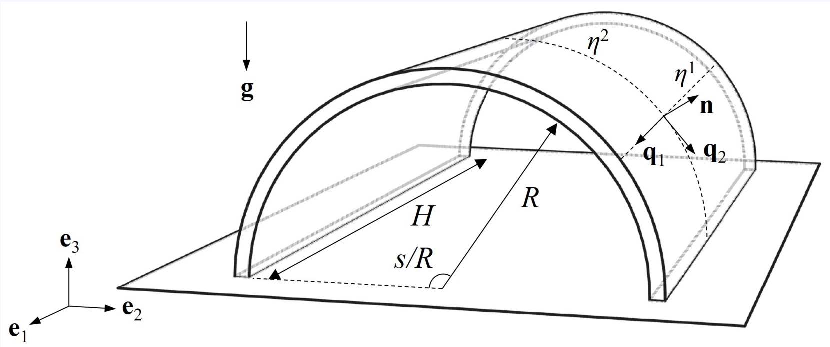
Studio 11: Membrane theory of shells

Exercise 11.1 In this studio, we will use the membrane theory of shells to analyze the membrane forces in thin shell structures under various loading conditions.

S 11.0.1 Problem 1: Cylindrical shell roofs

In this problem, we consider a cylindrical roof structure, as depicted in the figure below, which is an open circular cylindrical shell supported along its two straight edges. We will derive the membrane forces in the roof under its self-weight.



- **Parametrize the mid-surface of a cylindrical roof of radius R and height H in terms of $\eta^1 = x \in [0, H]$ and $\eta^2 = s \in [0, \pi R]$.**
  From the mapping

$$\mathring{\mathbf{r}}(x, s) = \left(x, -R\cos\frac{s}{R}, R\sin\frac{s}{R}\right), \tag{11.1}$$

  we can derive the covariant base vectors via their definition:

$$\mathbf{q}_1 = \frac{\partial \mathring{\mathbf{r}}}{\partial x} = (1, 0, 0), \tag{11.2}$$

$$\mathbf{q}_2 = \frac{\partial \mathring{\mathbf{r}}}{\partial s} = \left(0, \sin\frac{s}{R}, \cos\frac{s}{R}\right). \tag{11.3}$$

The normal vector is just the vector product between $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$\mathbf{n} = \frac{\mathbf{q}_1 \times \mathbf{q}_2}{|\mathbf{q}_1 \times \mathbf{q}_2|} = \left(0, -\cos\frac{s}{R}, \sin\frac{s}{R}\right) . \tag{11.4}$$

The covariant metric is defined as

$$(\mathring{a}_{\alpha\beta}) = (\mathbf{q}_\alpha \cdot \mathbf{q}_\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{11.5}$$

With this metric, there is no distinction between covariant and contravariant components, and the contravariant base vectors are

$$\mathbf{q}^1 = \mathbf{q}_1 \, , \tag{11.6}$$
$$\mathbf{q}^2 = \mathbf{q}_2 \, . \tag{11.7}$$

- **Equilibrium equations of the membrane theory for cylindrical shells:**

$$\boxed{-N^{\alpha 1}{}_{,\alpha} - f^1 = 0} \, , \tag{11.8}$$
$$\boxed{-N^{\alpha 2}{}_{,\alpha} - f^2 = 0} \, , \tag{11.9}$$
$$\boxed{\frac{N_2^2}{R} - p = 0} \, , \tag{11.10}$$

where $\mathbf{N}$ is the membrane force tensor, and f^1, f^2 and p are the load components along the covariant base vectors, $\mathbf{q}^1$ and $\mathbf{q}^2$, and the normal vector, $\mathbf{n}$, respectively.

Question:

Membrane forces in the roof under its self-weight. $q = \rho g t$ is the load per unit area of the shell surface, where ρ is the density of the material, t is the thickness of the shell, and g is the gravitational acceleration.

