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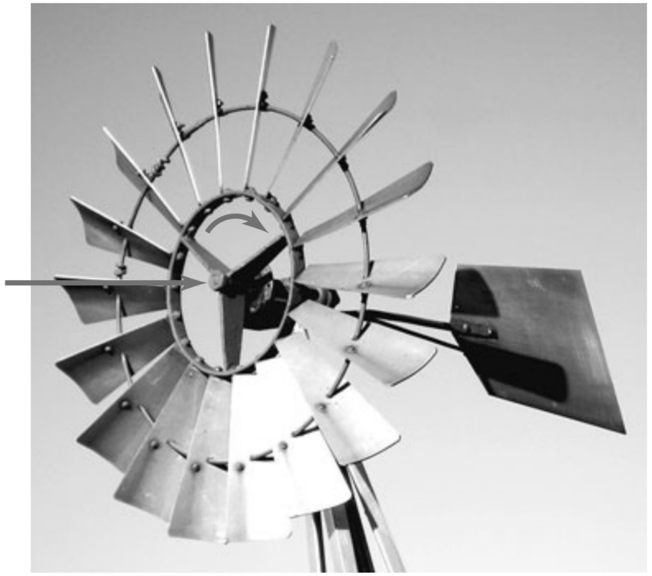

Chapter 5: Euler turbomachine equation

ME-342 Introduction to turbomachinery

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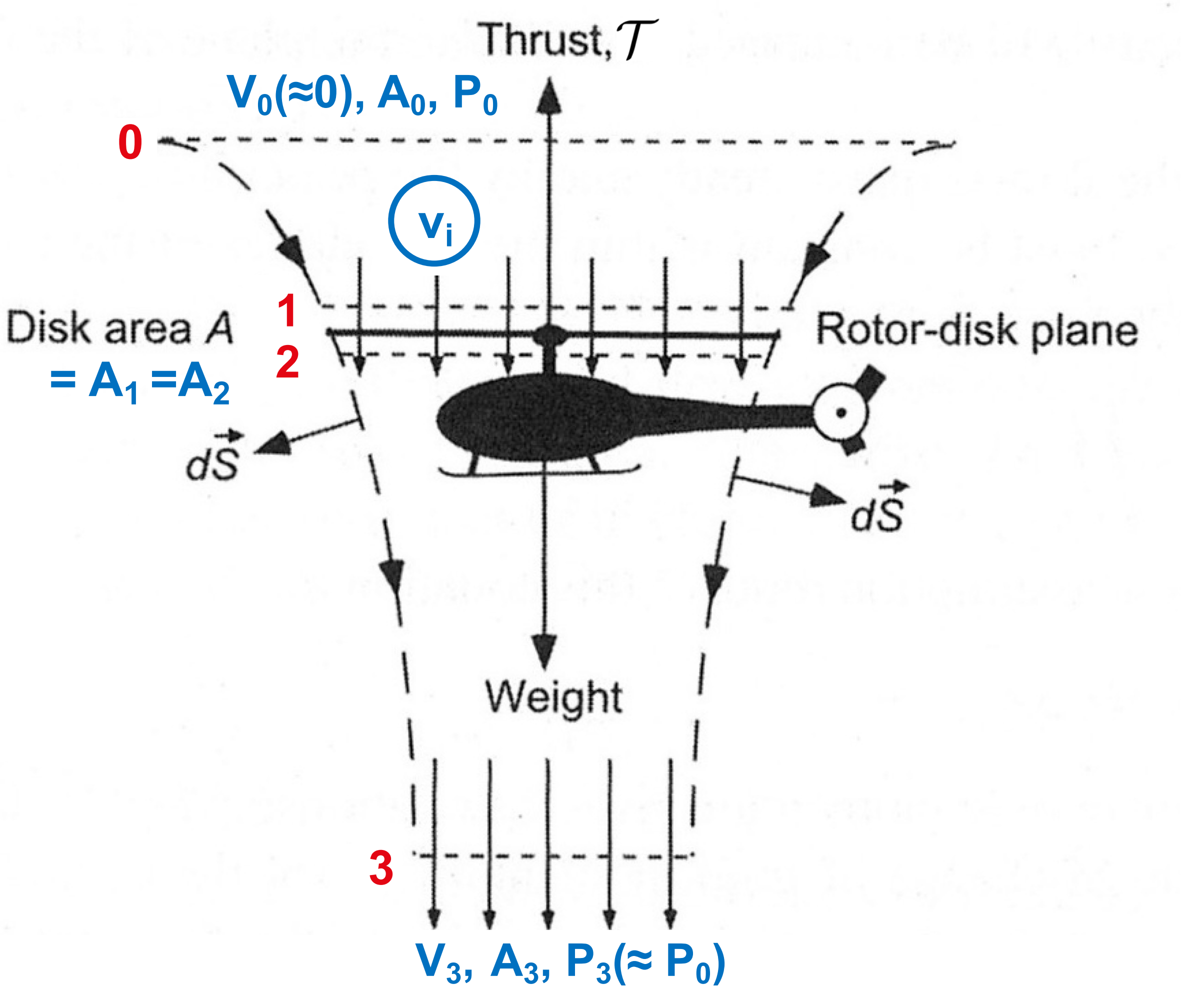
Recall: Helicopters- conservation law for hovering rotor



v_i : induced air velocity

V_3 : far wake velocity

- **Continuity & linear momentum equations:**

$$\frac{\partial}{\partial t}\int_{cv}\rho d\mathcal{V} + \int_{cs}\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = 0$$

$$\frac{\partial}{\partial t}\int_{cv}\mathbf{V}\rho d\mathcal{V} + \int_{cs}\mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = \sum\mathbf{F}_{\text{contents of the control volume}}$$

Steady-state approximation

$$\int_{cs}\rho\vec{V}\cdot d\vec{S} = 0$$

$$\int_{cs}\vec{V}\cdot\rho\cdot\vec{V}d\vec{S} = \sum F$$

$$\dot{m} = \int_A \rho\mathbf{V}\cdot\hat{\mathbf{n}}dA$$

Thrust, T (reaction force)

$$-F = T = \int_3 \vec{V}\rho\vec{V}d\vec{S} - \int_0 \vec{V}\rho\vec{V}d\vec{S}$$

$$\mathcal{T} = \dot{m}V_3$$

- **Rotor power** (work done by rotor per unit time)

$$P_{\text{rotor}} = \mathcal{T}v_i$$

Recall: Helicopters- conservation law for hovering rotor

- Energy equation (the gain of energy of the fluids per unit time)

adiabatic steady No internal energy change

$$\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{shaft\\net\ in}} = \frac{\partial}{\partial t}\int_{cv} e\rho d\forall + \int_{cs}\left(\check{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)\rho \mathbf{V}\cdot\hat{\mathbf{n}}dA$$

$d\vec{S}$

Mass flow rate

$$\dot{m} = \int_A \rho \mathbf{V}\cdot\hat{\mathbf{n}}dA$$

$$\dot{W}_{\substack{shaft\\net\ in}} = P_{rotor} = \int_{cs}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right)\rho\vec{V}\, d\vec{S}$$

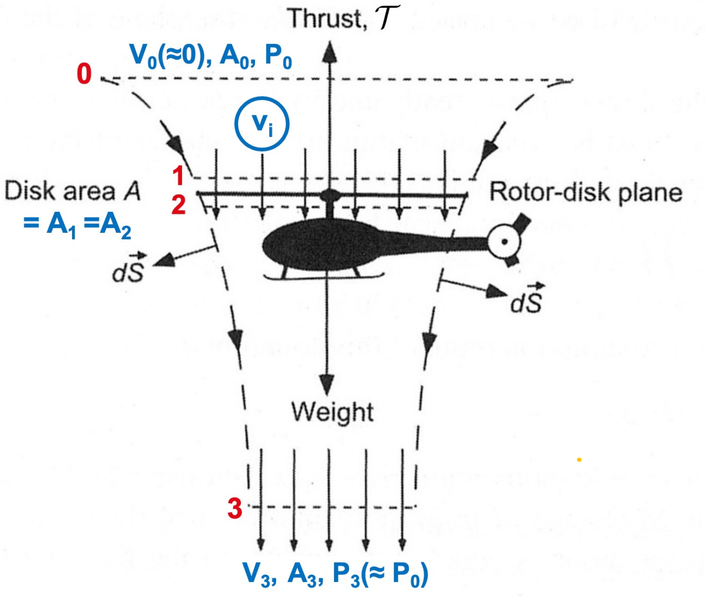
$= \dot{m}$

$$= \sum_{\text{flow out}}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right)\dot{m} - \sum_{\text{flow in}}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right)\dot{m}$$

Potential energy change ≪ kinetic energy

$$= \left(\frac{P_3}{\rho} + \frac{V_3^2}{2}\right)\dot{m} - \left(\frac{P_0}{2} + \frac{V_0^2}{2}\right)\dot{m}$$

$$= \frac{V_3^2}{2}\dot{m}$$



Recall: Helicopters- conservation law for hovering rotor

- **Continuity equation:**

$$\mathcal{T} = \dot{m}V_3, \quad P_{\text{rotor}} = \mathcal{T} v_i$$

- **Energy equation:** $P_{\text{rotor}} = \dfrac{V_3^2}{2}\dot{m}$

$$P_{rotor} = \frac{V_3^2}{2}\dot{m} = \dot{m} V_3 v_i$$

Induced velocity: $v_i = \frac{1}{2} V_3$

Far field velocity is twice of the induced velocity

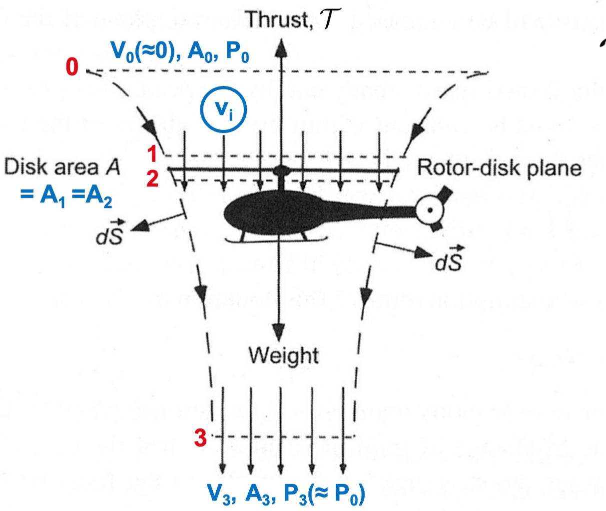
VERY

- **Mass flowrate:** $\dot{m} = \rho A_3 V_3 = \rho A_2 v_i = \rho A v_i$

$$\dot{m} = \rho A_3 V_3 = \rho A v_i \ \rightarrow A_3 = \frac{1}{2} A$$

$$\mathcal{T} = \dot{m}V_3 = \dot{m}(2v_i) = (\rho A v_i)(2vi) = 2\rho A v_i^2$$

Remember! The energy equation is valid along the streamlines.



The induced velocity:

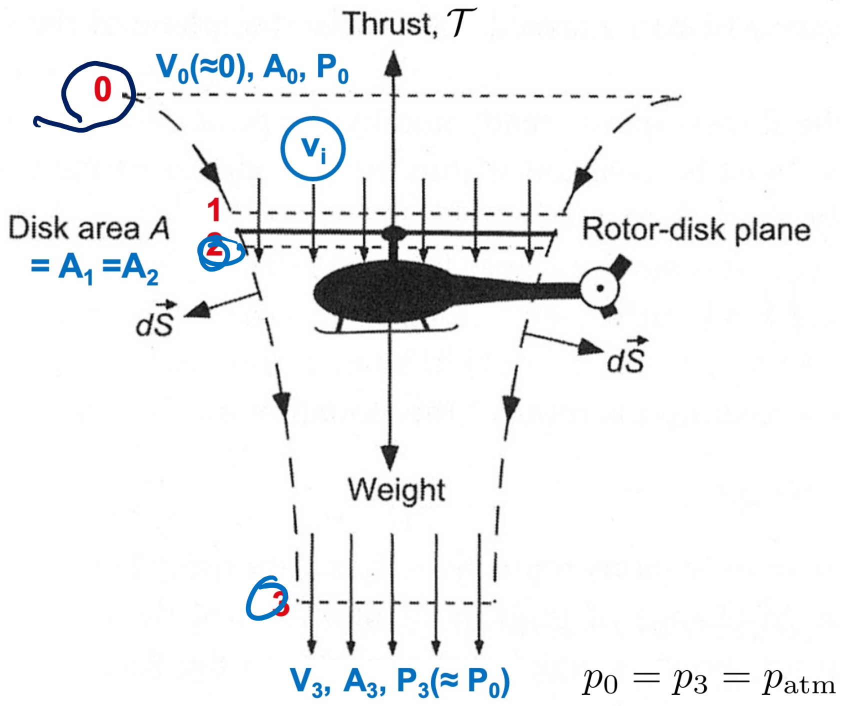
$$v_h \equiv v_i = \sqrt{\frac{\mathcal{T}}{2\rho A}} = \sqrt{\left(\frac{\mathcal{T}}{A}\right)\left(\frac{1}{2\rho}\right)}$$

