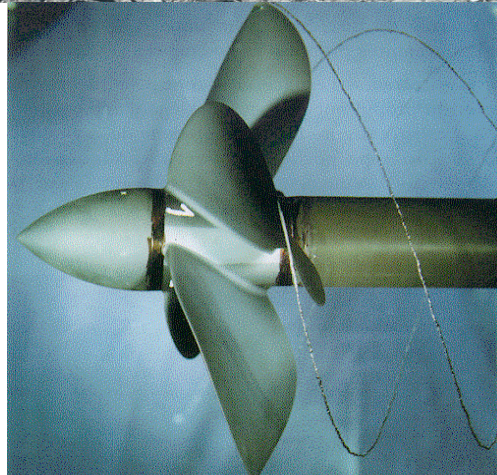




Chapter 4: Propellers & Vortex

ME-342 Introduction to
turbomachinery

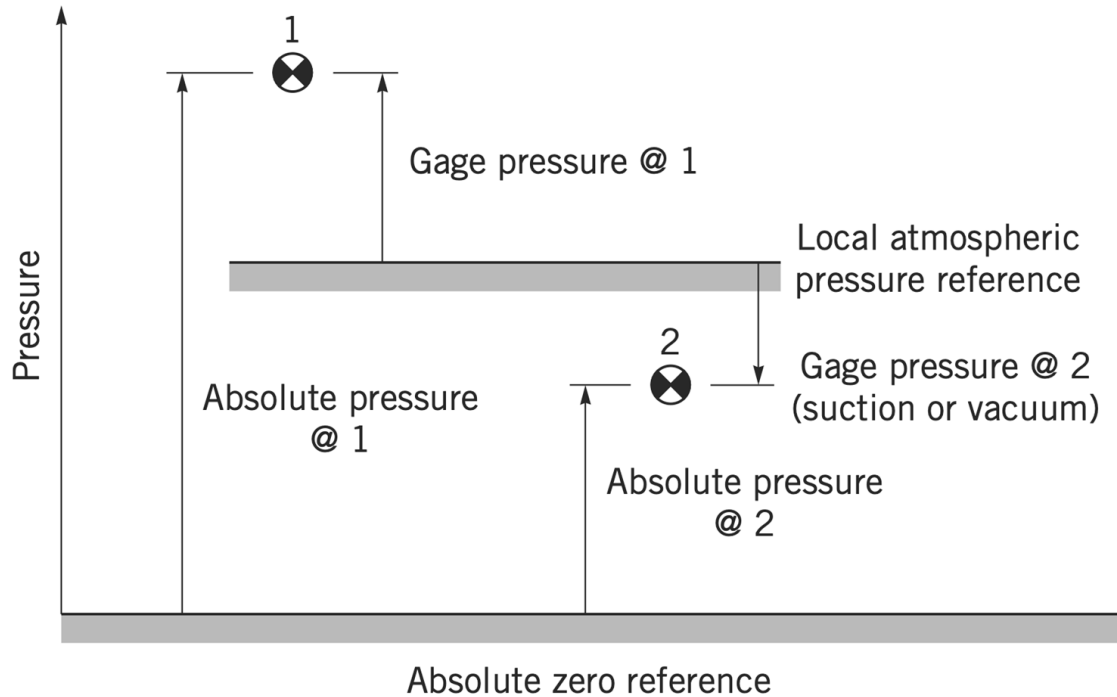
Prof. Eunok Yim, HEAD-lab.

Questions from last exercise session

- Pressure coefficient, C_p

Pressure coefficient	$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aerodynamics, hydrodynamics
Lift coefficient	$C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Drag coefficient	$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics

The pressure coefficient is a dimensionless form of the pressure



- Ideal gas law → absolute pressure

$$\rho = \frac{p}{RT}$$

The pressure in the ideal gas law must be expressed as an **absolute pressure**, denoted (abs), which means that it is measured relative to absolute zero pressure (a pressure that would only occur in a perfect vacuum).

$$P_{abs} = P_{atm} + P_{gage}$$

The pressure at a point within a fluid mass is designated as either an **absolute pressure** or a **gage pressure**.

measured relative to a perfect vacuum (absolute zero pressure)

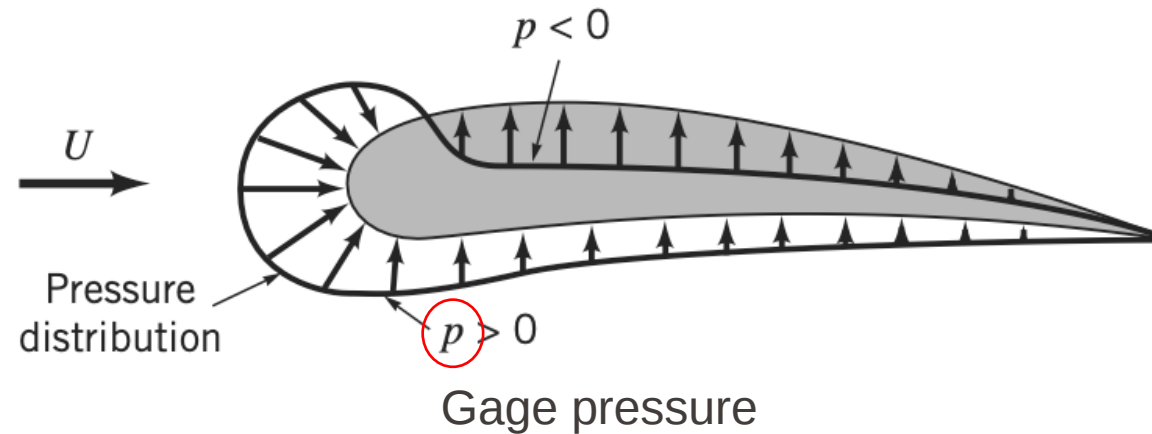
measured relative to the local atmospheric pressure

- Absolute pressures are always positive
- Gage pressure can be positive (above atm) or negative (below atm)
- Zero gage pressure: equal to the local atmospheric pressure
- Negative gage pressure is also referred to as a **suction** or **vacuum pressure**

When flow is involved...

- **Static Pressure**

- Exerted by a fluid when it is at rest or when there is no directional motion effect at a given point
- Measured perpendicular to the fluid flow in motion
- Can be expressed as absolute pressure or gage pressure



- **Absolute static pressure = Static pressure + Atmospheric pressure**

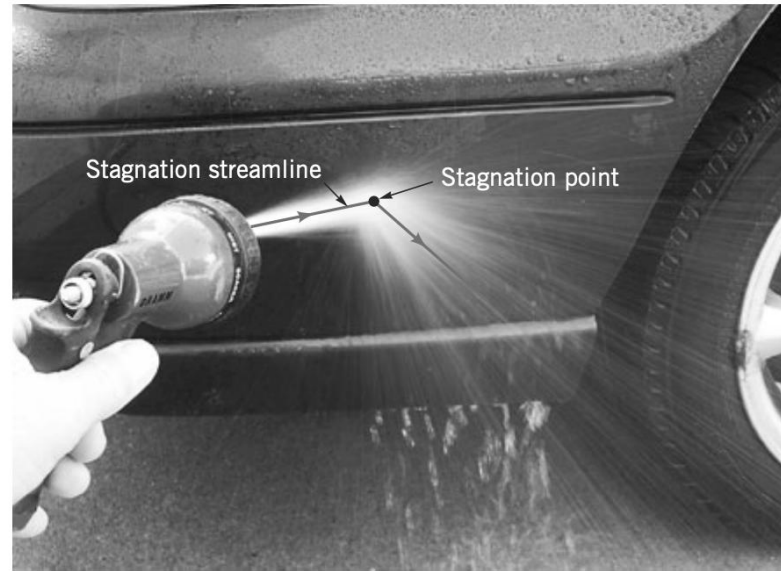
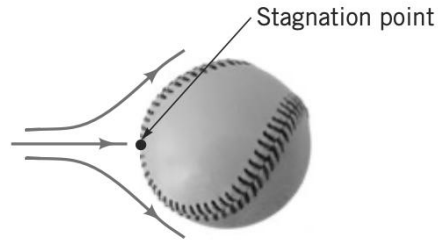
Total absolute pressure = Static (absolute) pressure + Dynamic pressure

- If static pressure is measured **relative to atmospheric pressure**, then it is the same as **gage** pressure.
- If static pressure is measured **relative to a vacuum**, then it is **absolute** pressure.

Stagnation, total pressure

If elevation effects are neglected, the **stagnation pressure**, $p + \rho V^2/2$ is the **largest pressure** obtainable along a given streamline.