Solution:

We resolve the load q into components along the covariant base vectors, $\mathbf{q}^1$ and $\mathbf{q}^2$, and the normal vector, $\mathbf{n}$:

$$f^1 = -q\,\mathbf{e}_3 \cdot \mathbf{q}^1 = 0 \, , \tag{11.11}$$
$$f^2 = -q\,\mathbf{e}_3 \cdot \mathbf{q}^2 = -q\cos\frac{s}{R} \, , \tag{11.12}$$
$$p = -q\,\mathbf{e}_3 \cdot \mathbf{n} = -q\sin\frac{s}{R} \, . \tag{11.13}$$

From Eqs.(11.10) and (11.13), we can get the circumferential normal force

$$\boxed{N_2^2 = pR = -qR\sin\frac{s}{R}} \, . \tag{11.14}$$

Substituting for f^2 from Eq.(11.12) into Eq.(11.9), we obtain

$$N^{12}{}_{,1} = -N^{22}{}_{,2} - f^2 = -N^2_{2,2} - f^2 = 2q\cos\frac{s}{R}\,. \tag{11.15}$$

The integral with respect to x of the above equation reads

$$N^1_2 = N^{12} = \int 2q\cos\frac{s}{R}\mathrm{d}x = 2q\cos\frac{s}{R}x + f(s)\,. \tag{11.16}$$

Considering the symmetry with respect to the shell midspan at $x = \frac{H}{2}$, we have

$$N^1_2|_{x=\frac{H}{2}} = qH\cos\frac{s}{R} + f(s) = 0\,. \tag{11.17}$$

Then, we can get

$$f(s) = -qH\cos\frac{s}{R}\,. \tag{11.18}$$

Therefore, the in-plane shear force is

$$\boxed{N^1_2 = 2qH\cos\frac{s}{R}\Big(\frac{x}{H} - \frac{1}{2}\Big)}\,. \tag{11.19}$$

Substituting for f^1 from Eq.(11.11) into Eq.(11.8), we can get

$$N^{11}{}_{,1} + N^{21}{}_{,2} = 0\,, \tag{11.20}$$

which reads

$$N^{11}{}_{,1} = -N^1_{2,2} = 2q\frac{H}{R}\sin\frac{s}{R}\Big(\frac{x}{H} - \frac{1}{2}\Big)\,. \tag{11.21}$$

The integral with respect to x of the above equation yields

$$N^1_1 = N^{11} = \int 2q\frac{H}{R}\sin\frac{s}{R}\Big(\frac{x}{H} - \frac{1}{2}\Big)\mathrm{d}x = q\frac{H}{R}\sin\frac{s}{R}\Big(\frac{x}{H} - 1\Big)x + f(s)\,. \tag{11.22}$$

Applying the boundary conditions at $x = 0$ and $x = H$ to the above equation,

$$N^1_1|_{x=0} = 0\,, \tag{11.23}$$
$$N^1_1|_{x=H} = 0\,, \tag{11.24}$$

we can get

$$f(s) = 0\,. \tag{11.25}$$

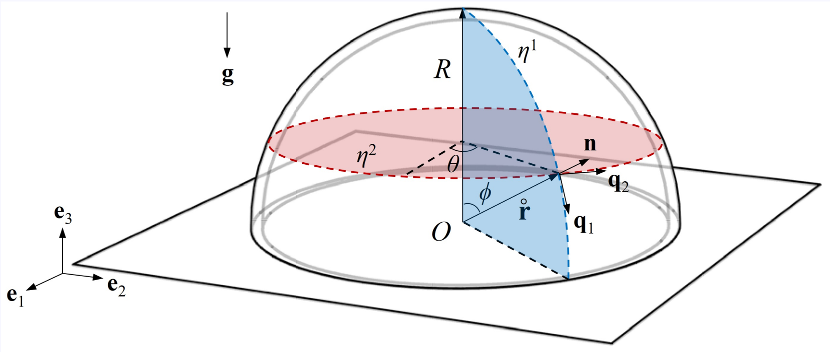
Therefore, the axial normal force N^1_1 is

$$\boxed{N^1_1 = q\frac{H}{R}\sin\frac{s}{R}\Big(\frac{x}{H} - 1\Big)x}\,. \tag{11.26}$$

S 11.0.2

Problem 2: Spherical shell roofs

In this problem, we consider a hemispherical roof structure supported along its equator, as depicted in the figure below. We will analyze the membrane forces in the roof under its self-weight.