Disk loading

Pressure around the helicopter

$$\Delta p = p_2 - p_1 = \frac{\mathcal{T}}{A}$$

Ignore the hydrostatic pressure : $\rho \Delta z \sim \epsilon$



$$p_0 = p_1 + \frac{1}{2}\rho v_i^2 \longrightarrow p_1 = p_0 - \frac{1}{2}\rho v_i^2$$

$$p_2 + \frac{1}{2}\rho v_i^2 = p_3 + \frac{1}{2}\rho V_3^2 \rightarrow p_2 = p_3 + \frac{1}{2}\rho V_3^2 - \frac{1}{2}\rho v_i^2$$

$$\Delta p = p_2 - p_1 = \frac{T}{A} = p_3 + \frac{1}{2}\rho V_3^2 - \frac{1}{2}\rho v_i^2 - p_0 + \frac{1}{2}\rho v_i^2$$

$$= \frac{1}{2}\rho V_3^2$$

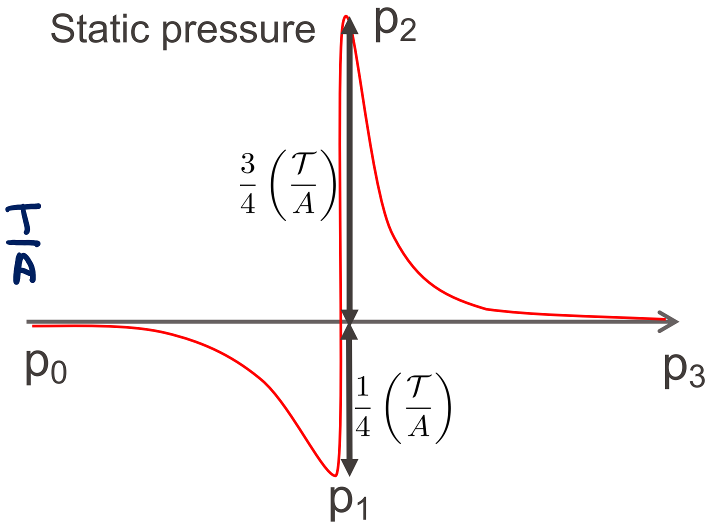
$$v_i = \frac{1}{2} V_3$$

$$p_1 = p_0 - \frac{1}{2}\rho v_i^2 = p_0 - \frac{1}{2}\rho\left[\frac{1}{2}V_3\right]^2 = p_0 - \frac{1}{4}\cdot\frac{1}{2}\rho V_3^2 = p_0 - \frac{1}{4}\frac{T}{A}$$

$$\left(\frac{1}{2}\rho V_3^2 = \frac{T}{A}\right)$$

$$p_2 = p_3 + \frac{3}{4}\frac{T}{A}$$

$$p_1 = p_{\text{atm}} - \frac{1}{4}\left(\frac{\mathcal{T}}{A}\right), \quad p_2 = p_{\text{atm}} + \frac{3}{4}\left(\frac{\mathcal{T}}{A}\right)$$



Dynamic pressure?

The ideal and actual powers

- Last week, we learned ideal power
  - It can be estimated the thrust needed for hover

- Actual power

$$P_{\text{actual}} = P_{\text{induced}} + P_{\text{profile}} + P_{\text{parasitic}} + P_{\text{climb}}$$

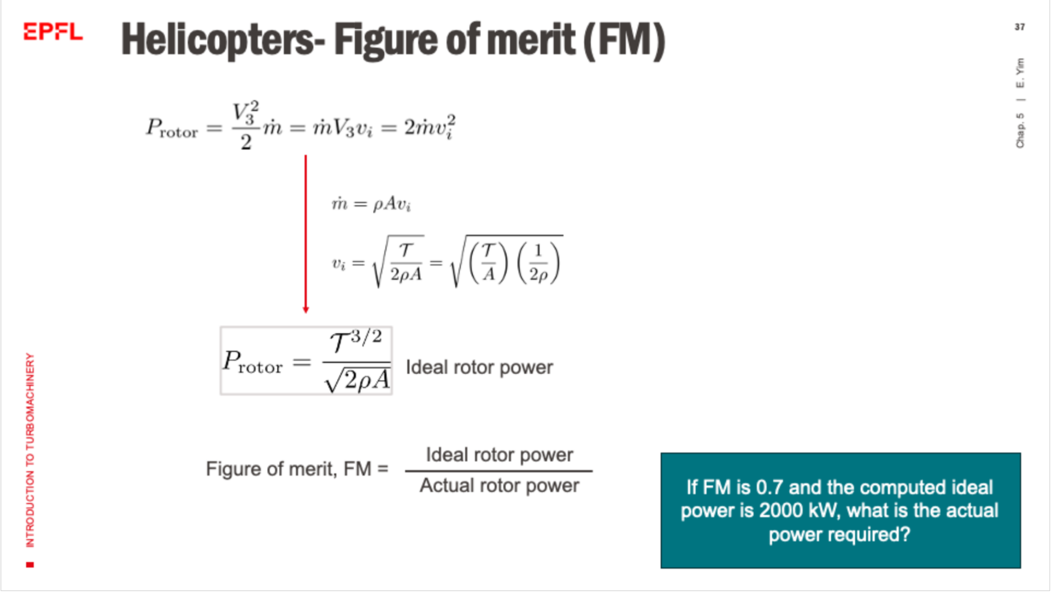
Induced power = modified ideal power with factor

Viscous effect

Climb power = Weight x V_c

P_{profile} : power required to overcome the rotor **drag** → Torque

$P_{\text{parasitic}}$: power required to overcome the helicopter body (fuselage) **drag** and **tail rotor power**

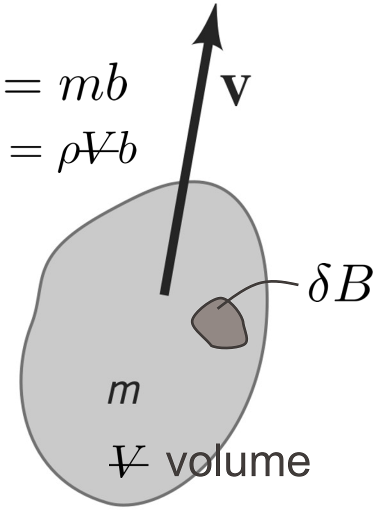


Reynolds transport theorem

Reynolds transport theorem

System: a collection of matter of fixed identity (always the same atoms or fluid particles), which may move, flow, and interact with its surroundings.
Control volume: a volume in space (a geometric entity, independent of mass) through which fluid may flow.

$$B = mb = \rho \mathcal{V} b$$



B	$b = B/m$
m	1
$m\mathbf{V}$	$\mathbf{V}$
$\frac{1}{2}mV^2$	$\frac{1}{2}V^2$

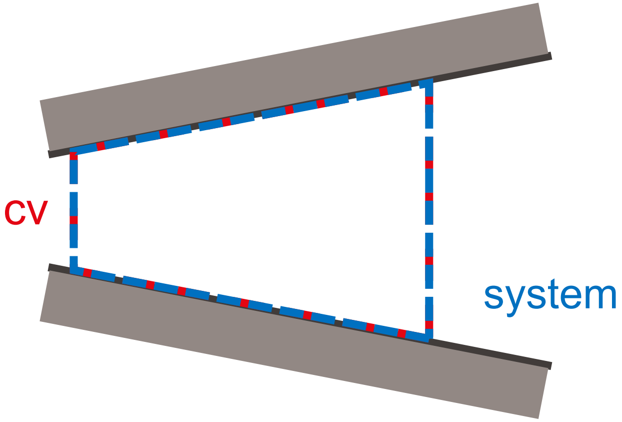
System vs. control volume (Lagrangian vs. Eulerian)

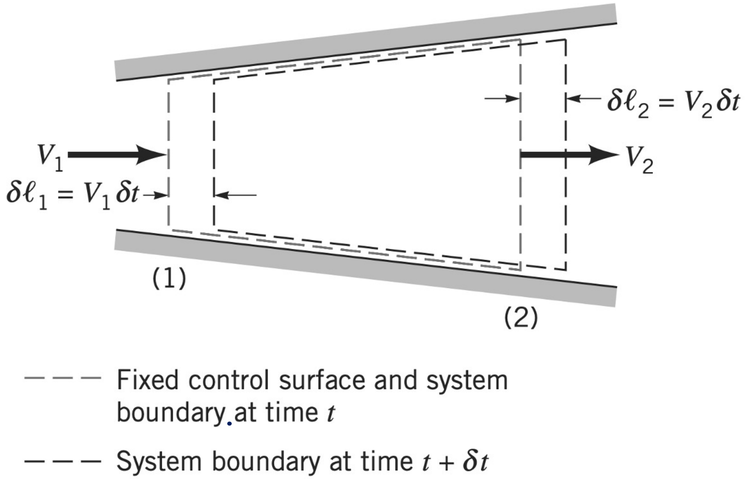
$$\delta B = b\rho\delta\mathcal{V} \longrightarrow B_{\text{sys}} = \lim_{\delta\mathcal{V}\to 0}\sum_i b_i\left(\rho_i\delta\mathcal{V}\right) = \int_{\text{sys}} \rho b\, d\mathcal{V}$$

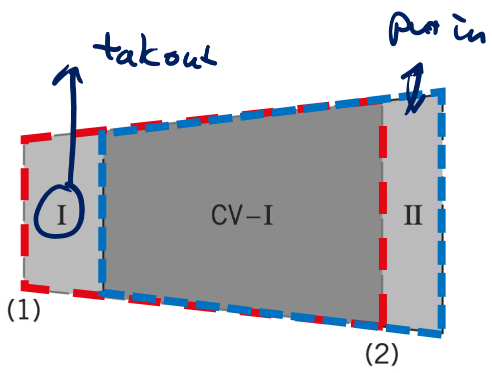
$$B_{\text{cv}} = \int_{\text{cv}} \rho b\, d\mathcal{V}$$

at $t = t_0$, $B_{sys}(t_0) = B_{cv}(t_0)$

at $t = t_0 + \delta t$, $B_{sys}(t_0+\delta t) = B_{cv}(t_0+\delta t) - B_{I}(t_0+\delta t) + B_{II}(t_0+\delta t)$







Reynolds transport theorem

$B = mb$

$$t = t_0, \quad B_{\mathrm{sys}}(t_0) = B_{\mathrm{cv}}(t_0)$$

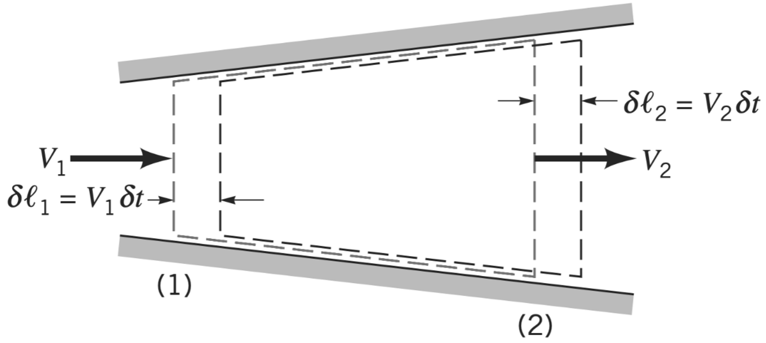
$$t = t_0 + \delta t, \quad B_{\mathrm{sys}}(t_0 + \delta t) = B_{\mathrm{cv}}(t_0 + \delta t) - B_{\mathrm{I}}(t_0 + \delta t) + B_{\mathrm{II}}(t_0 + \delta t)$$

$$\frac{\delta B_{sys}}{\delta t} = \frac{B_{sys}(t_0+\delta t) - B_{sys}(t_0)}{\delta t} = \frac{B_{cv}(t_0+\delta t) - B_I(t_0+\delta t) + B_{II}(t_0+\delta t) - B_{cv}(t_0)}{\delta t}$$

$$= \frac{B_{cv}(t_0+\delta t) - B_{cv}(t_0)}{\delta t} - \frac{B_I(t_0+\delta t)}{\delta t} + \frac{B_{II}(t_0+\delta t)}{\delta t}$$

$B_{II}(t_0+\delta t) \rightarrow \rho_2 b_2 A_2 V_2 \delta t$

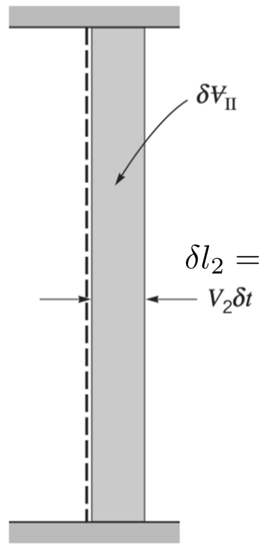
$B_I(t_0+\delta t) = \delta V_I \cdot \rho \cdot b = \rho_1 b_1 A_1 V_1 \delta t$