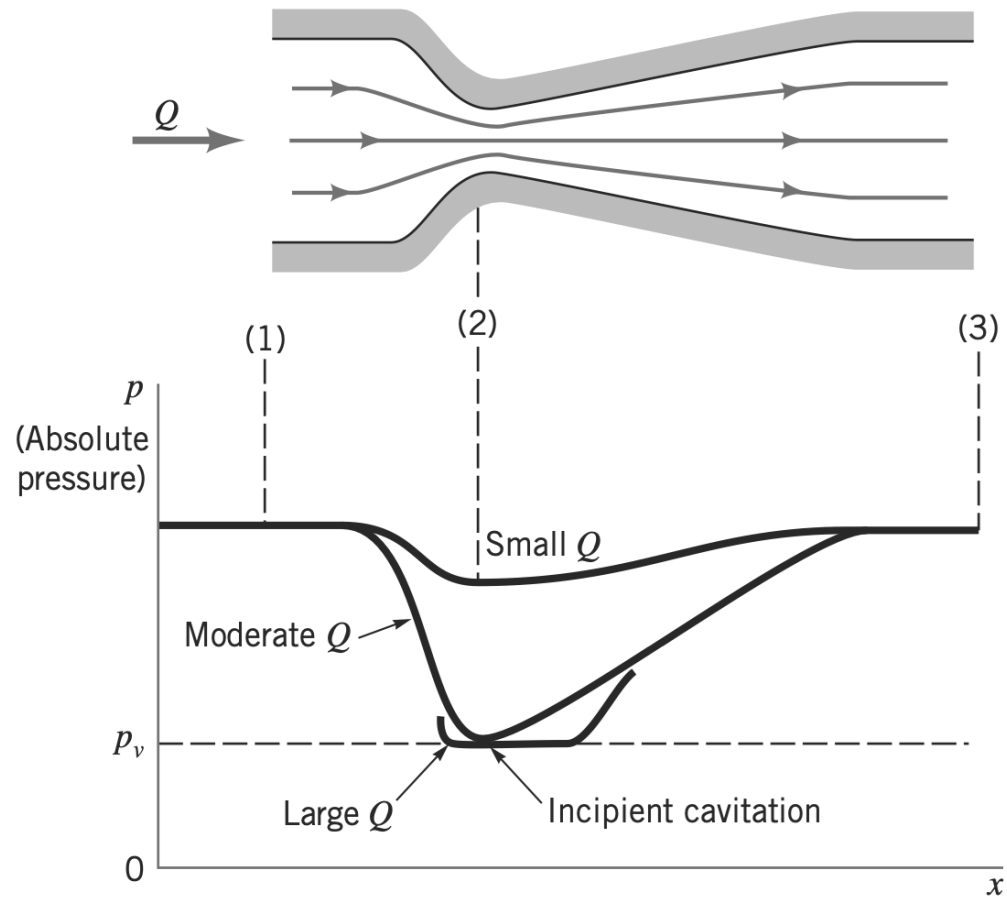
→ conversion of all of the kinetic energy into a pressure rise



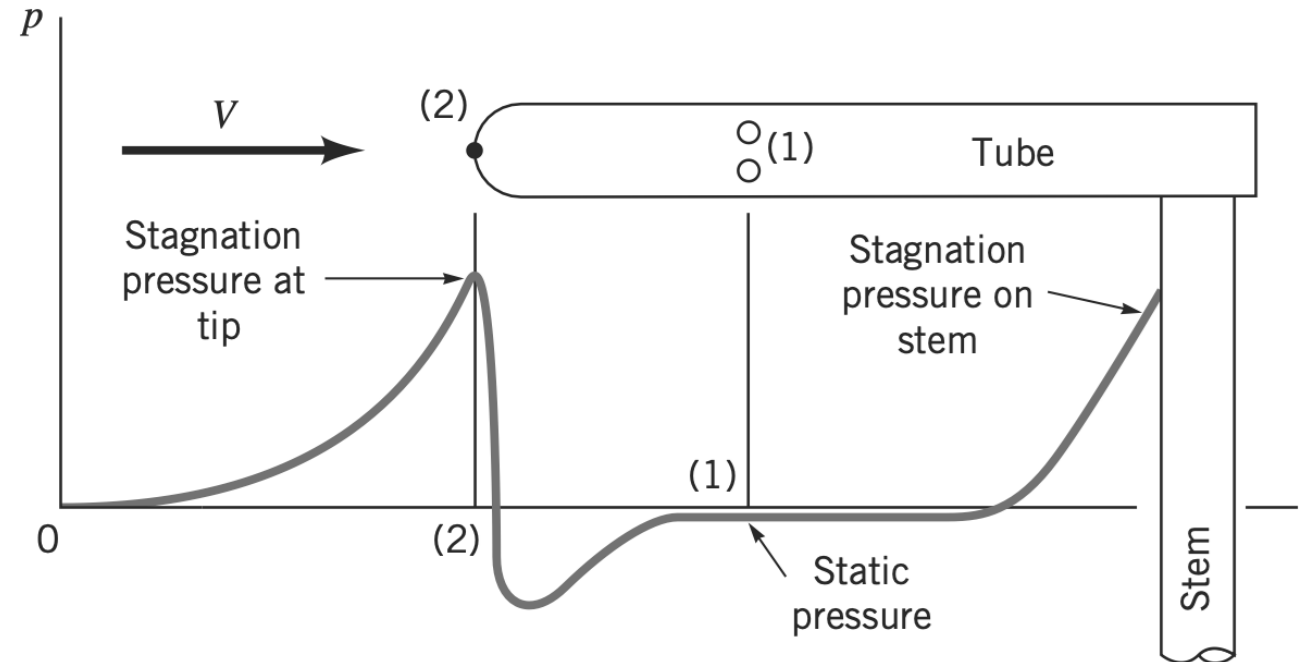
$$p + \frac{1}{2}\rho V^2 + \gamma z = p_T = \text{constant along a streamline}$$

Total pressure, p_T : sum of the static pressure, hydrostatic pressure, and dynamic pressure

• Venturi tube

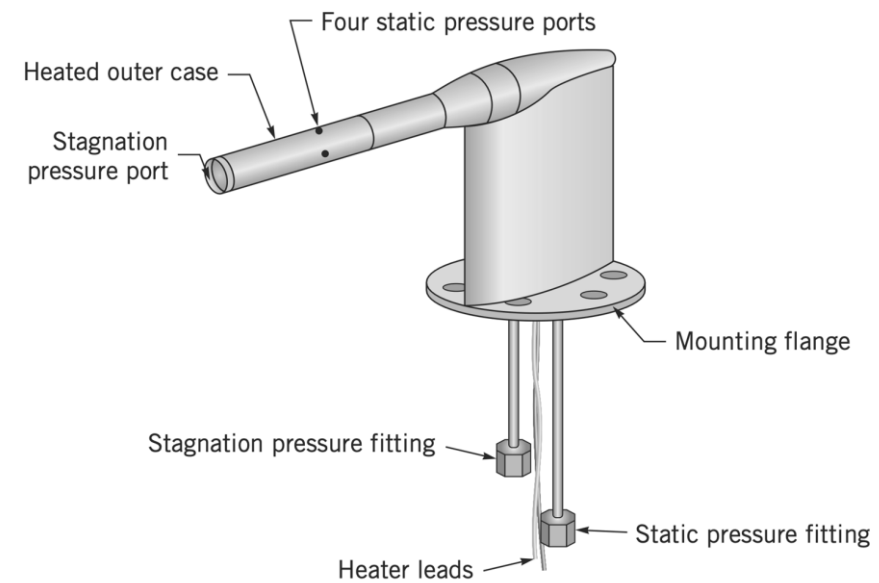
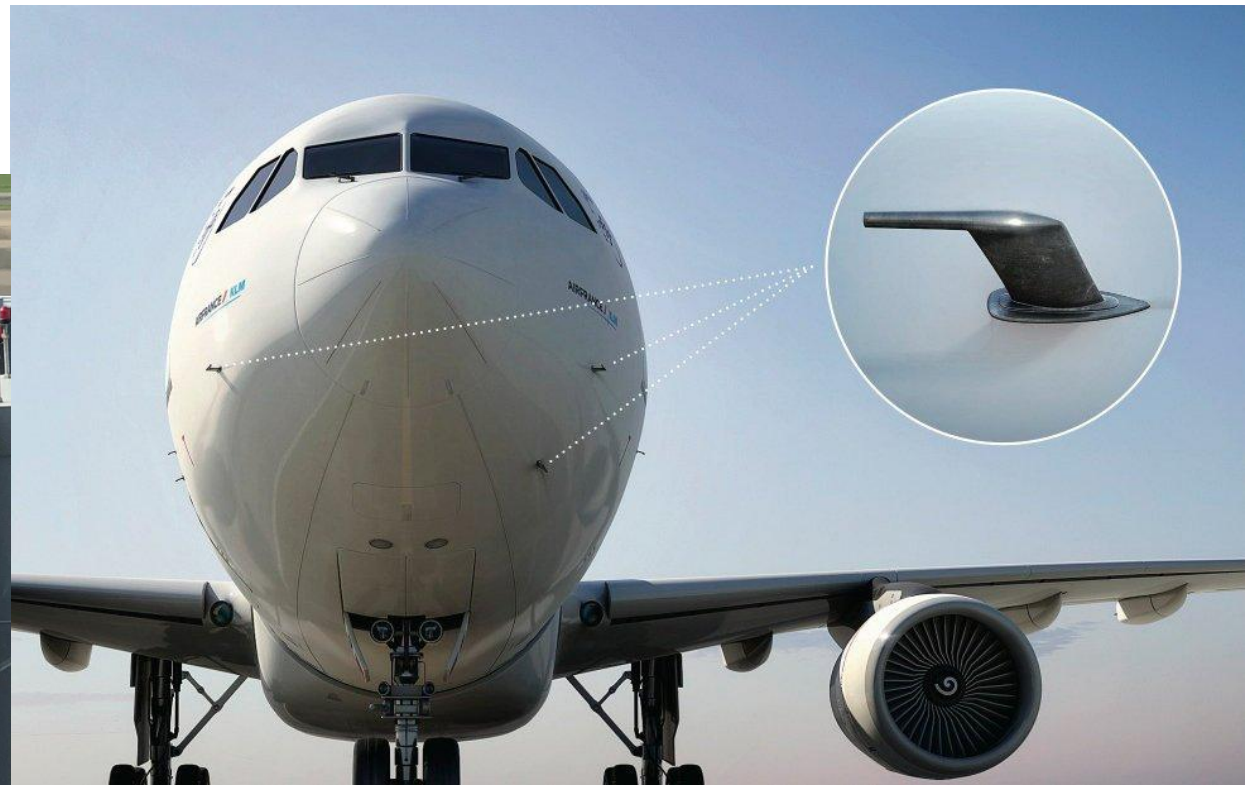