- **Parametrize the mid-surface of a hemispherical shell of radius R in terms of $\eta^1 = \phi \in [0, \frac{\pi}{2}]$ and $\eta^2 = \theta \in [0, 2\pi)$.**
  From the mapping

$$\mathring{\mathbf{r}}(\phi, \theta) = R\Big(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\Big)\,, \tag{11.27}$$

we can derive the covariant base vectors via their definition:

$$\mathbf{q}_1 = \frac{\partial\mathring{\mathbf{r}}}{\partial\phi} = R\Big(\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\Big)\,, \tag{11.28}$$

$$\mathbf{q}_2 = \frac{\partial\mathring{\mathbf{r}}}{\partial\theta} = R\Big(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0\Big)\,. \tag{11.29}$$

The normal vector is just the vector product between $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$\mathbf{n} = \frac{\mathbf{q}_1 \times \mathbf{q}_2}{|\mathbf{q}_1 \times \mathbf{q}_2|} = \Big(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\Big)\,. \tag{11.30}$$

The covariant metric is defined as

$$(\mathring{a}_{\alpha\beta}) = (\mathbf{q}_\alpha \cdot \mathbf{q}_\beta) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\phi \end{pmatrix}\,, \tag{11.31}$$

and the contravariant metric is

$$(\mathring{a}^{\alpha\beta}) = (\mathring{a}_{\alpha\beta})^{-1} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2\sin^2\phi} \end{pmatrix}\,. \tag{11.32}$$

The contravariant base vectors can be written as

$$\mathbf{q}^1 = \frac{1}{R}\Big(\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\Big)\,, \tag{11.33}$$

$$\mathbf{q}^2 = \frac{1}{R}\Big(-\frac{\sin\theta}{\sin\phi}, \frac{\cos\theta}{\sin\phi}, 0\Big)\,. \tag{11.34}$$

- **Equilibrium equations of the membrane theory for spherical shells:**

$$\boxed{-\nabla_\alpha N^{\alpha 1} - f^1 = 0}\,, \tag{11.35}$$

$$\boxed{-\nabla_\alpha N^{\alpha 2} - f^2 = 0}\,, \tag{11.36}$$

$$\boxed{\frac{N_1^1 + N_2^2}{R} - p = 0}\,, \tag{11.37}$$

where $\mathbf{N}$ is the membrane force tensor, and f^1, f^2 and p are the load components along the covariant base vectors, $\mathbf{q}^1$ and $\mathbf{q}^2$, and the normal vector, $\mathbf{n}$, respectively.
For axisymmetric systems, we have

$$\boxed{N_2^1 = N_1^2 = 0}\,. \tag{11.38}$$

Components of the divergence of a second-order tensor $\mathbf{T}$ in spherical coordinates (ϕ, θ), for an axisymmetric problem ($T_2^1 = T_1^2 = 0$), are given as

$$\nabla_\alpha T_1^\alpha = T^1_{1,1} + T_1^1 \frac{\cos\phi}{\sin\phi} - T_2^2 \frac{\cos\phi}{\sin\phi}\,, \tag{11.39}$$

$$\nabla_\alpha T_2^\alpha = T^2_{2,2}\,. \tag{11.40}$$

Question:

Compute the membrane forces in the roof under its self-weight, where $q = \rho g t$ is the load per unit area of the shell surface, with ρ denoting the density of the material, t the thickness of the shell, and g the gravitational acceleration.

Solution:
We resolve the load q into components along the covariant base vectors, $\mathbf{q}^1$ and $\mathbf{q}^2$, and the normal vector, $\mathbf{n}$:

$$f^1 = -q\,\mathbf{e}_3 \cdot \mathbf{q}^1 = \frac{q}{R}\sin\phi\,, \tag{11.41}$$

$$f^2 = -q\,\mathbf{e}_3 \cdot \mathbf{q}^2 = 0\,, \tag{11.42}$$

$$p = -q\,\mathbf{e}_3 \cdot \mathbf{n} = -q\cos\phi\,. \tag{11.43}$$

From Eqs.(11.36) and (11.42), we get

$$\nabla_\alpha N^{\alpha 2} = \nabla_1 N^{12} + \nabla_2 N^{22} = N^{22}{}_{,2} = 0\,. \tag{11.44}$$

Therefore, we have

$$N^2_{2,2} = 0\,, \tag{11.45}$$

which is natural for an axisymmetric system.
From Eqs.(11.35) and (11.41), we get

$$-\nabla_\alpha N^{\alpha 1} - \frac{q}{R}\sin\phi = -\nabla_\alpha(\mathring{a}^{11} N_1^\alpha) - \frac{q}{R}\sin\phi = 0\,. \tag{11.46}$$