- - - Fixed control surface and system boundary at time t

— — — System boundary at time $t + \delta t$

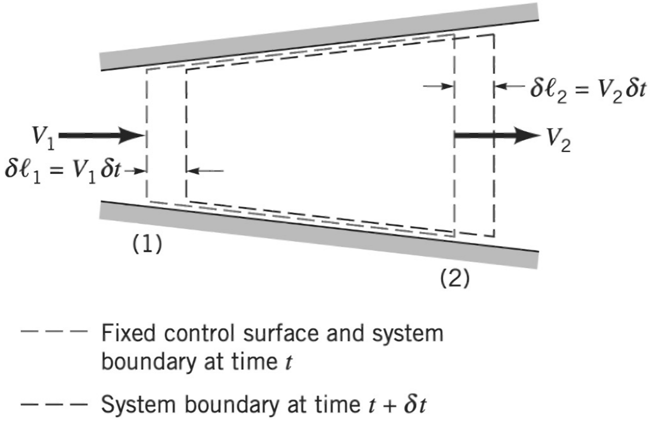
$$\frac{DB_{\mathrm{sys}}}{Dt} = \frac{\partial B_{\mathrm{cv}}}{\partial t} + \rho_2 A_2 V_2 b_2 - \rho_1 A_1 V_1 b_1$$

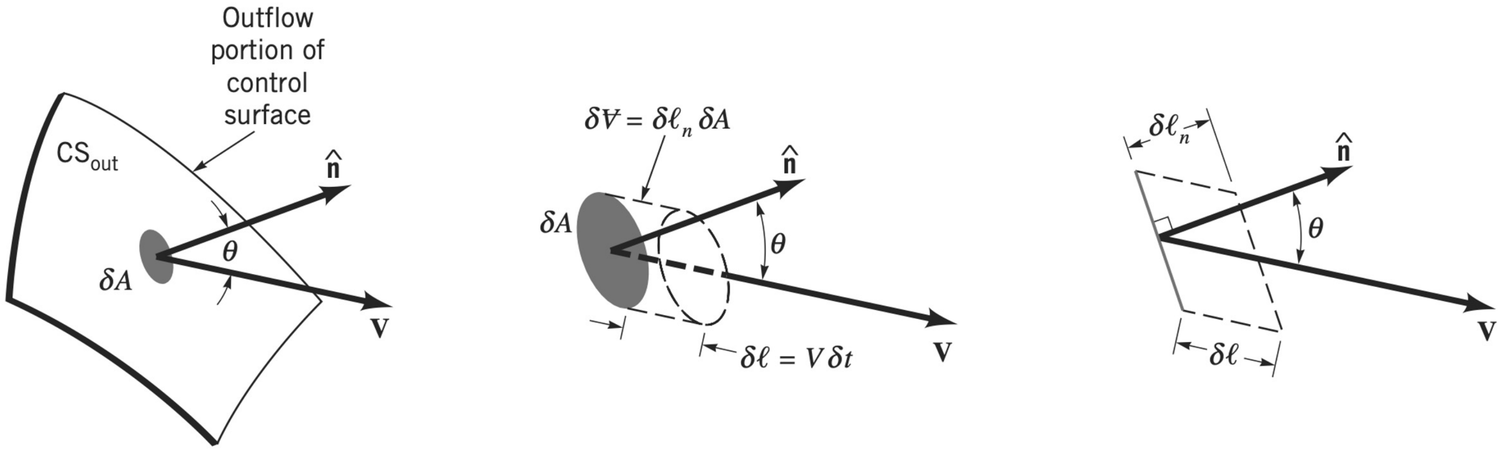


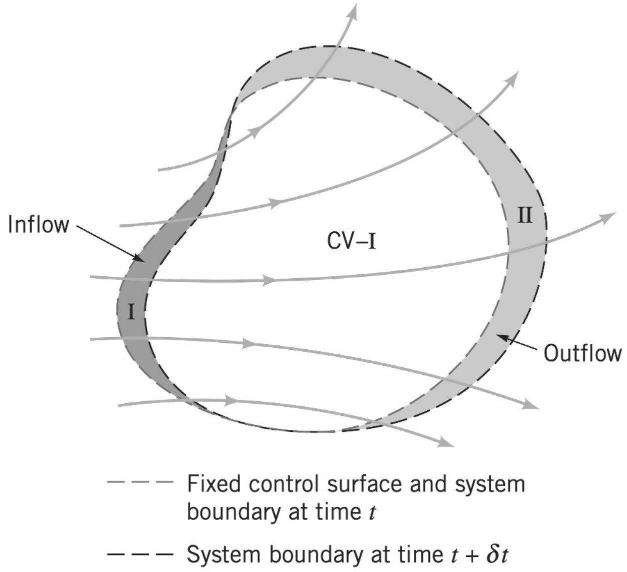
Reynolds transport theorem

E. Yim

- General case







$$\dot{B}_{out} = \int_{cs_{out}} d\dot{B}_{out} = \int_{cs_{out}} \rho b V \cos\theta \, dA$$

$$V\cos\theta = \mathbf{V}\cdot\hat{\mathbf{n}}$$

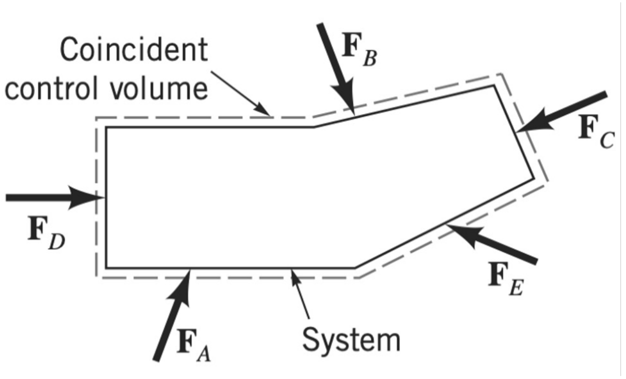
$$\dot{B}_{out} - \dot{B}_{in} = \int_{cs_{out}} \rho b \mathbf{V}\cdot\hat{\mathbf{n}}\, dA - \left(- \int_{cs_{in}} \rho b \mathbf{V}\cdot\hat{\mathbf{n}}\, dA \right)$$

$$= \int_{cs} \rho b \mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b \, dV\!\!\!\!- + \int_{cs} \rho b \mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

B	$b = B/m$
m	1
$m\mathbf{V}$	$\mathbf{V}$
$\frac{1}{2}mV^2$	$\frac{1}{2}V^2$

Recall Last lecture: linear momentum equation



Newton's second law of motion

The change of motion of an object is **proportional** to the **force** impressed; and is made in the direction of the straight line in which the force is impressed

$$\text{momentum} = \text{mass} \times \text{(its) velocity} = \int_{sys} \rho \vec{V} \, d\forall$$

$$\text{mass} = \int_{sys} \rho \, d\forall \qquad \text{velocity} = \vec{V}$$

$$\frac{D}{Dt}\left(\int_{sys} \rho \vec{V} \, d\forall\right) = \frac{\partial}{\partial t}\int_{cv} \rho \vec{V} \, d\forall + \int_{cs} \vec{V} \cdot \rho \vec{V} \, \hat{n} \, dA$$

Reynolds transport theorem

- Linear momentum equation

$$\frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V} \rho \, d\forall + \int_{\text{cs}} \mathbf{V} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA = \sum \mathbf{F}_{\substack{\text{contents of the} \\ \text{control volume}}}$$

Moment-of Momentum (angular momentum)

Moment-of-momentum

- Linear momentum equation

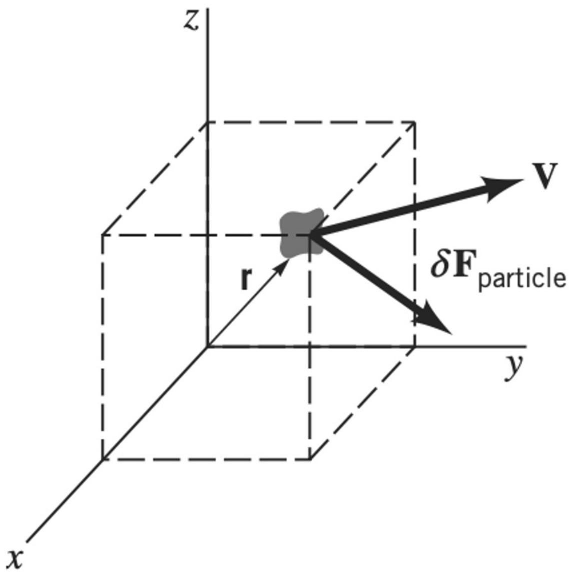
$$\frac{D}{Dt}\int_{sys} \mathbf{V}\rho d\mathcal{V} = \frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho d\mathcal{V} + \int_{\text{cs}} \mathbf{V}\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$$

- Linear momentum for a particle volume of δV

$$\frac{D}{Dt}(\mathbf{V}\rho\delta\mathcal{V}) = \delta\mathbf{F}_{\text{particle}}$$

- **Form the moment*** w.r.t. origin of an inertial coordinate system

$$\mathbf{r}\times\frac{D}{Dt}(\mathbf{V}\rho\delta\mathcal{V}) = \mathbf{r}\times\delta\mathbf{F}_{\text{particle}}$$



$$\frac{D}{Dt}(\vec{r}\times\vec{V}\rho\delta\mathcal{V}) = \frac{D\vec{r}}{Dt}\times\vec{V}\rho\delta\mathcal{V} + \vec{r}\times\frac{D(\vec{V}\rho\delta\mathcal{V})}{Dt}$$

$$\frac{D\vec{r}}{Dt} = \vec{V}, \quad \vec{V}\times\vec{V} = 0$$

*Apply force at a distance

Moment-of-momentum

Every particle of a system $$\frac{D}{Dt}[(\mathbf{r} \times \mathbf{V})\rho\delta V\!\!\!/] = \mathbf{r} \times \delta\mathbf{F}_{\text{particle}}$$

For a system (collection of fluid particles)

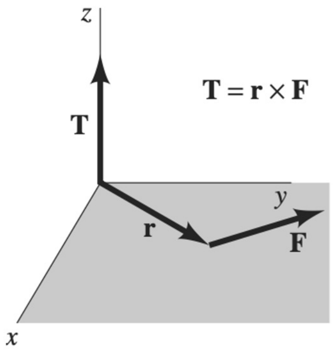
$$\int_{\text{sys}} \frac{D}{Dt}[(\mathbf{r} \times \mathbf{V})\rho dV\!\!\!/] = \sum(\mathbf{r} \times \mathbf{F})_{\text{sys}}$$

For a control volume that is **instantaneously coincident** with the system

$$\sum(\mathbf{r} \times \mathbf{F})_{\text{sys}} = \sum(\mathbf{r} \times \mathbf{F})_{\text{cv}}$$

- Moment-of-momentum equation

$$\frac{\partial}{\partial t}\int_{\text{cv}} (\mathbf{r} \times \mathbf{V})\rho dV\!\!\!/ + \int_{\text{cs}} (\mathbf{r} \times \mathbf{V})\rho\mathbf{V} \cdot \hat{\mathbf{n}} dA = \sum(\mathbf{r} \times \mathbf{F})_{\substack{\text{contents of the}\\ \text{control volume}}}$$



Moment-of-momentum equation is Torque

Linear momentum equation

$$\frac{D}{Dt}\int_{sys} \mathbf{V}\rho dV\!\!\!/ = \frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho dV\!\!\!/ + \int_{\text{cs}} \mathbf{V}\rho\mathbf{V} \cdot \hat{\mathbf{n}} dA = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$$

Moment-of-momentum

- Moment-of-momentum equation

$$\frac{\partial}{\partial t}\int_{\text{cv}} (\mathbf{r}\times\mathbf{V})\rho\, d\forall + \int_{\text{cs}} (\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA = \sum(\mathbf{r}\times\mathbf{F})_{\substack{\text{contents of the}\\ \text{control volume}}}$$

In turbomachine, a series of particles (a continuum) passes through the rotor.
For steady flow (or for turbomachine rotors with steady-in-the-mean or steady-on-average cyclical flow)

$$\sum(\mathbf{r}\times\mathbf{F}) = \int_{cs} (\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

external torques (moments) acting on the contents of the control volume

net rate of flow of moment-of-momentum (**angular momentum**) through the control surface

$$\int_{cs} (\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA = \int_{A_2} (\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA - \int_{A_1} (\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

Outlet   Inlet

Moment-of-momentum

Assume:
- The only external torque is the shaft torque
- The flow is steady and uniform at the inlet and outlet, meaning $(\mathbf{r}\times\mathbf{V})$ and $\rho\mathbf{V}\cdot\hat{\mathbf{n}}$ are constant over the areas A_1 & A_2
- The mass flow rate $\dot{m}=\rho\mathbf{V}\cdot\hat{\mathbf{n}}A$ is the same at inlet and outlet (steady flow)

- Derivation of Euler turbomachine equation

$$\sum(\mathbf{r}\times\mathbf{F})=\int_{cs}(\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA$$

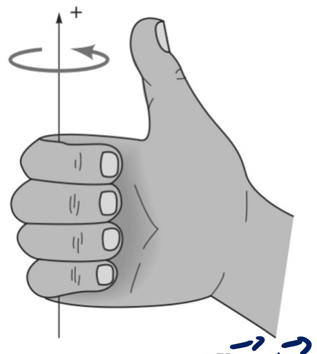
$$\mathbf{r}\times\mathbf{V}=\begin{vmatrix}\hat{e}_r & \hat{e}_\theta & \hat{e}_x\\ r & 0 & x\\ V_r & V_\theta & V_x\end{vmatrix}$$