• Pitot tube

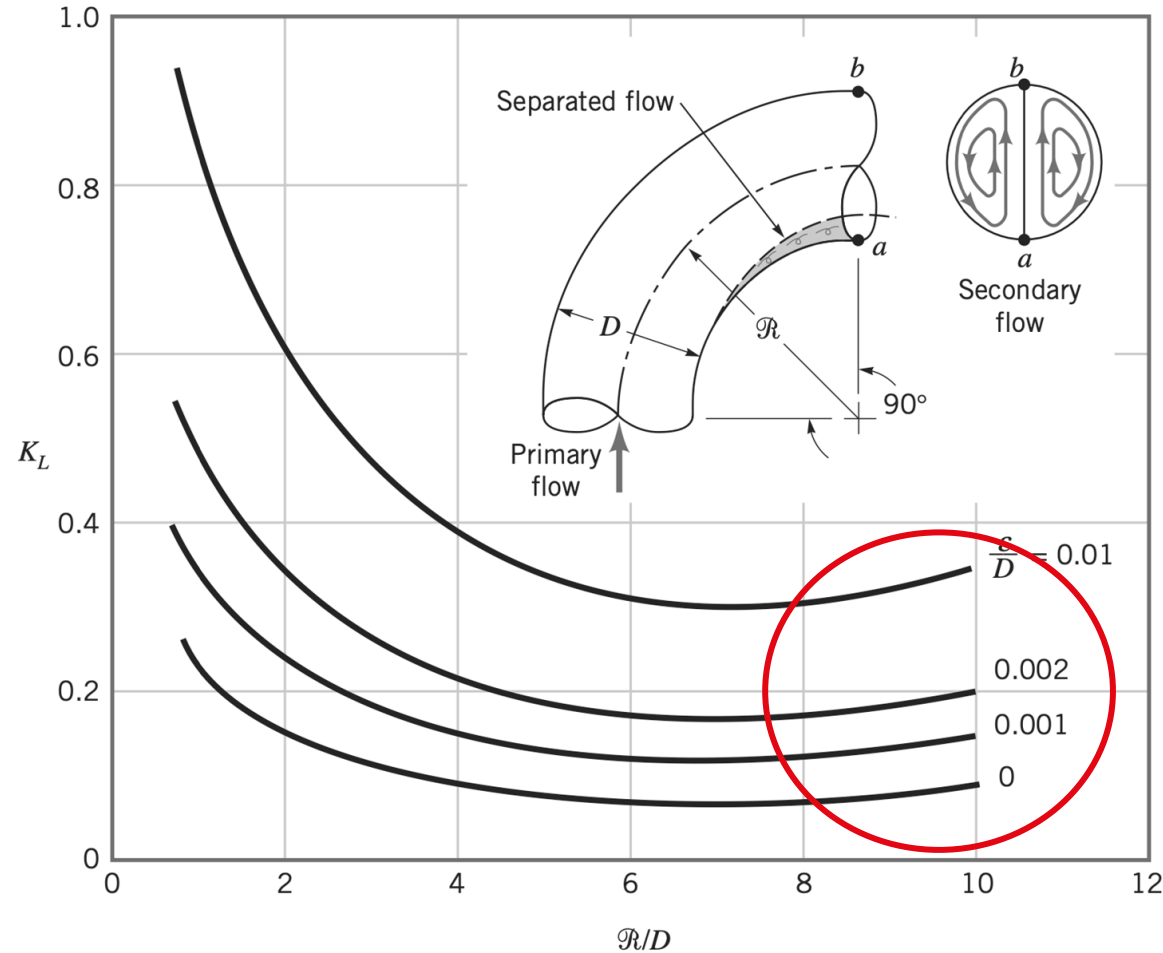


$$p_2 + \frac{1}{2}\rho V_2^2 = p_1$$

Do you remember?



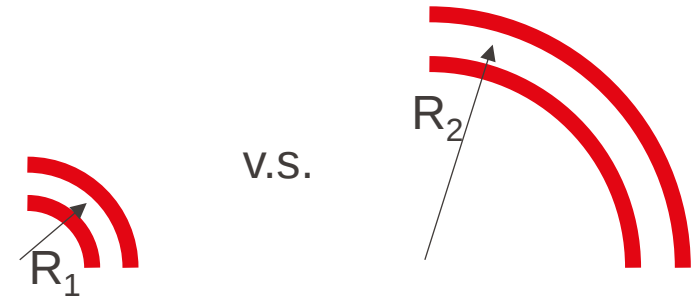
Why does the loss coefficient increase slightly?



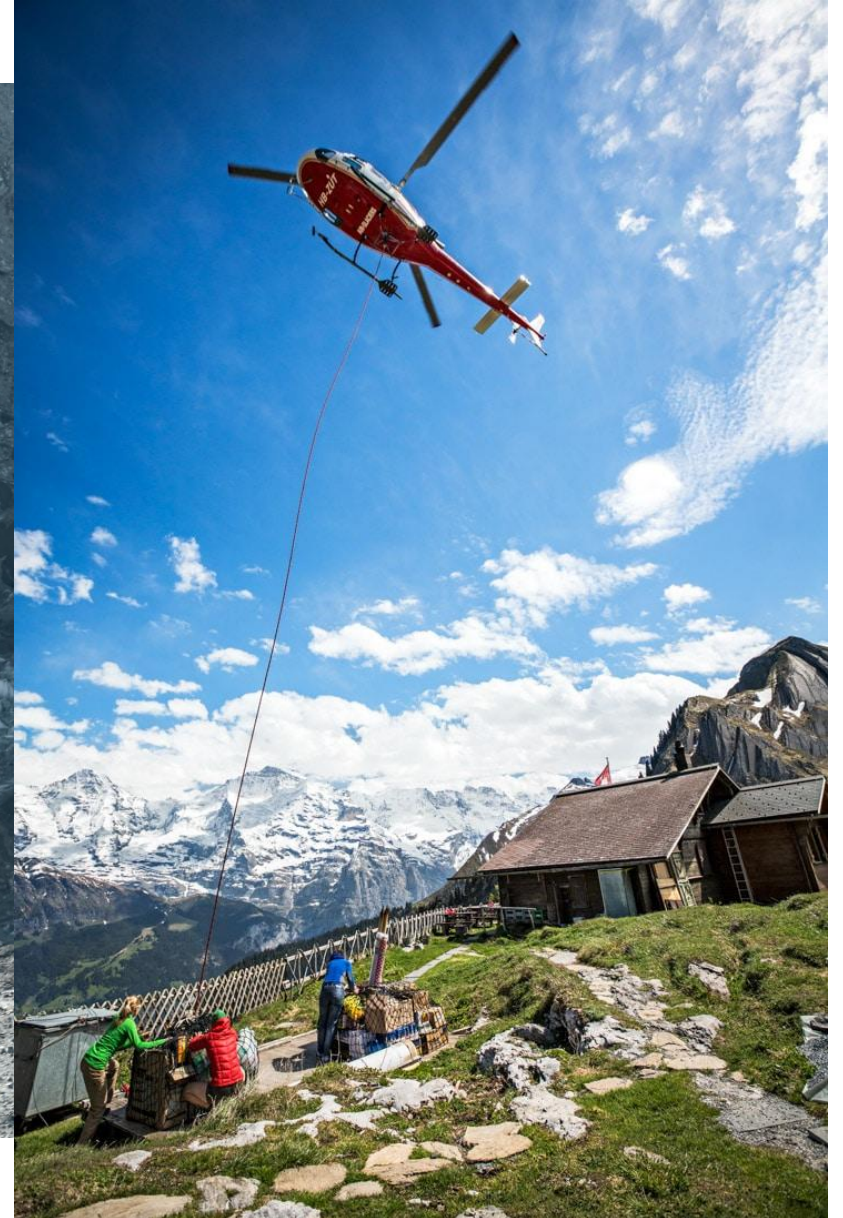
$$h_{L \text{ major}} = f \frac{\ell}{D} \frac{V^2}{2g}$$

$$h_{L \text{ minor}} = K_L \frac{V^2}{2g}$$

- Increased flow path
 - for a given D , the flow path is increased as R
 - Velocity profile difference between inner and outer bend is larger $\rightarrow$ secondary flow separation



One example turbomachinery, which is directly linked to the lift and drag of airfoils?



One example turbomachinery, which is directly linked to the lift and drag of airfoils?



Propellers (Helicopter)

1-Continuity equation (Mass conservation)

Time rate of change of the system mass = 0 :

$$\frac{DM_{\text{sys}}}{Dt} = 0$$

The system mass, M_{sys} is

$$M_{\text{sys}} = \int_{\text{sys}} \rho dV$$

Reynolds
transport
theory

- Continuity equation

$$\frac{D}{Dt} \int_{\text{sys}} \rho dV = \frac{\partial}{\partial t} \int_{\text{cv}} \rho dV + \int_{\text{cs}} \rho \vec{V} \cdot \hat{n} dA = 0$$

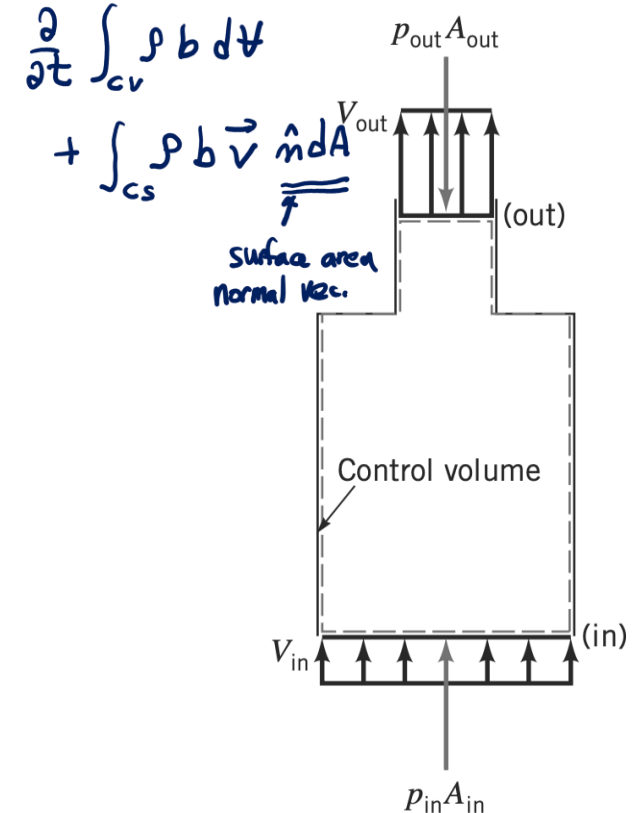
- Mass flow rate $\dot{m} = \rho Q = \rho AV$

$$\dot{m} = \int_A \rho \vec{V} \cdot \hat{n} dA$$

$$B = mb$$

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\partial t} + \dot{B}_{\text{out}} - \dot{B}_{\text{in}}$$

