$ is the height of the free end of the sheet at $s = L$. b) Plot of the dimensionless parameters $\Delta$ and $y_L/L$ for equilibrium shapes $\theta_E$ that satisfy Eq. (3.4). c) Table for some values of $\Delta$ versus $y_L/L$.

3. Non-dimensionalize Eq. (3.3) using $L$ and show that

$$\bar{\theta}_E''(\bar{s}) = \Delta(1 - \bar{s})\cos\bar{\theta}_E(\bar{s}), \tag{3.4}$$

where $\Delta = (L/L_c)^3$ and $L_c$ is often called the elasto-gravity length. What is $L_c$ in terms of the physical quantities in the problem? What is the physical significance of $L_c$?

4. Determine $L_c$ from scaling arguments and ensure that you get the same result as in Question 3.

5. Show that the differential equation in Eq. (3.4) is identical to

$$\frac{1}{2}\bar{\theta}_E'(\bar{s})^2 = \Delta\left[(1 - \bar{s})\sin\bar{\theta}_E + (\bar{y}(\bar{s}) - \bar{y}_L)\right], \tag{3.5}$$

where $\bar{y}(\bar{s}) = y(s)/L$, $\bar{y}_L = y_L/L$, and $y_L = y(s = L)$ is the height of the free end with respect to the horizontal (see the diagram in Figure (a)). (*hint: it will be easier to go backwards from Eq. (3.5) to Eq. (3.4).*)

6. The differential equation that describes the equilibrium shapes of the bent sheet under gravity – Eq. (3.5) – is nonlinear and therefore one has to solve it numerically to find $\theta_E(s)$ (that yields the equilibrium shapes). One of your friends has done this for you and plotted $\Delta$ as a function of $y_L/L$ in Figure (b) (some points of this graph are given in the table, Figure (c)). Develop an experimental method to determine the bending modulus, $B/w$, of a piece of paper (*hint: you may need only one of the points in the table of Figure (c)*).

7. Using your technique, determine the numerical value of the specific bending modulus $B/w$ for the A4 white paper sheet in the printers of EPFL (*hint: you will need the paper density that you will find written on the label packages of the paper rims. Most likely it will be* $\rho h = 75\ g/m^2$).

**General hints:**

1. The total energy functional is:

$$\mathcal{E}[\theta(s)] = \int_0^L \left[\frac{1}{2}B\theta'^2(s) + \rho_l g(L-s)\sin\theta(s)\right] \mathrm{d}s.$$

2. Show that the variation of total energy is given by:

$$\delta\mathcal{E} = \int_0^L \left[-B\,\theta'' + \rho_l g(L-s)\cos\theta\right] \delta\theta \mathrm{d}s.$$

3. Remember that $\theta'_E(s) = \bar{\theta}'_E(\bar{s})/L$.
4. Balance the bending and gravitational energies to obtain $L_c = (B/\rho_l g)^{1/3}$.
5. Differentiate both terms of the equation with respect to $s$ and use $\bar{y}'(\bar{s}) = \sin\bar{\theta}_E(\bar{s})$.
6. We have $y_L = 0$ for a sheet length $L^\star$ and therefore $B/w = \frac{L^{\star 3}\rho_l g}{13.66 w}$.
7. With the values given by the problem: $B/w = 3.7 \times 10^{-4}$ Nm.

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