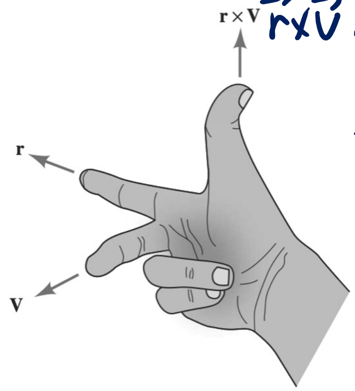
$$=(\vec{r}_2\times\vec{V}_2)\rho V_2\cdot\hat{n}\underbrace{\int_{A_2}dA}_{=A_2}-(\vec{r}_1\times\vec{V}_1)\rho V_1\cdot\hat{n}\underbrace{\int_{A_1}dA}_{=A_1}$$

($\dot{m}_2$, $\dot{m}_1$)

$$=(\vec{r}_2\times\vec{V}_2)\dot{m}_2-(\vec{r}_1\times\vec{V}_1)\dot{m}_1$$

$$\Sigma(r\times F)\Big|_{axial}=r_2V_{\theta 2}\dot{m}_2-r_1V_{\theta 2}\dot{m}_1$$





$$\vec{r}\times\vec{V}=\hat{e}_r[0\cdot V_x-V_\theta\cdot x]-\hat{e}_\theta[rV_x-xV_r]+\hat{e}_x[rV_\theta-0\cdot V_r]$$

Positive/negative according to the right-hand rule

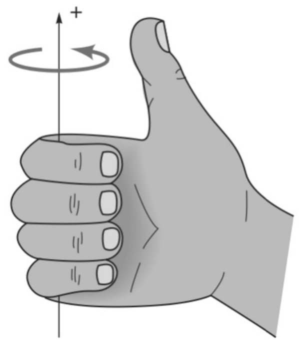
Moment-of-momentum

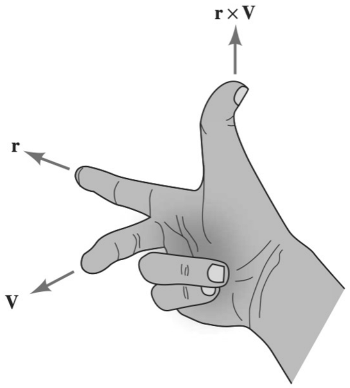
Assume:
- The only external torque is the shaft torque
- The flow is steady and uniform at the inlet and outlet, meaning $(\mathbf{r} \times \mathbf{V})$ and $\rho\mathbf{V}\cdot\hat{\mathbf{n}}$ are constant over the areas A_1 & A_2
- The mass flow rate $\dot{m} = \rho\mathbf{V}\cdot\hat{\mathbf{n}}A$ is the same at inlet and outlet (steady flow)

- Derivation of Euler turbomachine equation

$$\sum(\mathbf{r} \times \mathbf{F}) = \int_{cs} (\mathbf{r} \times \mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA$$

$$\mathbf{r} \times \mathbf{V} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_x \\ r & 0 & x \\ V_r & V_\theta & V_x \end{vmatrix}$$





Positive/negative according to the right-hand rule

The axial component is interesting for turbomachinery as it rotates along the **shaft** axis

$$\sum\left[(\mathbf{r} \times \mathbf{F})_{\substack{\text{contents of the}\\ \text{control volume}}}\right]_{\text{axial}} = T_{\text{shaft}}$$

$$[(\mathbf{r}_2 \times \mathbf{V}_2)\dot{m}_2 - (\mathbf{r}_1 \times \mathbf{V}_1)\dot{m}_1]_{\text{axial}} = \dot{m}_2(r_2V_{\theta 2}) - \dot{m}_1(r_1V_{\theta 1})$$

$$T_{\text{shaft}} = -\dot{m}_1\left(r_1V_{\theta 1}\right) + \dot{m}_2\left(r_2V_{\theta 2}\right)$$

Euler turbomachine equation

= axial component of moment-of-momentum equation

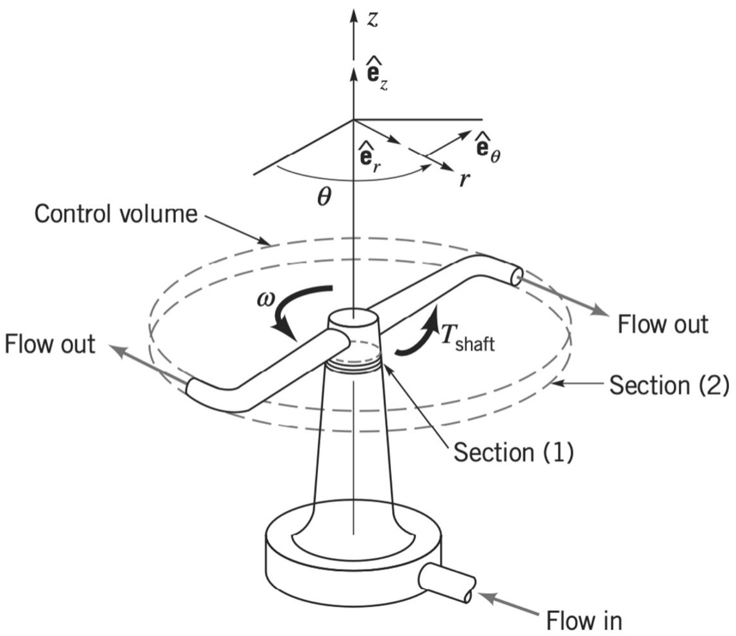
Application of moment-of-momentum equation

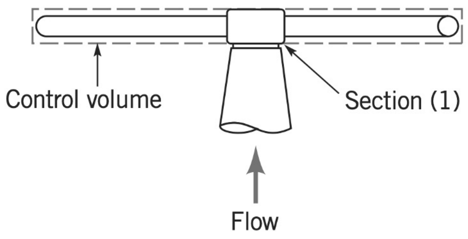
Select a fixed and nondeforming control volume that includes the rotating parts and the fluid in an instant.
The flow within this control volume is cyclical, but steady in the mean.
The only torque we consider is the driving shaft torque, T_{shaft}.

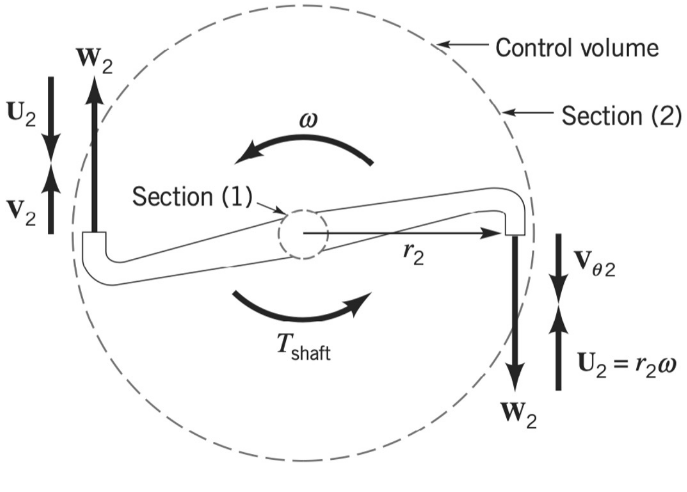
$$\int_{cs} (\mathbf{r} \times \mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = \sum(\mathbf{r}\times\mathbf{F})_{\substack{\text{contents of the}\\ \text{control volume}}}$$

$$\sum\left[(\mathbf{r}\times\mathbf{F})_{\substack{\text{contents of the}\\ \text{control volume}}}\right]_{\text{axial}} = T_{\text{shaft}}$$

$$\left[\int_{cs} (\mathbf{r} \times \mathbf{V})\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA\right]_{\text{axial}} = (-r_2V_{\theta 2})(+\dot{m})$$







(b)

$$\mathbf{V} = \mathbf{W} + \mathbf{U}$$

$\mathbf{W}$: Relative velocity

$\mathbf{V}$: Absolute velocity

$\mathbf{U}$: Moving nozzle velocity

Sign of the axial component of $\mathbf{r}\times\mathbf{V}$

$+$: If $\mathbf{V}_\theta$ and $\mathbf{U}$ are same direction

$-$: If $\mathbf{V}_\theta$ and $\mathbf{U}$ are opposite direction

$$-r_2V_{\theta 2}\dot{m} = T_{\text{shaft}}$$

Application of moment-of-momentum equation

- **Shaft power**

  ω : Angular velocity

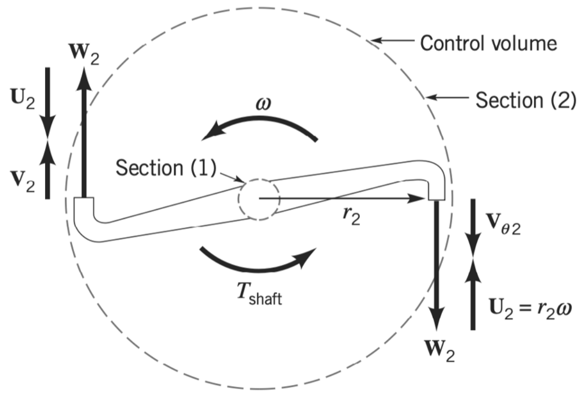
$$\dot{W}_{\text{shaft}} = T_{\text{shaft}}\,\omega = -r_2 V_{\theta 2}\dot{m}\omega$$

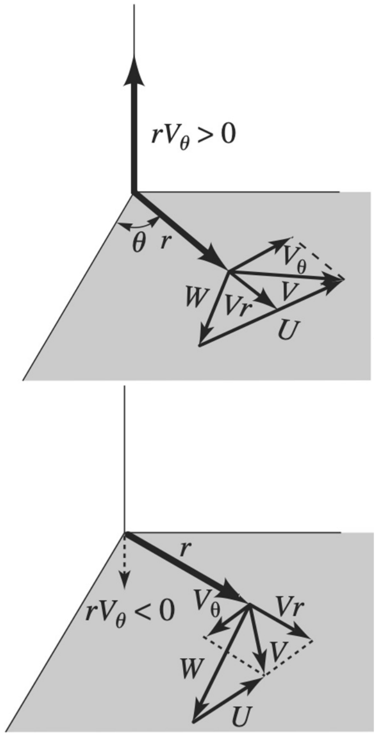
U_2

$$\dot{W}_{\text{shaft}} = -U_2 V_{\theta 2}\dot{m}$$

- **Shaft work per unit mass (shaft power per unit mass flow rate)**

$$w_{\text{shaft}} = -U_2 V_{\theta 2}$$





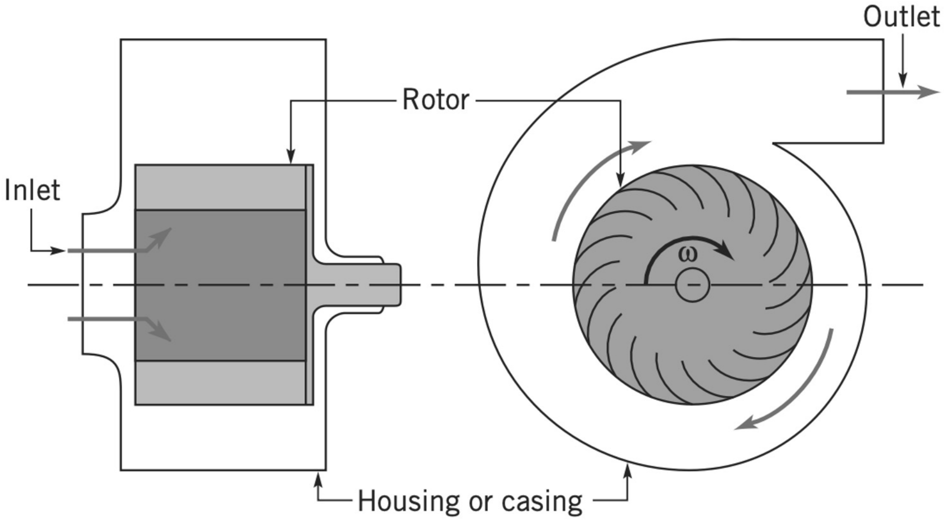
When the shaft **torque** and shaft **rotation**

- are in the **same** direction, power is *transferred* into the fluid (pump)
- are in **opposite** directions, power is *extracted* from the fluid (turbine)

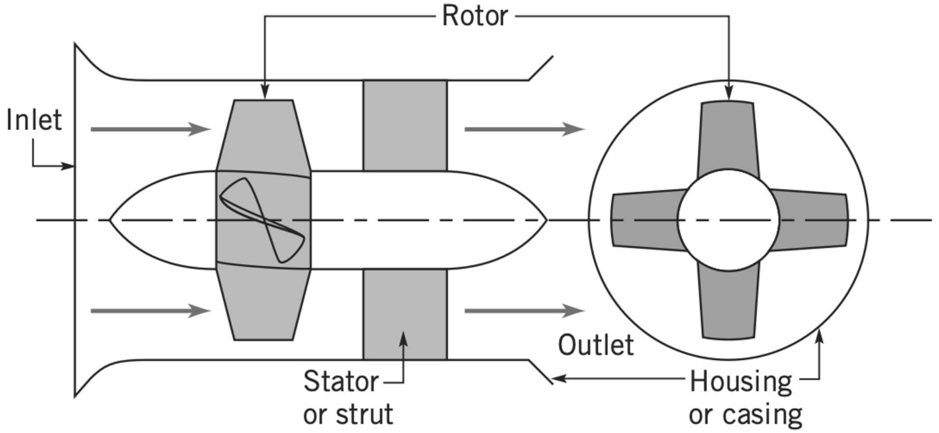
Basic energy conversion

Axial and radial flows

- Depending on the predominant direction of the fluid motion relative to the rotor axis as the fluid passes the blades