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\partial t} + \rho_2 A_2 V_2 b_2 - \rho_1 A_1 V_1 b_1$$



- cs: control **s**urface
- cv: control **v**olume

2-Linear momentum equation

Newton's second law of motion

The change of **motion** of an object is **proportional** to the **force** impressed; and is made in the direction of the straight line in which the force is impressed

$$\frac{D}{Dt}$$

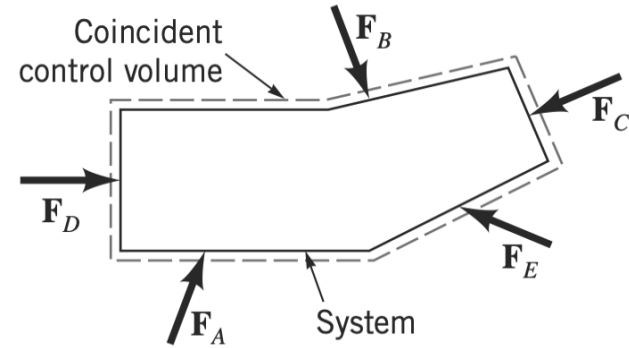
$$\text{momentum} = \text{mass} \times (\text{its}) \text{velocity} = \int_{\text{sys}} \rho \vec{V} dV$$

$$= \int_{\text{sys}} \rho dV = \vec{V}$$

$$\frac{D}{Dt} \left(\int_{\text{sys}} \rho \vec{V} dV \right) = \frac{\partial}{\partial t} \int_{\text{cv}} \rho \vec{V} dV + \int_{\text{cs}} \vec{V} \cdot \rho \vec{V} \hat{n} dA$$

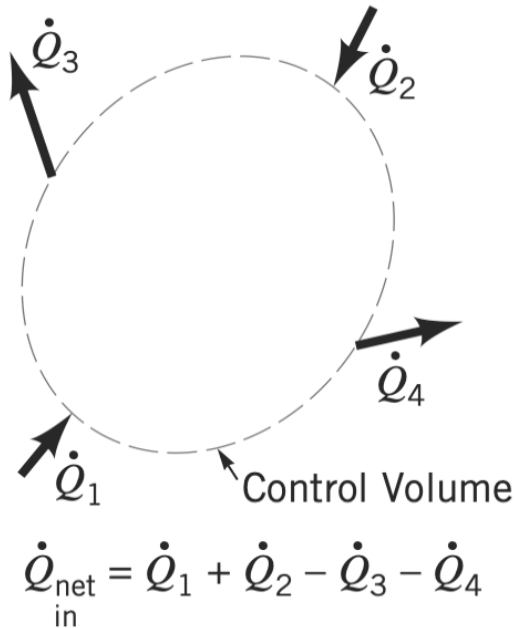
- Linear momentum equation

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho \vec{V} dV + \int_{\text{cs}} \rho \vec{V} \cdot \hat{n} dA = \sum \mathbf{F}_{\text{contents of the control volume}}$$



3-Energy equation

- cs: control **surface**
- cv: control **volume**



The first law of thermodynamics

$$\frac{D}{Dt} \int_{\text{sys}} e \rho dV = \frac{\partial}{\partial t} \int_{\text{cv}} e \rho dV + \int_{\text{cs}} e \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

Time rate
of increase
of the total
stored energy
of the system

=

time rate of increase
of the total stored
energy of the contents
of the control volume

net rate of flow
of the total stored energy
+ out of the control
volume through the
control surface

where $e = \underbrace{\check{u}}_{\text{internal energy}} + \frac{V^2}{2} + gz$ total stored energy per unit mass

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \frac{\partial}{\partial t} \int_{\text{cv}} e \rho dV + \int_{\text{cs}} \left(\check{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz \right) \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

Heat
transfer ratio

Work transfer
rate, power

Three governing equations

- Continuity equation

$$\frac{\partial}{\partial t} \int_{\text{cv}} \rho dV + \int_{\text{cs}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = 0$$

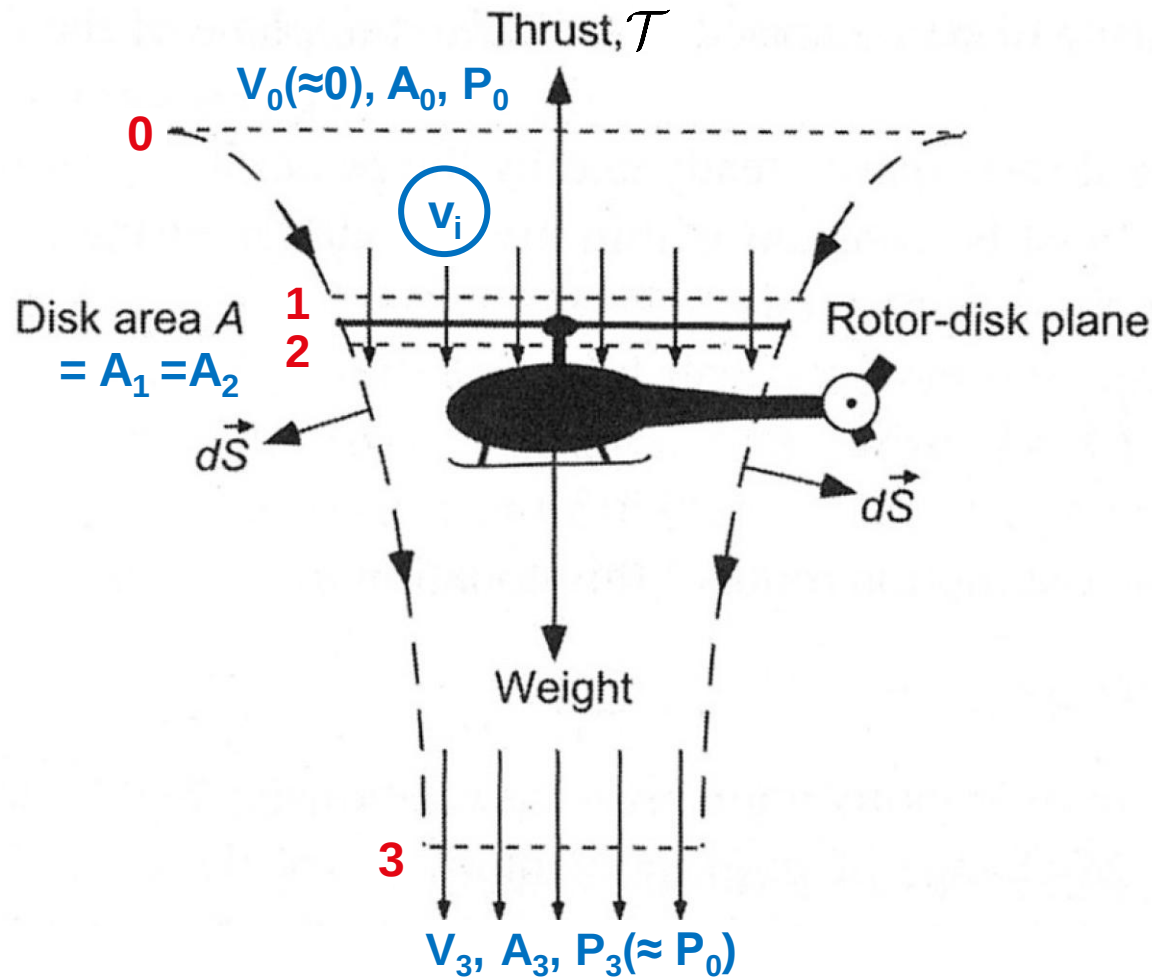
- Linear momentum equation

$$\frac{\partial}{\partial t} \int_{\text{cv}} \mathbf{V} \rho dV + \int_{\text{cs}} \mathbf{V} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = \sum \mathbf{F}_{\text{contents of the control volume}}$$

- Energy equation

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \frac{\partial}{\partial t} \int_{\text{cv}} e \rho dV + \int_{\text{cs}} \left(\check{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz \right) \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

Helicopters- conservation law for hovering rotor



v_i : induced air velocity

V_3 : far wake velocity

- Continuity & linear momentum equations:

$$\frac{\partial}{\partial t} \int_{cv} \rho dV + \int_{cs} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = 0$$

$$\frac{\partial}{\partial t} \int_{cv} \mathbf{V} \rho dV + \int_{cs} \mathbf{V} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = \sum \mathbf{F}_{\text{contents of the control volume}}$$

Steady-state approximation

$$\int_{cs} \rho \vec{V} \cdot d\vec{S} = 0$$

$$\int_{cs} \vec{V} \cdot \rho \vec{V} d\vec{S} = \sum \mathbf{F}$$

Thrust, T (reaction force)

$$-F = T = \int_3 \vec{V} \rho \vec{V} d\vec{S} - \int_0 \vec{V} \rho \vec{V} d\vec{S}$$

$$\mathcal{T} = \dot{m} V_3$$

$$\dot{m} = \int_A \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

- Rotor power** (work done by rotor per unit time)

$$P_{\text{rotor}} = \mathcal{T} v_i$$

Helicopters- conservation law for hovering rotor

- Energy equation (the gain of energy of the fluids per unit time)

adiabatic

steady

No internal energy change

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft net in}} = \frac{\partial}{\partial t} \int_{\text{cv}} e \rho dV + \int_{\text{cs}} \left(\cancel{\check{u}} + \frac{p}{\rho} + \frac{V^2}{2} + gz \right) \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