Since $\mathring{a}^{11} = R^{-2}$, we have

$$-\nabla_\alpha N_1^\alpha - qR\sin\phi = 0\,, \tag{11.47}$$

which reads

$$N_{1,1}^1 + N_1^1 \frac{\cos\phi}{\sin\phi} - N_2^2 \frac{\cos\phi}{\sin\phi} + qR\sin\phi = 0\,. \tag{11.48}$$

By multiplying the previous equation by $\sin\phi$, we get

$$(N_1^1 \sin\phi)_{,1} - N_2^2 \cos\phi + qR\sin^2\phi = 0\,. \tag{11.49}$$

From Eqs.(11.37) and (11.43), we obtain

$$N_2^2 = -N_1^1 - qR\cos\phi = 0\,. \tag{11.50}$$

Substituting the above equation to Eq.(11.49), we have

$$(N_1^1 \sin\phi)_{,1} + N_1^1 \cos\phi + qR = 0\,. \tag{11.51}$$

Then,

$$(N_1^1 \sin^2\phi)_{,1} = -qR\sin\phi\,. \tag{11.52}$$

The integral with respect to ϕ of the above equation reads

$$N_1^1 = -\frac{1}{\sin^2\phi}\int qR\sin\phi \mathrm{d}\phi = \frac{1}{\sin^2\phi}\big(qR\cos\phi + \mathrm{C}\big)\,. \tag{11.53}$$

Applying the boundary condition at the equator ($\phi = \frac{\pi}{2}$) to the above equation,

$$N_1^1|_{\phi=\frac{\pi}{2}} = -qR\,, \tag{11.54}$$

we get

$$\mathrm{C} = -qR\,. \tag{11.55}$$

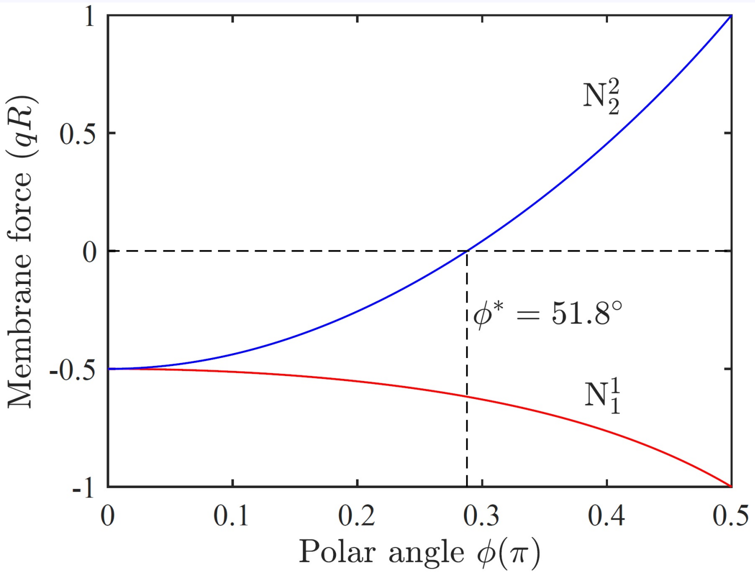
Therefore, the meridional force is

$$\boxed{N_1^1 = -\frac{qR}{1+\cos\phi}}\,. \tag{11.56}$$

From Eq.(11.50), the circumferential force is

$$\boxed{N_2^2 = -N_1^1 - qR\cos\phi = qR\Big(\frac{1}{1+\cos\phi} - \cos\phi\Big)}\,. \tag{11.57}$$

The distribution of the membrane forces, N_1^1 and N_2^2, along the meridian of the dome is shown in the figure below. The meridional force, N_1^1, is compressive, increasing from $-\frac{qR}{2}$ at the apex to $-qR$ at the bottom of the dome. The circumferential force, N_2^2, which is compressive near the apex, decreases gradually with the increasing polar angle and changes sign at $\phi^* = 51.8°$. ϕ^* denotes the polar angle of the latitude line on which $N_2^2 = 0$. Below this line, the circumferential force, N_2^2, is tensile. Spherical domes whose opening angle is less than ϕ^* are free from tensile stresses.



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