Radial-flow fan

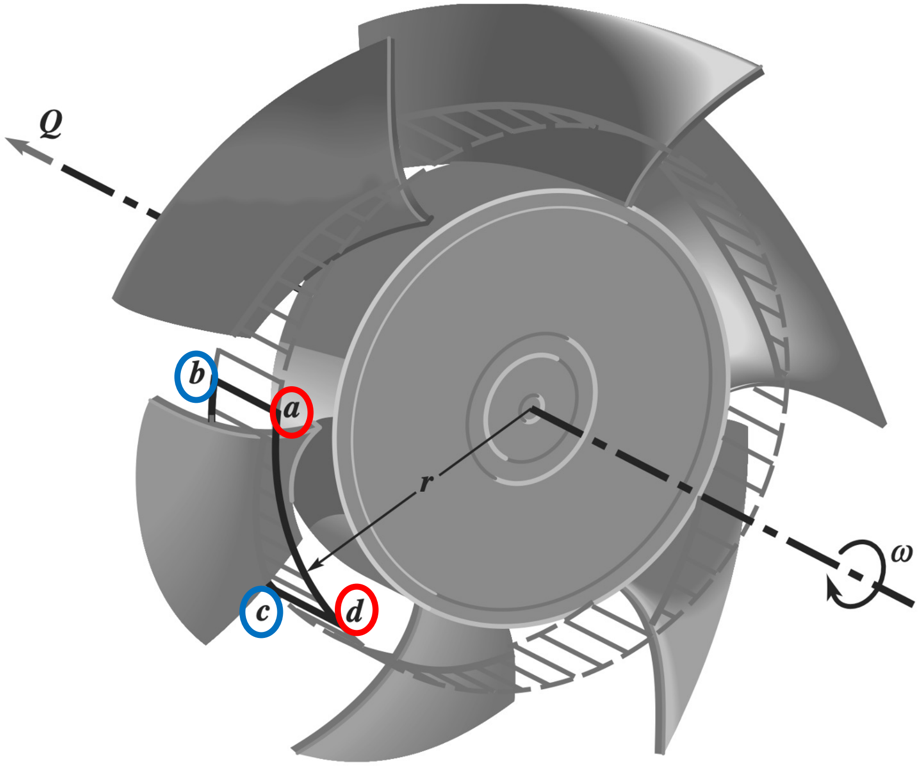


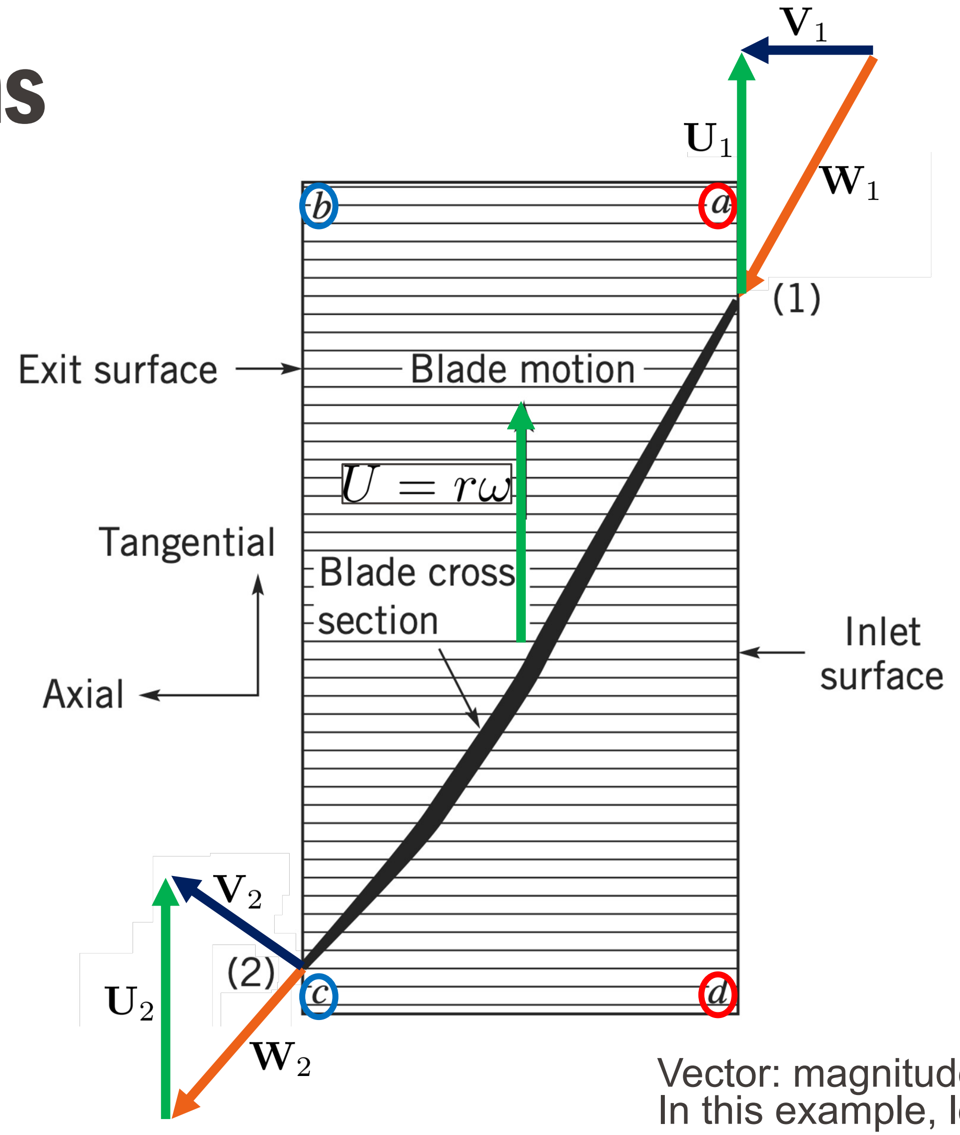
Axial-flow fan

EPFL

Basic Energy Considerations

- **Velocity diagram (fan)**



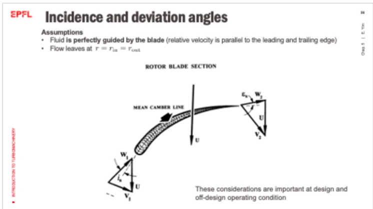


Vector: magnitude and angle
In this example, let's assume we know the magnitude of **W**

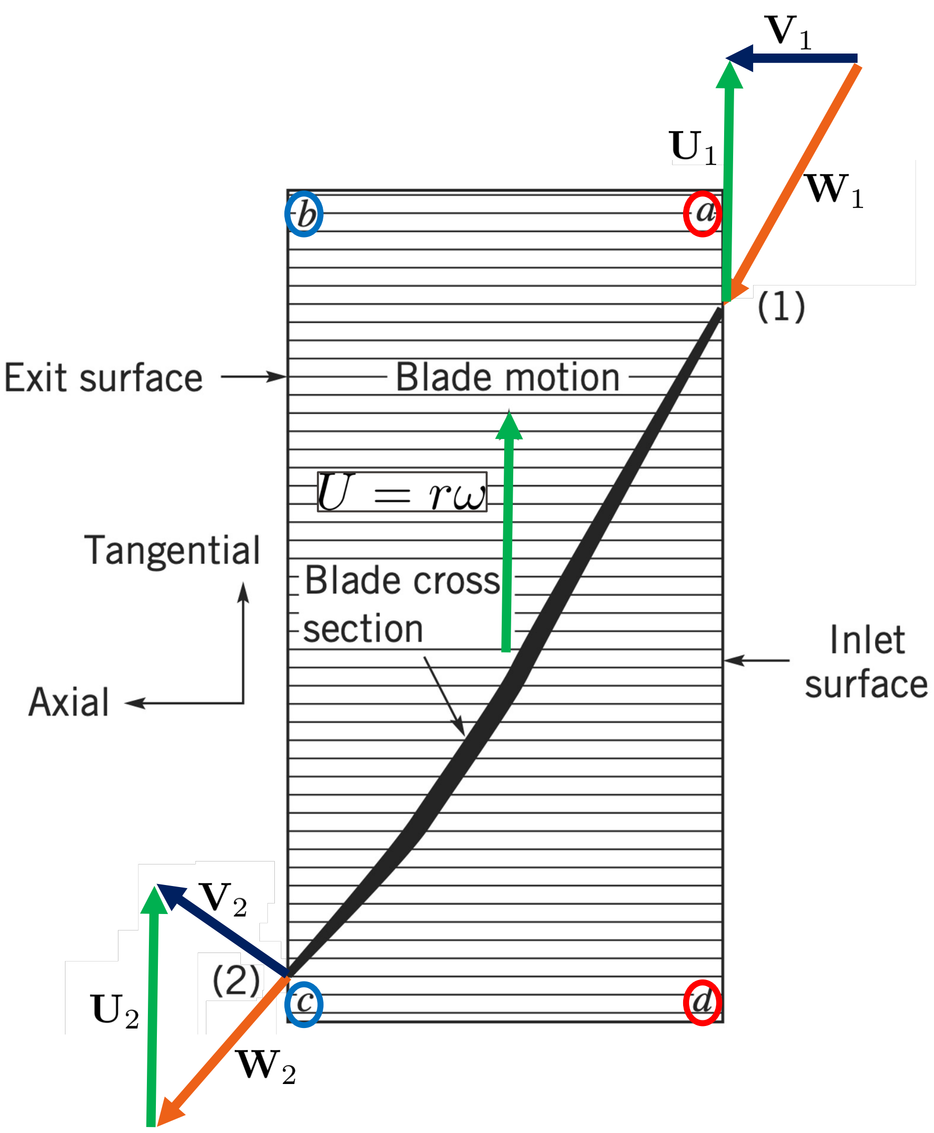
Assumptions

- Fluid is **perfectly guided by the blade** (relative velocity is parallel to the leading and trailing edge)*
- Flow leaves at $r = r_{\text{in}} = r_{\text{out}}$

*sometimes leading edge (inlet) **V** is prescribed



Basic Energy Considerations



- **Velocity diagram:**
  The actual (**absolute**) velocity is the vector sum of the **relative** and **blade** velocities

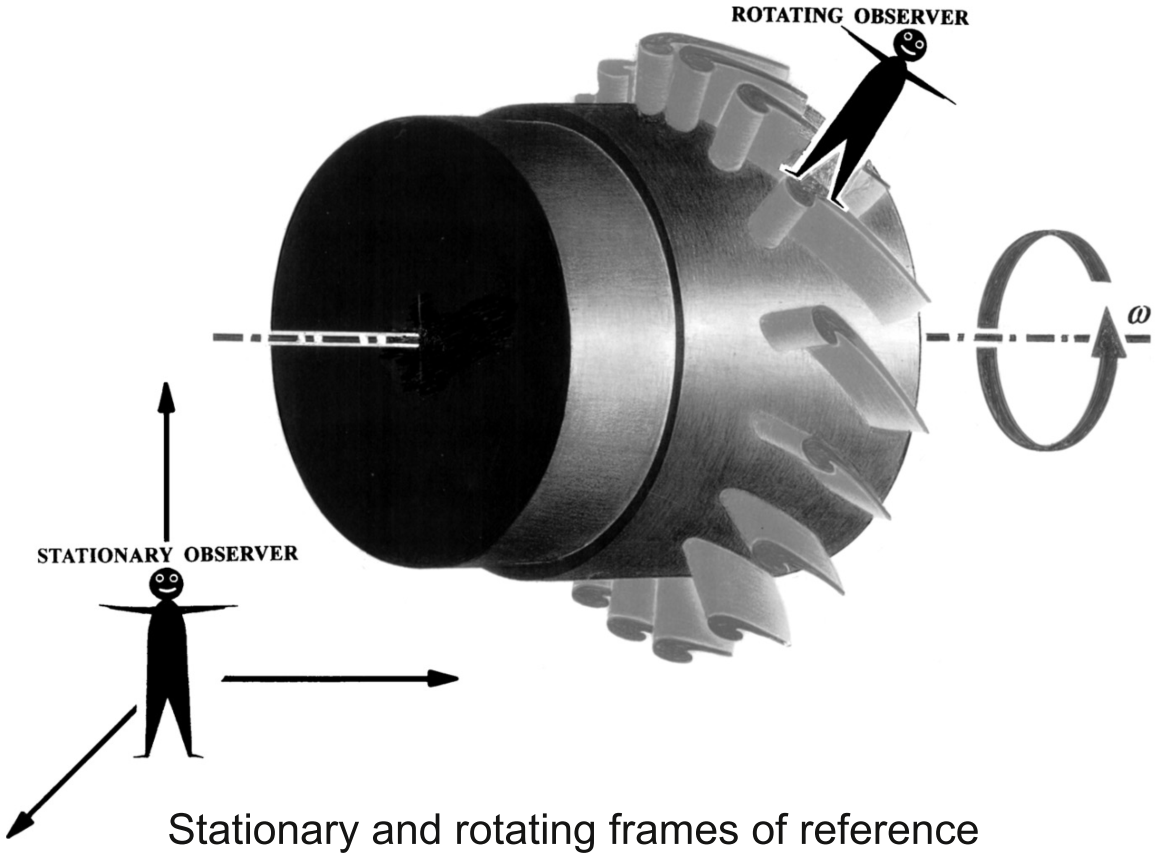
$$\mathbf{V} = \mathbf{W} + \mathbf{U}$$