$d\vec{S}$

Mass flow rate

$$\dot{m} = \int_A \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

$$\dot{W}_{\text{shaft net in}} = P_{\text{rotor}} = \int_{\text{cs}} \left(\frac{p}{\rho} + \frac{V^2}{2} + gz \right) \rho \vec{V} \cdot d\vec{S}$$

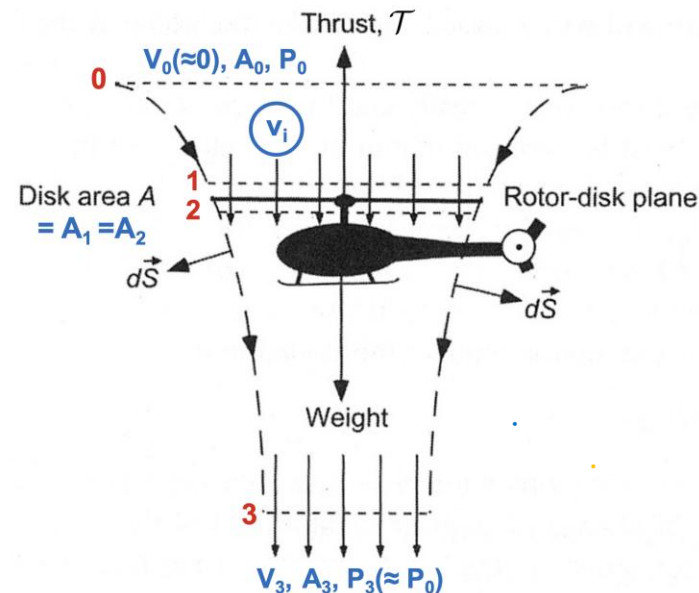
$\underbrace{\rho \vec{V} \cdot d\vec{S}}_{=\dot{m}}$

$$= \sum_{\text{flow out}} \left(\frac{p}{\rho} + \frac{V^2}{2} + gz \right) \dot{m} - \sum_{\text{flow in}} \left(\frac{p}{\rho} + \frac{V^2}{2} + gz \right) \dot{m}$$

$$= \left(\frac{p_3}{\rho} + \frac{V_3^2}{2} \right) \dot{m} - \left(\frac{p_0}{\rho} + \frac{V_0^2}{2} \right) \dot{m}$$

Potential energy change $\ll$ kinetic energy

$$= \frac{V_3^2}{2} \dot{m}$$



Helicopters- conservation law for hovering rotor

- Continuity equation:

$$\mathcal{T} = \dot{m} V_3, \quad P_{\text{rotor}} = \mathcal{T} v_i$$

- Energy equation: $P_{\text{rotor}} = \frac{V_3^2}{2} \dot{m}$

$$P_{\text{rotor}} = \frac{V_3^2}{2} \dot{m} = \dot{m} V_3 v_i$$

Induced velocity:

$$v_i = \frac{1}{2} V_3$$

Far field velocity is twice of the induced velocity

- Mass flowrate: $\dot{m} = \rho A_3 V_3 = \rho A_2 v_i = \rho A v_i$

$$\mathcal{T} = \dot{m} V_3 = \dot{m} (2v_i) = (\rho A v_i) (2v_i) = 2\rho A v_i^2$$

The induced velocity:

$$v_h \equiv v_i = \sqrt{\frac{\mathcal{T}}{2\rho A}} = \sqrt{\left(\frac{\mathcal{T}}{A}\right) \left(\frac{1}{2\rho}\right)}$$

Disk loading

Helicopters- Figure of merit (FM)

$$P_{\text{rotor}} = \frac{V_3^2}{2} \dot{m} = \dot{m} V_3 v_i = 2 \dot{m} v_i^2$$

$$\dot{m} = \rho A v_i$$

$$v_i = \sqrt{\frac{\mathcal{T}}{2\rho A}} = \sqrt{\left(\frac{\mathcal{T}}{A}\right) \left(\frac{1}{2\rho}\right)}$$

$$P_{\text{rotor}} = \frac{\mathcal{T}^{3/2}}{\sqrt{2\rho A}} \quad \text{Ideal rotor power}$$

$$\text{Figure of merit, FM} = \frac{\text{Ideal rotor power}}{\text{Actual rotor power}}$$

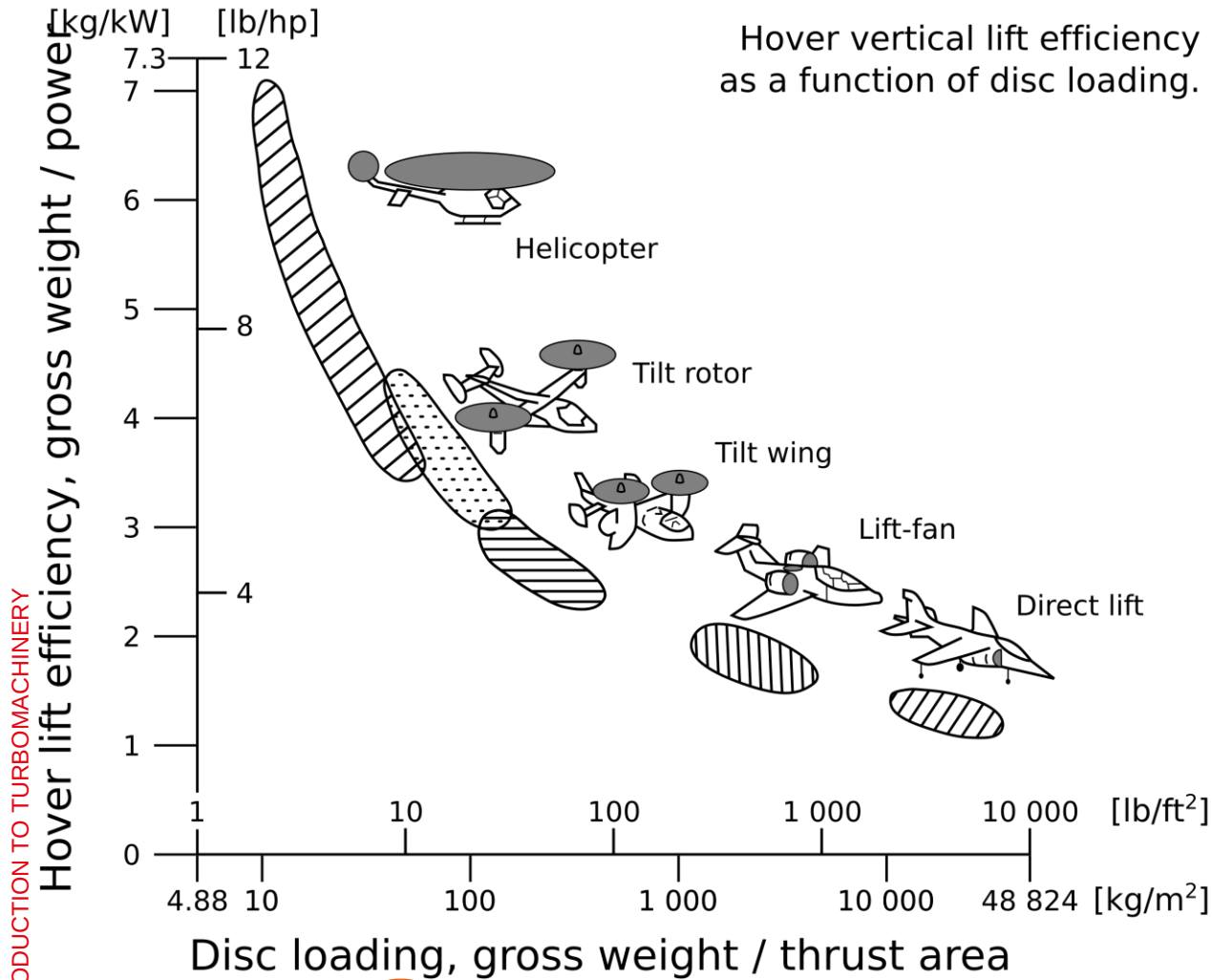
If FM is 0.7 and the computed ideal power is 2000 kW, what is the actual power required?