$\mathbf{V}$ = Absolute fluid velocity

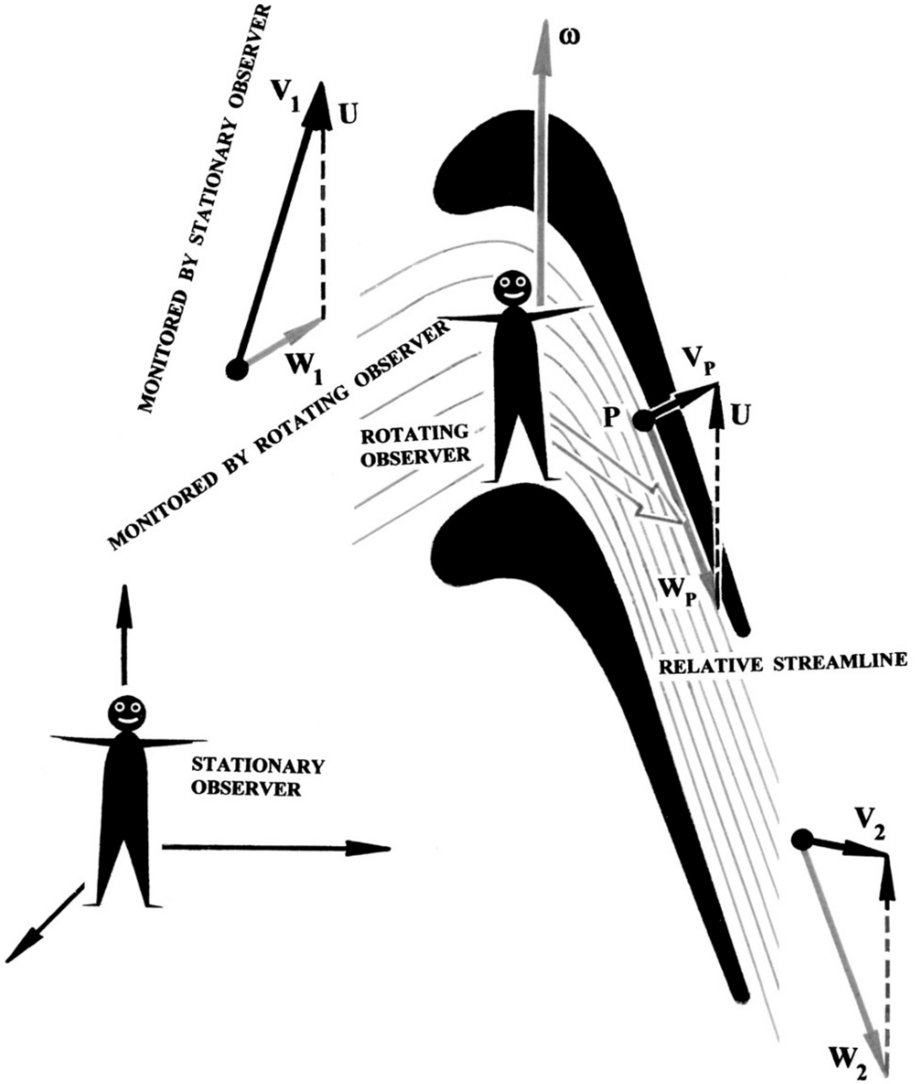
$\mathbf{W}$ = Relative velocity

$\mathbf{U}$ = Blade speed, ωr

Absolute/relative velocities



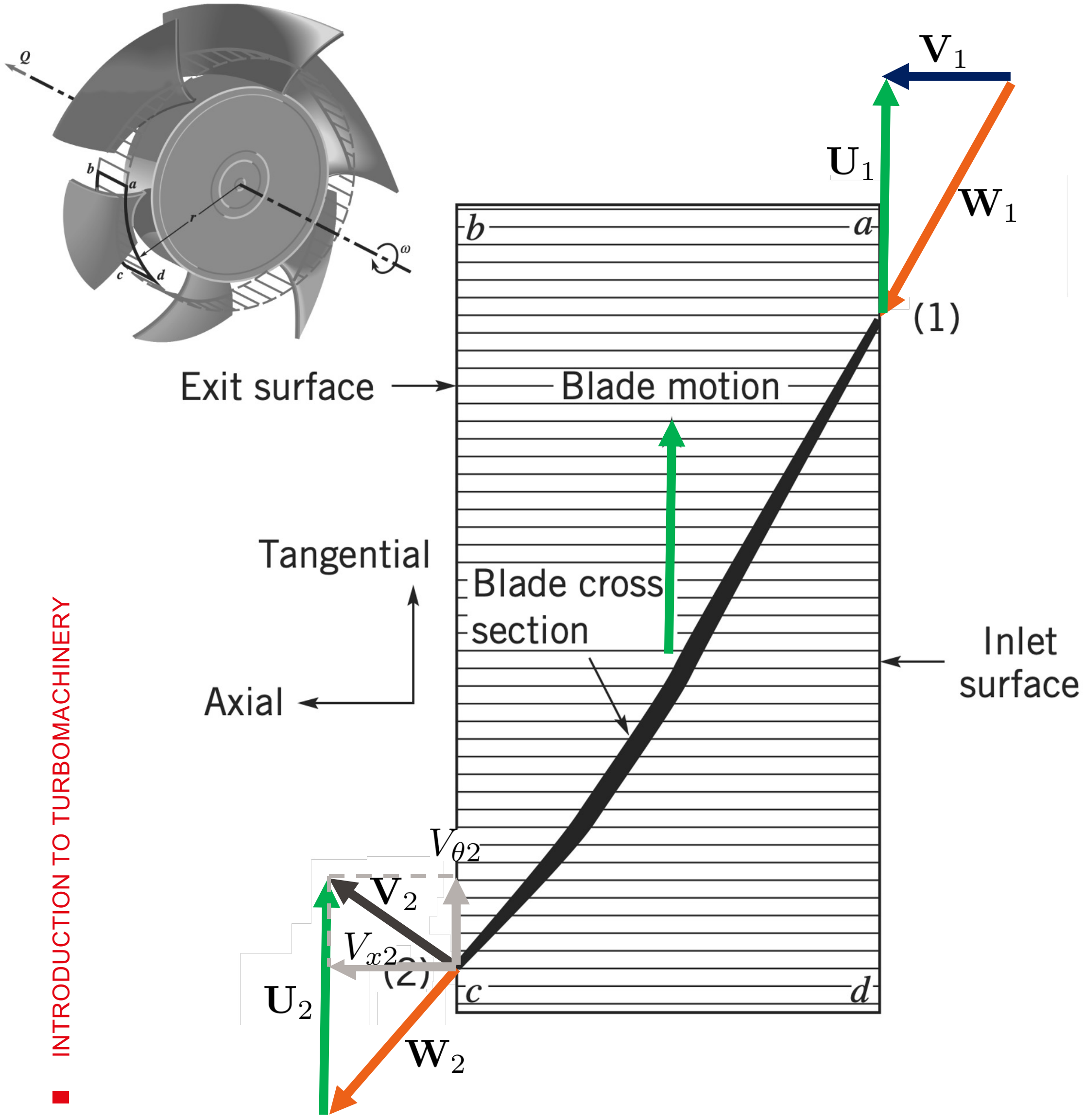
Stationary and rotating frames of reference



Relative streamlines in the rotating frame of reference

Basic Energy Considerations

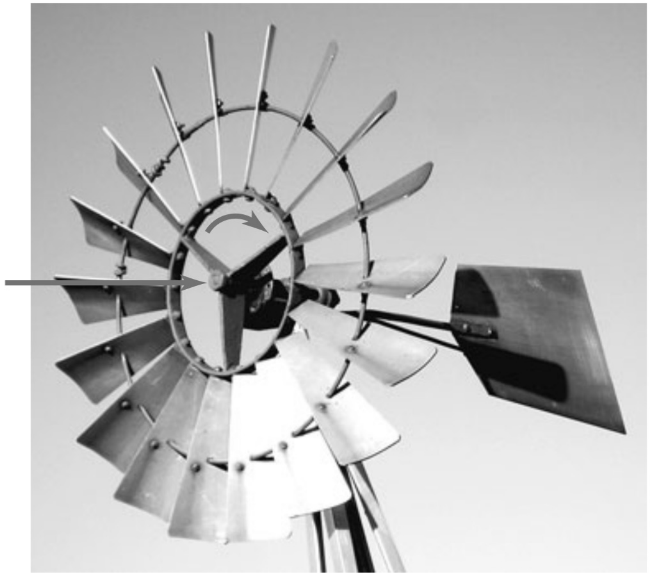
- Velocity triangle/diagram (fan)



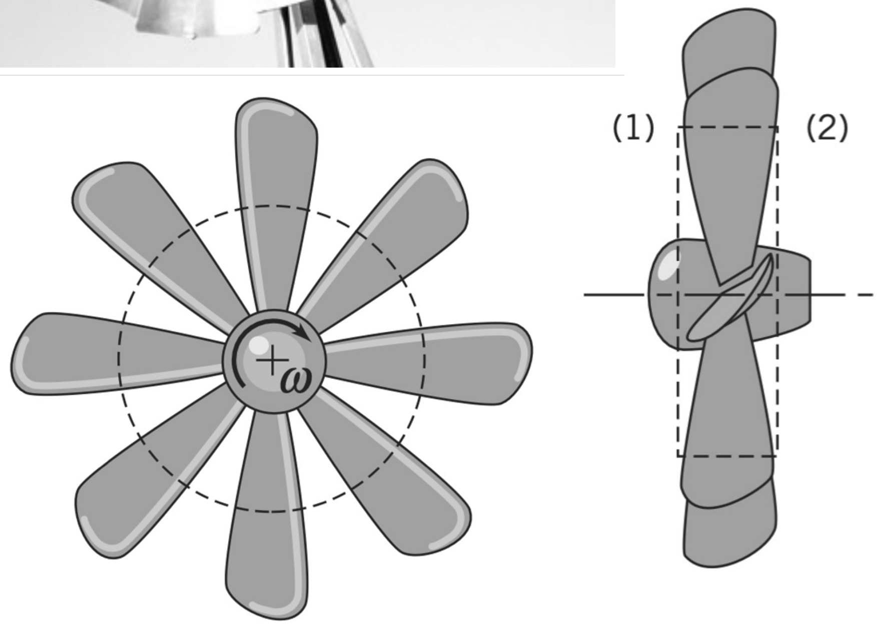
- The fan blade (because of its shape and motion) “pushes” the fluid, causing it to change direction. The absolute velocity vector, **V**, is turned during its flow across the blade from section (1) to section (2) :
  - Inlet (1): the fluid had **no tangential** component of **absolute** velocity (in the direction of the motion of the blade, θ)
  - Outlet (2): the **tangential** component of absolute velocity is **nonzero**. For this to occur, the blade must push on the fluid in the tangential direction. That is, the blade exerts a tangential force component on the fluid in the direction of the motion of the blade.
- This tangential force component and the blade motion are in the same direction—the blade does work on the fluid. **This device is a pump**

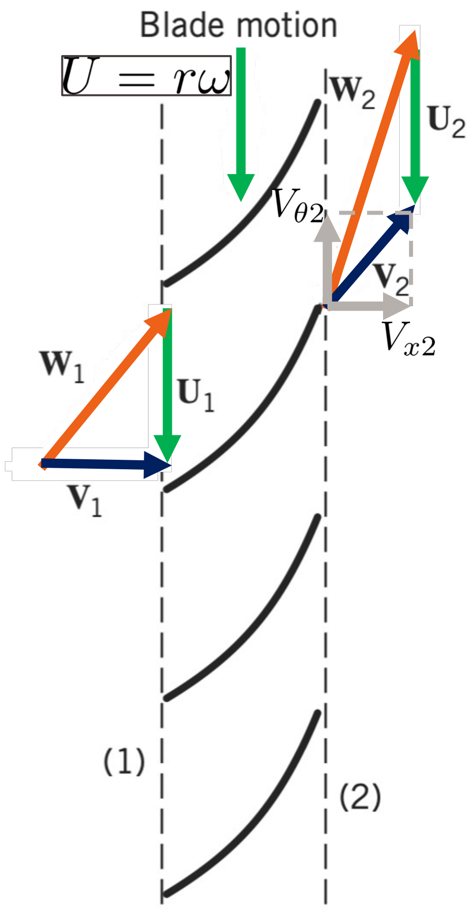
Basic Energy Considerations

- **Velocity diagram (turbine)**



Blades move in the direction of the lift force exerted on each blade by the wind blowing through the rotor





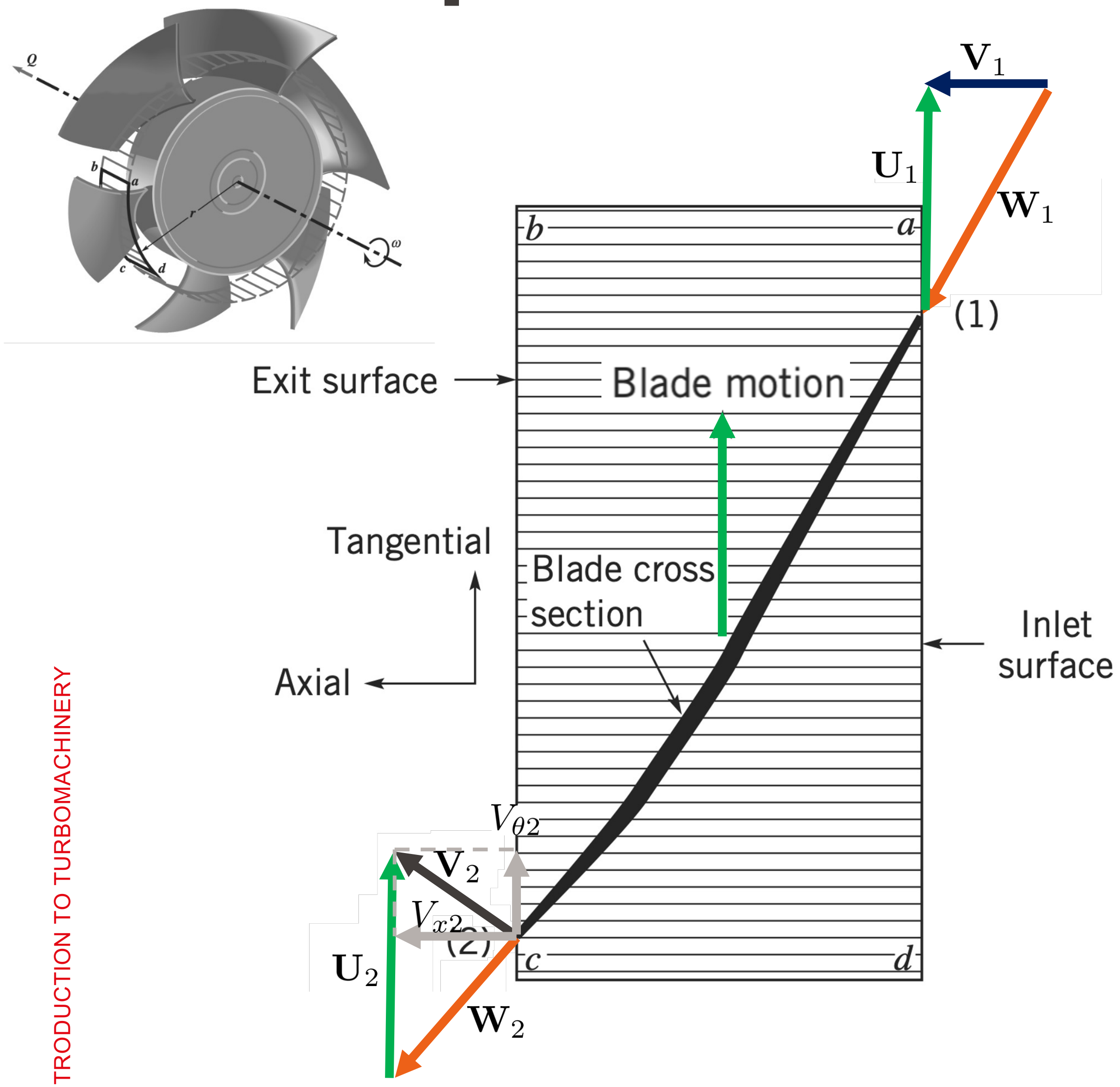
Due to the blade shape and motion:

- Absolute velocity vectors at (1) and (2), $\mathbf{V}_1$ and $\mathbf{V}_2$, have different directions

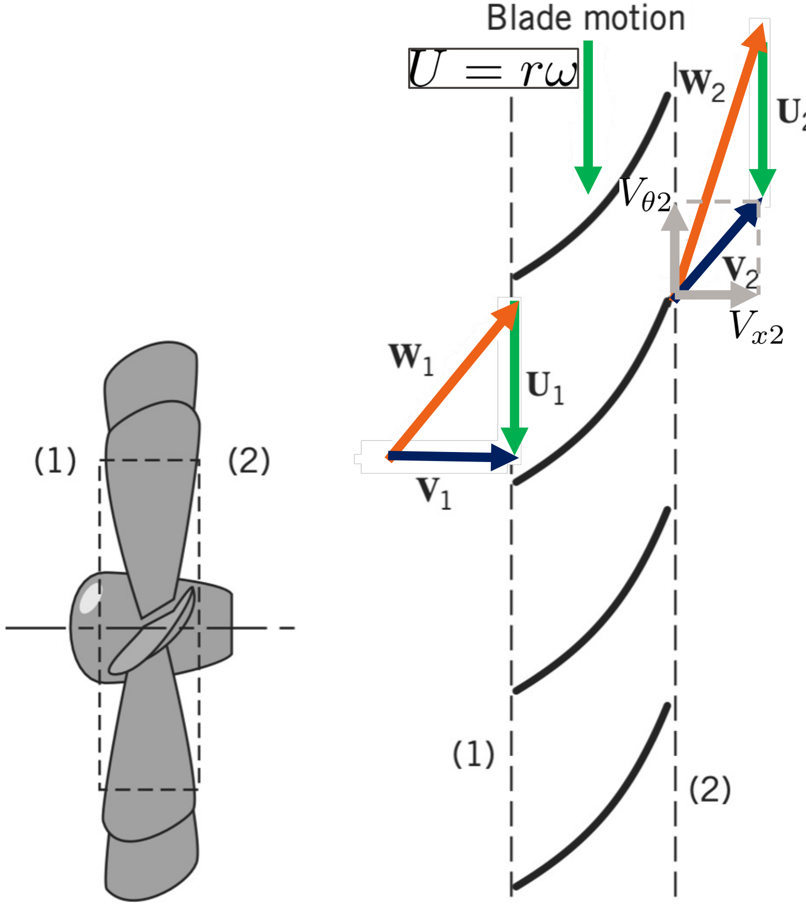
- The blades must have pushed up on the fluid → opposite to the direction of blade motion.
- Because of equal and opposite forces action-reaction, the fluid must have pushed on the blades in the direction of their motion—the **fluid does work on the blade**

→ Extraction of energy from the fluid, purpose of a **turbine**

Pump and turbine



$U = r\omega$



Euler turbomachine equation

Euler turbomachine equation

$$T_{\text{shaft}} = -\dot{m}_1 (r_1 V_{\theta 1}) + \dot{m}_2 (r_2 V_{\theta 2})$$

mass flowrate into CV

Outflow from CV

$$\mathbf{V} = [V_r, V_\theta, V_x]$$

Sign of the axial component of $\mathbf{r} \times \mathbf{V}$ or tangential component of $\mathbf{V}_{\text{tangential}} = \mathbf{V}_\theta = V_\theta$

$+$: If $\mathbf{V}_\theta$ and $\mathbf{U}$ are same direction

$-$: If $\mathbf{V}_\theta$ and $\mathbf{U}$ are opposite direction

T_{shaft} Positive same direction as rotation

T_{shaft} Negative opposite direction of rotation

- Shaft torque is directly proportional to the mass flowrate $\dot{m} = \rho Q$

  → for the same volume flowrate, ~1000 more torque is required to pump water than air
- **Tangential** component of the **absolute velocity** is important

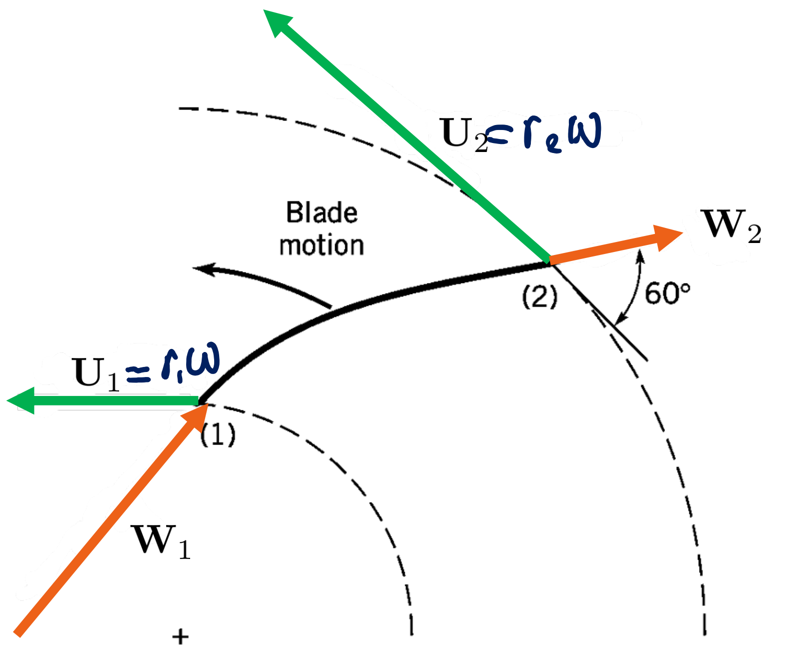
Basic Energy Considerations

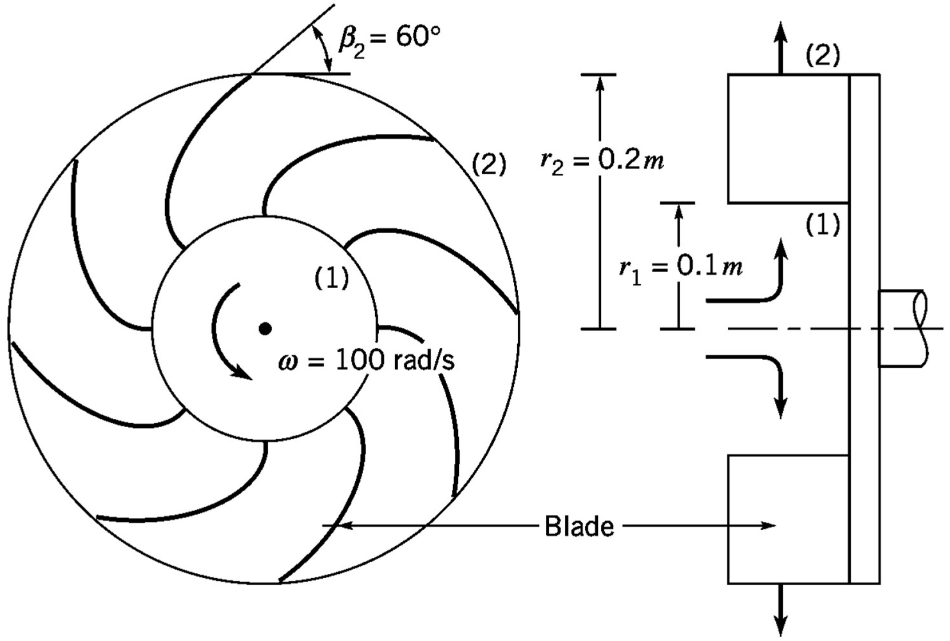
- **Exercise: Turbine or pump?**

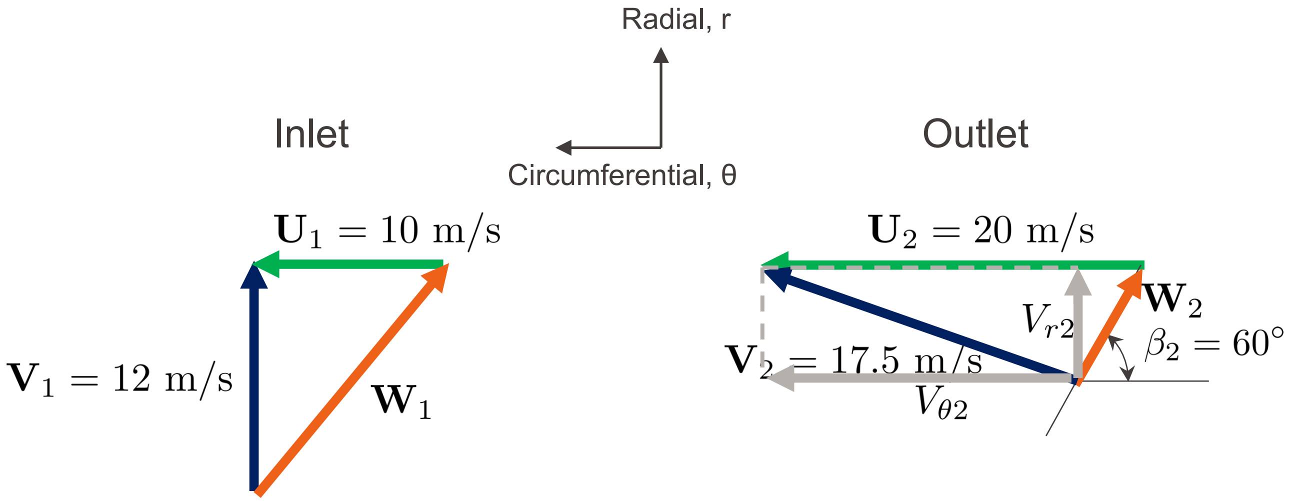
Known:

- Rotation speed ω = 100 rad/s
- r_1 = 0.1 m, r_2 = 0.2 m
- Absolute velocity at the inlet is purely radial
- Absolute velocities are measured V_1 = 12 m/s, V_2 = 17.5 m/s

Tell if it is a pump or turbine







Basic governing equations for turbomachine

- **Shaft torque**

$$T_{\text{shaft}} = -\dot{m}_1 \left(r_1 V_{\theta 1}\right) + \dot{m}_2 \left(r_2 V_{\theta 2}\right)$$

- **Shaft power**

$$\dot{W}_{\text{shaft}} = T_{\text{shaft}}\, \omega = -\dot{m} r_1 V_{\theta 1} \omega + \dot{m} r_2 V_{\theta 2} \omega$$

U_1 $\qquad$ U_2

$$\dot{W}_{\text{shaft}} = \left(-\dot{m}_1\right)\left(U_1 V_{\theta 1}\right) + \dot{m}_2 \left(U_2 V_{\theta 2}\right)$$ [W] = [kg·m²/s³]

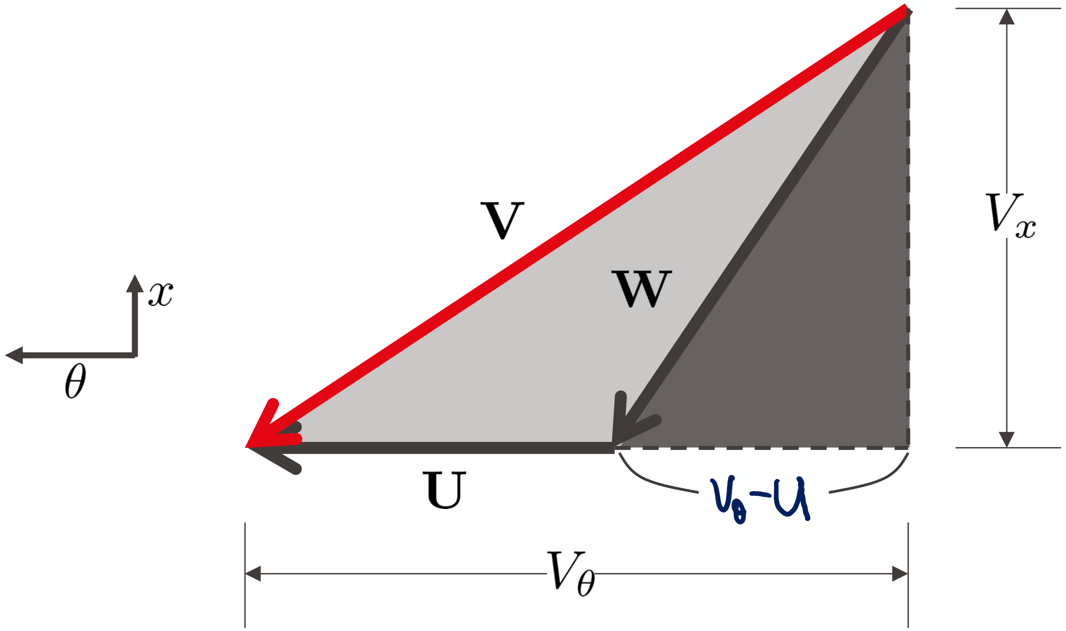
- **Shaft work per unit mass (shaft power per unit mass flow rate),** $\dot{m}_1 = \dot{m}_2$

$$w_{\text{shaft}} = -\left(U_1 V_{\theta 1}\right) + \left(U_2 V_{\theta 2}\right)$$ [m²/s²]

- Basic governing equations for pumps or turbines whether the machines are radial-, mixed-, or axial-flow devices and for compressible and incompressible flows
- Note it is only the function of tangential component of velocity, no V_r, V_x

Basic governing equations for turbomachine

$$\mathbf{V} = \mathbf{W} + \mathbf{U}$$



Velocity triangle: **V** absolute velocity, **W** relative velocity, **U** blade velocity

- From the big triangle (grey)

$V^2 = V_\theta^2 + V_x^2$ OR $V_x^2 = V^2 - V_\theta^2$

- From the small triangle (dark grey)

$W^2 = (V_\theta - U)^2 + V_x^2$

$= V_\theta^2 - 2V_\theta U + U^2 + V_x^2$

$W^2 = V_\theta^2 - 2V_\theta U + U^2 + V^2 - V_\theta^2$

$V_\theta U = \dfrac{-W^2 + U^2 + V^2}{2}$

$$w_{\text{shaft}} = -\left(U_1 V_{\theta 1}\right) + \left(U_2 V_{\theta 2}\right)$$

$$w_{\text{shaft}} = \frac{V_2^2 - V_1^2 + U_2^2 - U_1^2 - \left(W_2^2 - W_1^2\right)}{2}$$

Turbomachine work is related to changes in absolute, relative, and blade velocities.

About torques in helicopter

Two main rotors in a helicopter
- horizontal lift-producing rotor
- vertical tail rotor

Without the **tail rotor** to **cancel** the **torque** from the main rotor, the helicopter body would spin out of control in the direction opposite to that of the main rotor.

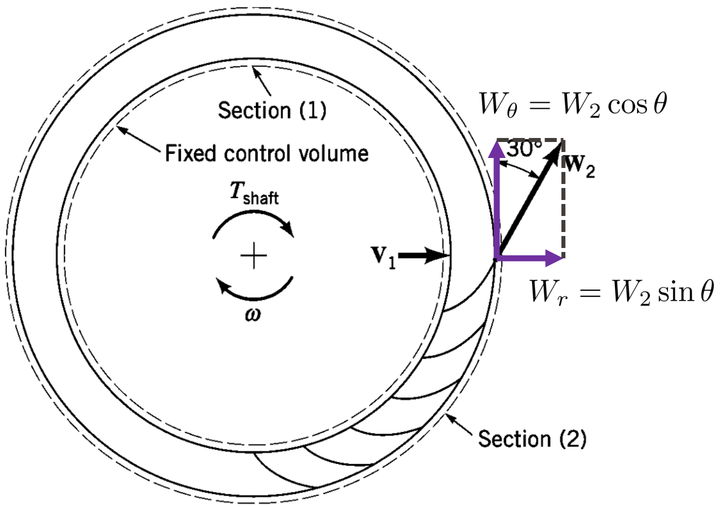
Tailless jet-rotor
Jet-rotor concept removes the need for a tail rotor in helicopters.
Gas (hot exhaust gases or compressed air) is directed through the blades/Exhausted from nozzles at blade tips, perpendicular to blade axis.
Uses the **angular momentum principle** for rotation. Similar to a rotating lawn sprinkler or dishwasher arm. No drive shaft needed to turn the main rotor, eliminating torque issues

Hiller YH-32 Hornet

Used **peroxide-fueled tip jets** to rotate the blades
A catalyst (silver or platinum) decomposed the **hydrogen peroxide** into **high-temperature steam and oxygen**.
Not efficient compare d to conventional helicopter, fuel consumption, too noisy

Exercise

An air fan has a bladed rotor of 30 cm. outside diameter and 25 cm. inside diameter. The height of each rotor blade is constant at 2.5 cm from blade inlet to outlet. The flowrate is steady, on a time-average basis, at 0.1 m^3/s and the absolute velocity of the air at blade inlet, $\mathbf{V}_1$, is radial. The blade discharge angle is 30° from the tangential direction. The rotor rotates at a constant speed of 1725 rpm. Air density is ρ = 1.2 kg/m^3, the only torque we consider is the driving shaft torque.



- Estimate the power required to run the fan

$$\dot{W}_{\text{shaft}} = (-\dot{m}_{\text{in}})(U_{\text{in}} V_{\theta\ \text{in}}) + \dot{m}_{\text{out}} (U_{\text{out}} V_{\theta\ \text{out}})$$
$$= (-\dot{m}_1)(U_1 V_{\theta 1}) + \dot{m}_2 (U_2 V_{\theta 2})$$

$$\mathbf{V} = \mathbf{W} + \mathbf{U}$$

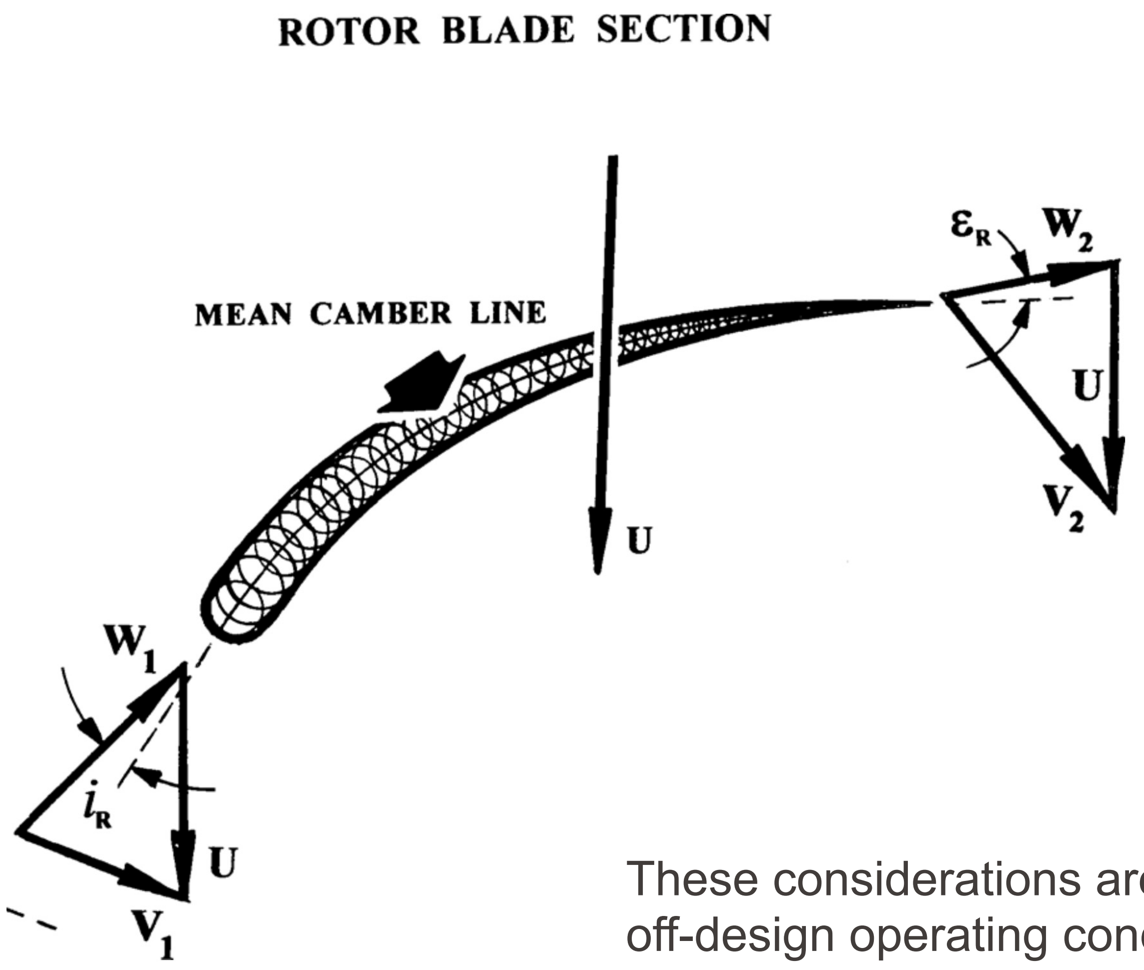
Power that needs to be delivered through the fan shaft. Ideally, all of this power would go into the flowing air.

appendix

Incidence and deviation angles

Assumptions

- Fluid **is perfectly guided by the blade** (relative velocity is parallel to the leading and trailing edge)
- Flow leaves at $r = r_{\text{in}} = r_{\text{out}}$



These considerations are important at design and off-design operating condition

Helicopters- Figure of merit (FM)

$$P_{\text{rotor}} = \frac{V_3^2}{2}\dot{m} = \dot{m}V_3 v_i = 2\dot{m}v_i^2$$

$$\dot{m} = \rho A v_i$$

$$v_i = \sqrt{\frac{\mathcal{T}}{2\rho A}} = \sqrt{\left(\frac{\mathcal{T}}{A}\right)\left(\frac{1}{2\rho}\right)}$$

$$P_{\text{rotor}} = \frac{\mathcal{T}^{3/2}}{\sqrt{2\rho A}}$$

Ideal rotor power

$$\text{Figure of merit, FM} = \frac{\text{Ideal rotor power}}{\text{Actual rotor power}}$$

If FM is 0.7 and the computed ideal power is 2000 kW, what is the actual power required?