Hovering efficiency vs. disk loading

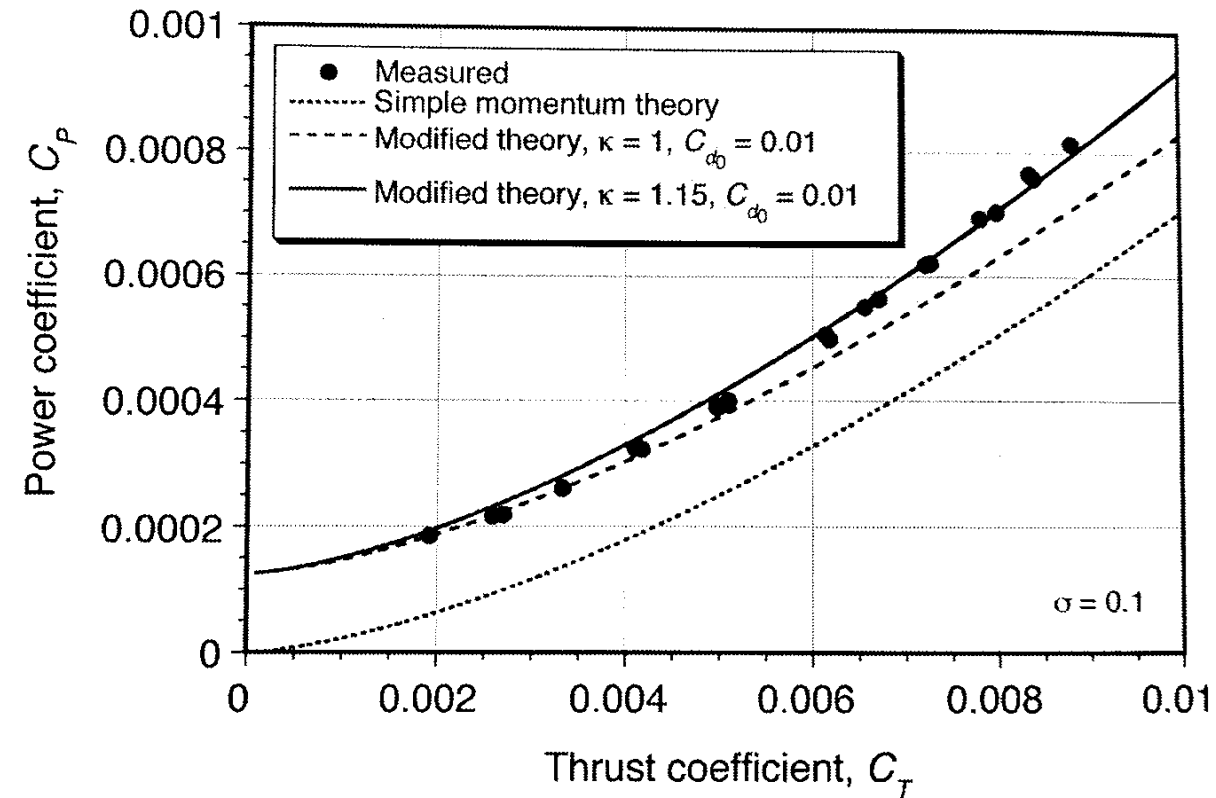
Ideal power:

$$P_{\text{rotor}} = \frac{\mathcal{T}^{3/2}}{\sqrt{2\rho A}}$$

Hover vertical lift efficiency
as a function of disk loading.



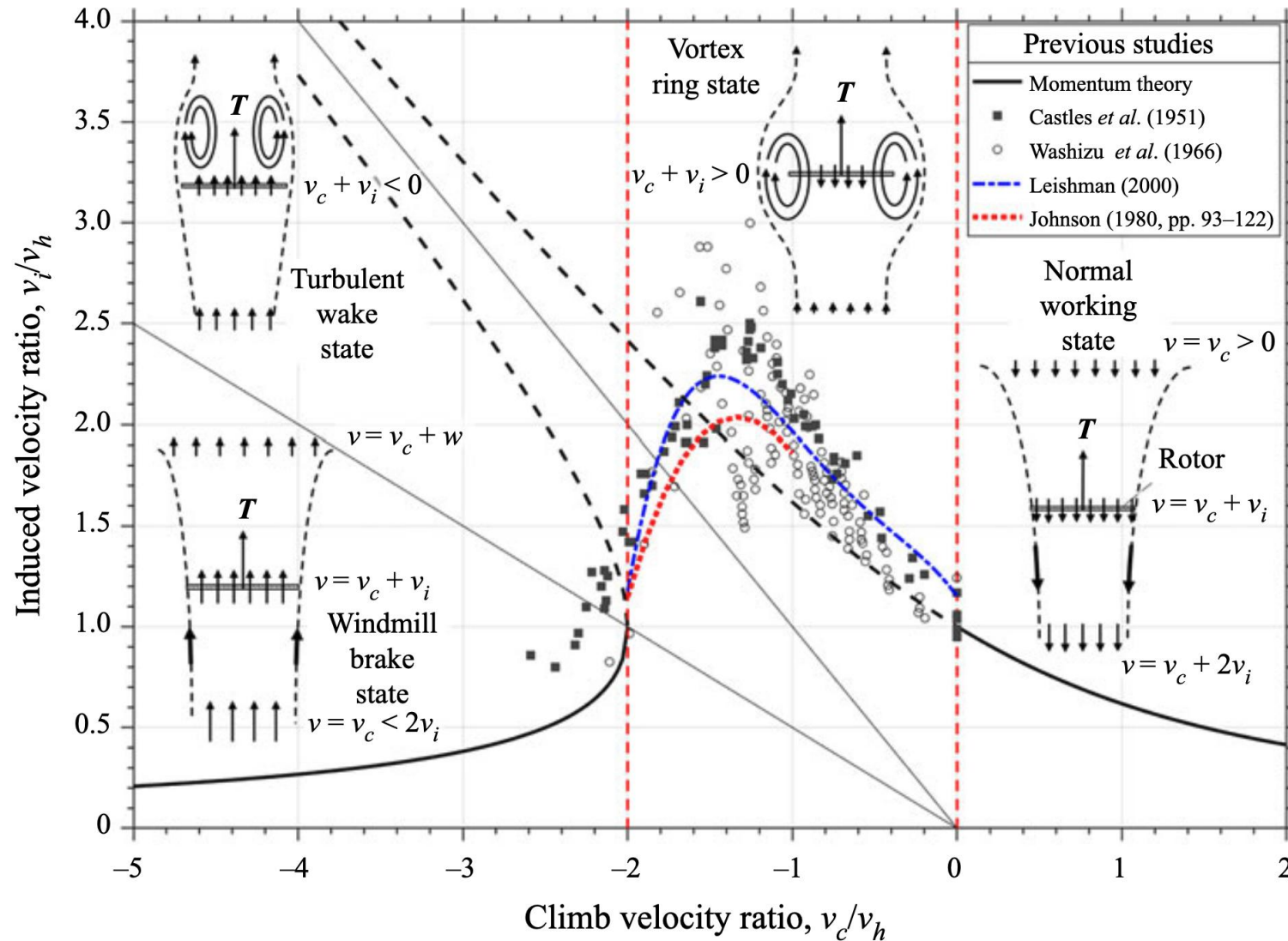
$$\frac{\mathcal{T}}{A}$$

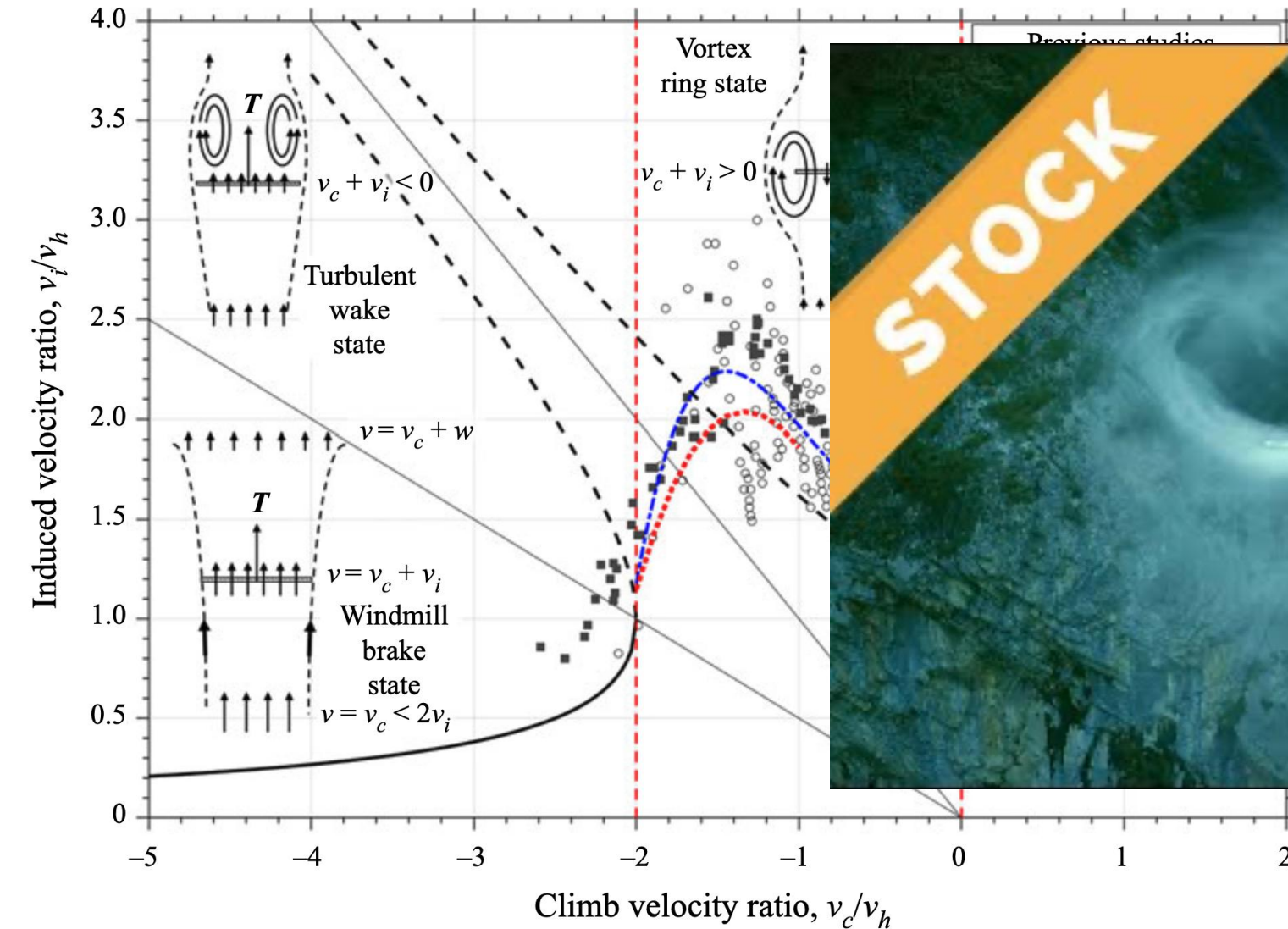


*Note, here C_p is not the pressure coefficient (C_p)

Real helicopter dynamics are much more complex..

Vortex ring state





A helicopter is helping to the construction of a mountain cabane.

- (1) Estimate the power required for the helicopter to hover. 7022 kW
- (2) What is the amount of fuel required for 10 min hover? (ignore mass change)

- Gross mass of helicopter 20 tonnes
- Rotor diameter 12 m, $R=6\text{ m}$
- Density of air 1.2 kg/m^3
- Figure of merit of the rotor 0.75
- Fuel for helicopter (BP Avgas 80), 30 MJ/litre (= 8.3 kWh/litre)



$$(1) T = 20000\text{ kg} \times 9.8\text{ m/s}^2 \\ = 196000\text{ N}$$

$$P_{\text{rotor}} = T v_i$$

$$v_i = \sqrt{\frac{T}{2\rho A}}$$

$$A = \pi R^2 = 3.14 \times 6^2 = 113.04\text{ m}^2$$

$$v_i = \sqrt{\frac{196000}{2 \cdot 1.2 \cdot 113.04}} = 26.87\text{ m/s}$$

$$P_{\text{rotor, ideal}} = T \cdot v_i = 196000 \times 26.87 = 5266520\text{ W} \approx 5266.5\text{ kW}$$

$$P_{\text{actual}} = \frac{P_{\text{ideal}}}{\text{FM}} = \underline{\underline{7022\text{ kW}}}$$

$$(2) \text{ hover time } 10\text{ min} = \frac{1}{6}\text{ hour} = 0.167\text{ hours} \\ \text{Energy req. } 10\text{ min hover} = 7022\text{ kW} \cdot 0.167\text{ h} \\ = 1173\text{ kWh}$$

$$\text{Fuel req} = \frac{1173\text{ kWh}}{8.3\text{ kWh/litre}} \\ \approx \underline{\underline{141\text{ litre}}}$$

Vortex

Vortex flow: A vortex may be seen as region of a flow in which the fluid rotates around a straight or curved line

- Free vortex (potential flow):
 - Velocity field:

$$\vec{u} = (0, u_\theta, 0), \quad u_\theta(r) = \frac{\Gamma}{2\pi r}$$

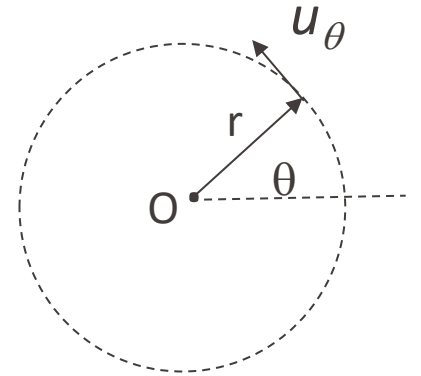
Vortex intensity Γ

$$\Gamma = \oint_C \vec{u} \cdot d\vec{s} = cst.$$

$d\vec{s}$ = infinitesimal path element along the closed curve C

- Irrotational flow

$$\vec{\zeta} = \nabla \times \vec{u} = \left(0, 0, \frac{1}{r} \frac{\partial (ru_\theta)}{\partial r} \right) = (0, 0, 0) \quad \forall r > 0$$



- Free vortex (potential flow):

- Pressure field :

- Navier-Stokes (radial equilibrium): $\frac{\partial p}{\partial r} = \rho \frac{u_\theta^2}{r} = \rho \frac{\Gamma^2}{4\pi^2 r^3}$

$$\text{Integration} \rightarrow p(r) = p_\infty - \frac{\rho \Gamma^2}{8\pi^2 r^2} \quad u_\theta(r) = \frac{\Gamma}{2\pi r}$$

- Alternate method - Bernoulli equation:

Since the flow is irrotational and steady, Bernoulli equation reads:

$$\forall r > 0, \quad p(r) + \rho \frac{u_\theta^2}{2} = \text{constant}$$

$$\Rightarrow \forall r > 0, \quad p(r) = p_\infty - \rho \frac{u_\theta^2}{2} = p_\infty - \frac{\rho \Gamma^2}{8\pi^2 r^2}$$

- Limitations:

$$\lim_{r \rightarrow 0} u_\theta(r) = +\infty, \quad \lim_{r \rightarrow 0} p(r) = -\infty$$

- Rankine model:
 - Close to the axis, the viscous forces limits the velocity
 - Velocity field:

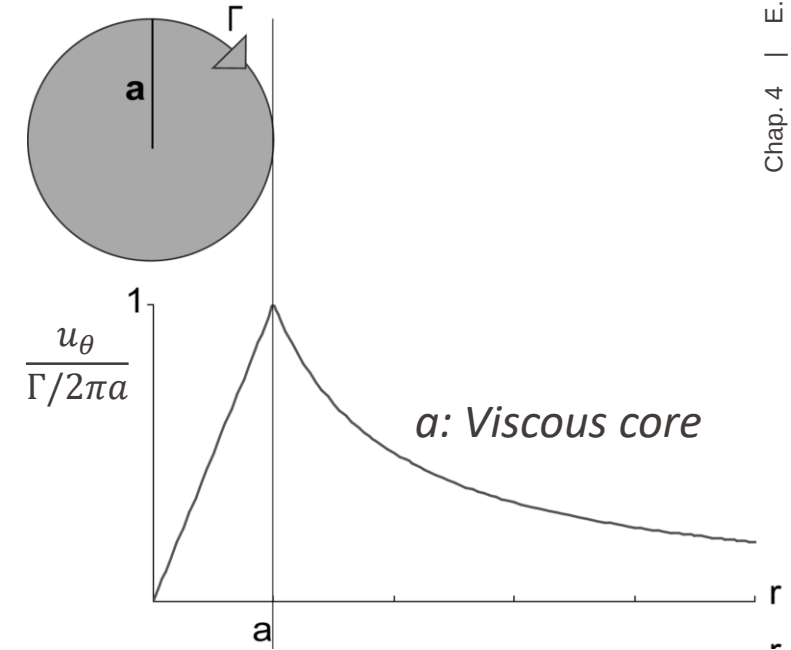
$$\begin{cases} r \geq a, & u_\theta = \frac{\Gamma}{2\pi r} \\ r \leq a, & u_\theta = \omega r \end{cases}$$

where Γ is the circulation:

$$\Gamma = \int_0^{2\pi} u_\theta r d\theta$$

- Continuity of the velocity at $r = a$

$$\omega a = \frac{\Gamma}{2\pi a} \rightarrow \omega = \frac{\Gamma}{2\pi a^2} \quad \text{or} \quad \Gamma = 2\pi\omega a^2$$



- Rankine model:

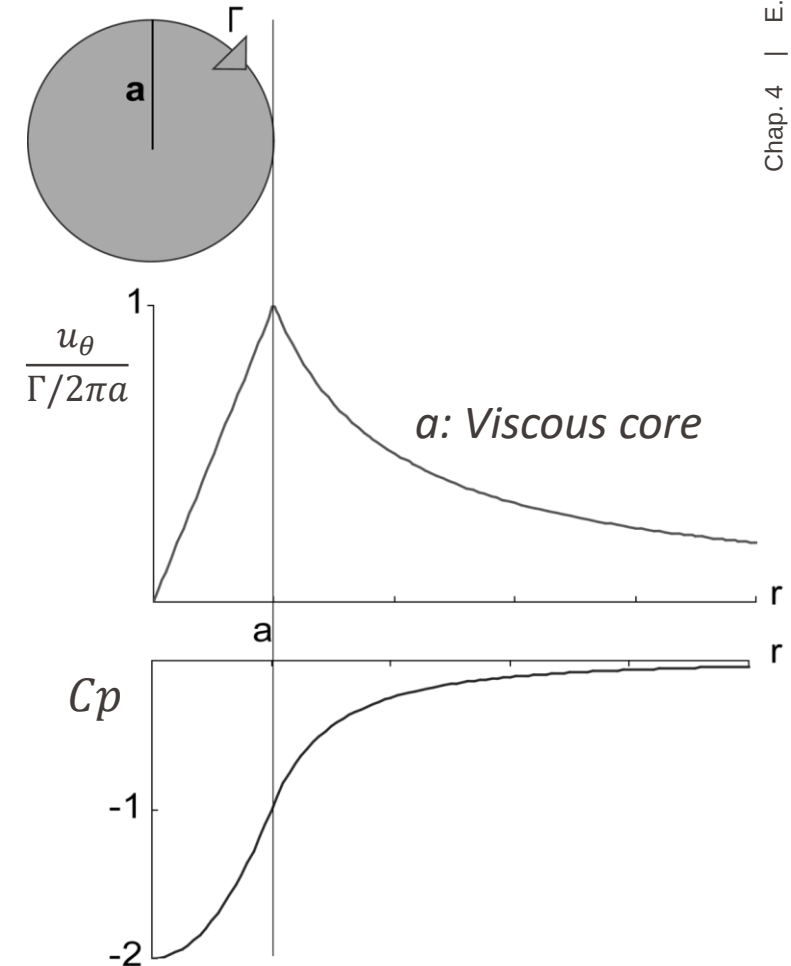
- Pressure field (Navier Stokes)

$$\frac{\partial p}{\partial r} = \rho \frac{u_\theta^2}{r}$$

- Integration:

$$\left\{ \begin{array}{l} r \geq a, \quad p(r) = p_\infty - \frac{\rho \Gamma^2}{8\pi^2 r^2} \\ r \leq a, \quad p(r) = p_\infty - \frac{\rho \Gamma^2}{8\pi^2 a^2} \left(2 - \frac{r^2}{a^2} \right) \end{array} \right.$$

$$C_p(r) = \frac{p(r) - p_\infty}{\frac{1}{2} \rho \left(\frac{\Gamma}{2\pi a} \right)^2} \quad C_{p,\min} = -2$$



- Discontinuity of the velocity derivative at $r = a$

- More sophisticated models are available (e.g. Oseen model, Vestas model, ...)

EPFL Vortex flow

- Example of vortex flow: Super Typhoon Haiyan, Philippines, November 2013

- Maximum wind speed ~ 360 km/h
 - Pressure in the vortex axis ? $\hookrightarrow \frac{360000 \text{ m/h}}{3600 \text{ s/h}} = 100 \text{ m/s}$
- Assumptions:
 - Air density $\rho = 1 \text{ kg/m}^3$
 - Atm. pressure = 1000 hPa $\hookrightarrow 100000 \text{ Pa}$
 - 2D flow

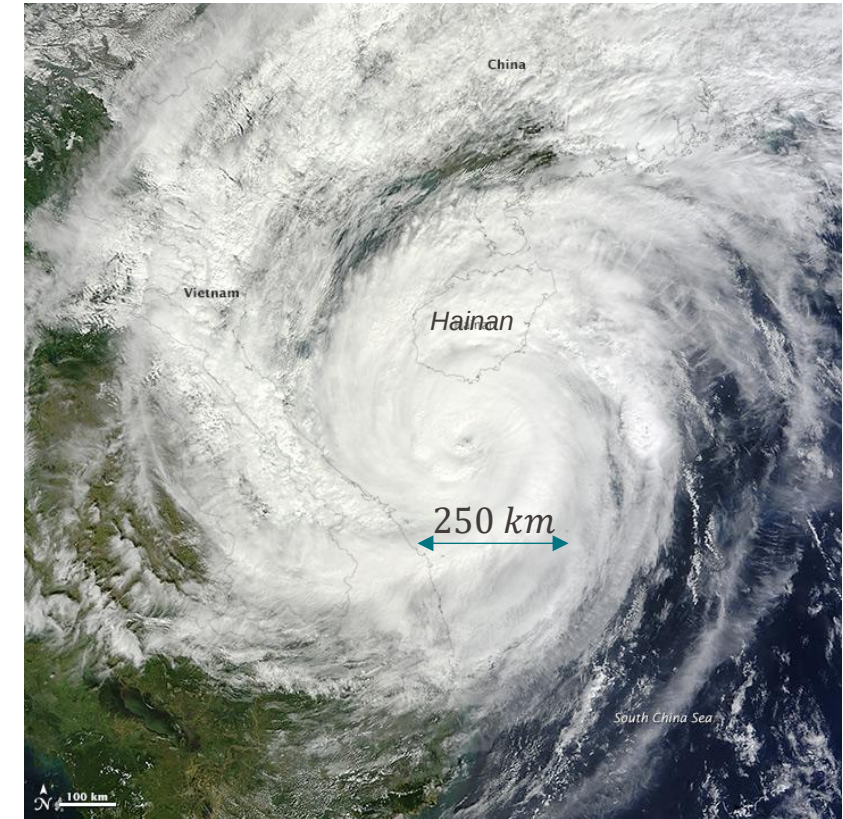
Pressure $r \leq a$

$$p_{\infty} - \frac{\rho \Gamma^2}{8\pi^2 a^2} \left(2 - \frac{r^2}{a^2} \right)$$

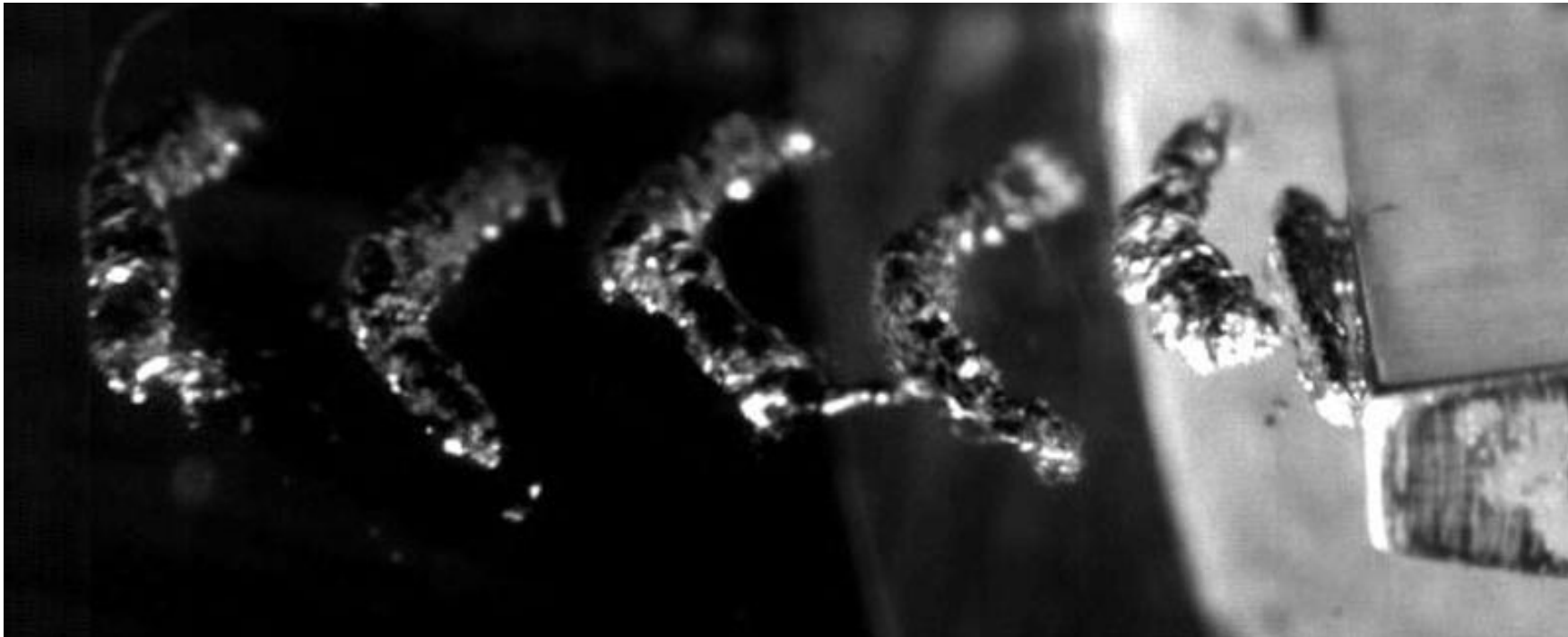
$$\text{at } r=0, \quad p = p_{\infty} - \frac{\rho \Gamma^2}{4\pi^2 a^2}$$

$$v_{\theta \text{ max}} = \frac{\Gamma}{2\pi a} = u_{\theta}^2$$

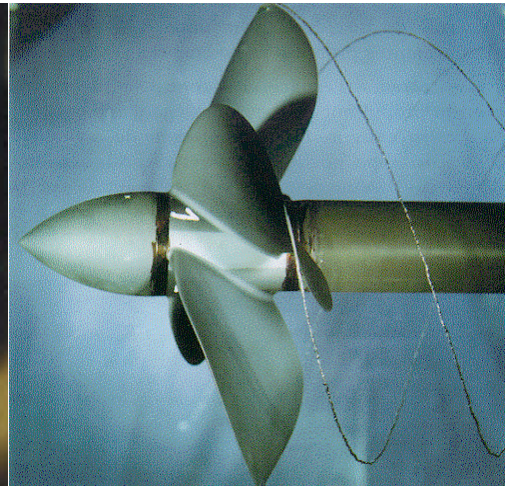
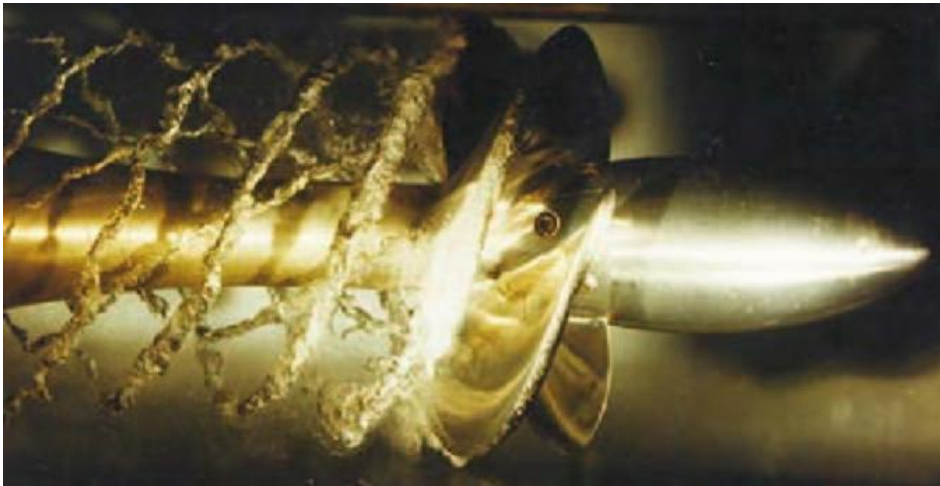
$$p(r=0) = p_{\infty} - \frac{\rho \Gamma^2}{4\pi^2 a^2} = p_{\infty} - \rho u_{\theta}^2 = 100000 - 1 \cdot 100^2 = 90000 \text{ Pa} = 900 \text{ hPa}$$



- Example of vortex flow: Karman vortices in the wake of a blunt hydrofoil
 - Flow instability in the wake → Alternate shedding of discrete vortices
 - Low pressure in the core of the vortices → Cavitation
 - Hydro-elastic coupling (Strong induced vibration):
Vortex shedding frequency $\sim$ resonance frequency of the hydrofoil (torsional mode)



- Example of vortex flow: Tip vortices in propellers
- Lifting blade with finite span (ducted or non-ducted impeller):
 - A swirl develop at the tip as the flow escapes from high to low pressure sides
 - Risk of cavitation → Source of noise and vibration



- Example of vortex flow: Rope in Francis turbines
 - The rope is a cavitating vortex that develops at the outlet of a Francis turbine
 - Due to the residual kinetic moment at part load condition
 - Source of strong hydraulic instabilities (noise and vibration)

