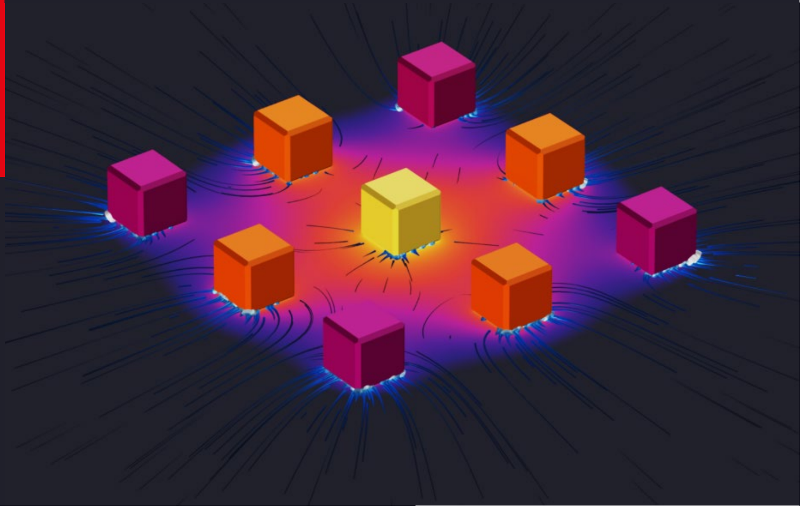
Heat and Mass Transfer ME-341

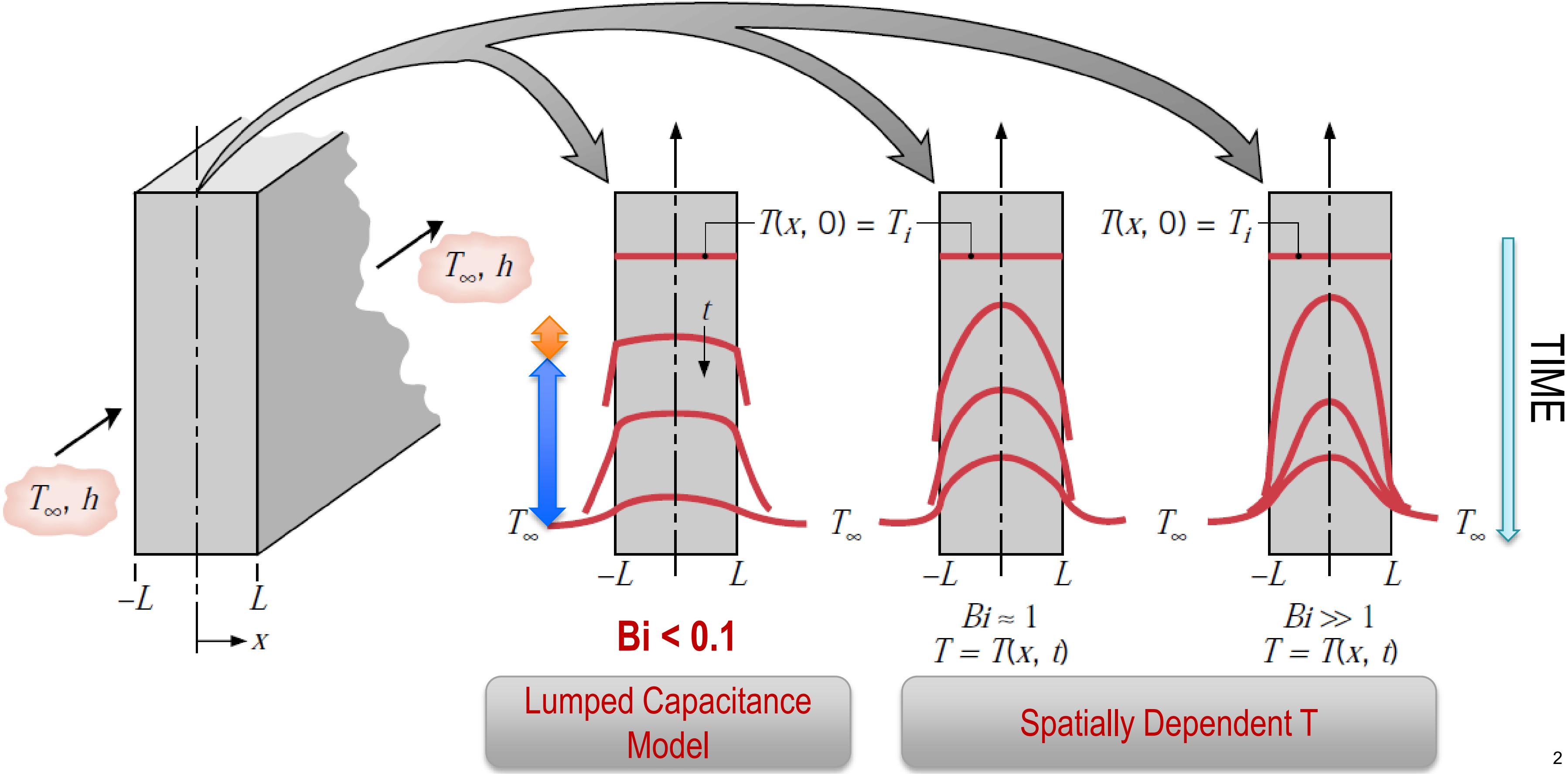
Instructor: Giulia Tagliabue



École polytechnique fédérale de Lausanne

Spring Semester

Transient Heat Conduction



Previously

- ❑ Transient Heat Diffusion
  - ✔ Generalized solution for planar/radial/spherical geometries

Learning Objectives:

- ❑ Solve transient heat conduction under convective cooling

Transient Heat Conduction – Bi>0.1, Fo>0.2

$$\theta^* = \frac{\theta}{\theta_i} = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

Planar Wall

$$\theta^* = A_1 \exp(-\lambda_1^2 Fo) \cos\left(\frac{\lambda_1 x}{L}\right)$$

$$\theta^* = \theta_0 \cos\left(\frac{\lambda_1 x}{L}\right)$$

$$\theta_0 = A_1 \exp(-\lambda_1^2 Fo)$$

θ_0= temperature evolution at the center of the slab

$$Fo = \frac{\alpha t}{L^2} \qquad Bi = \frac{hL}{k}$$

Half thickness of the layer!!

$$\frac{Q}{Q_0} = 1 - \frac{\sin \lambda_1}{\lambda_1}\theta_0$$

Cylinder

$$\theta^* = A_1 \exp(-\lambda_1^2 Fo) J_0\left(\frac{\lambda_1 r}{r_0}\right)$$

$$\theta^* = \theta_0 J_0\left(\frac{\lambda_1 r}{r_0}\right)$$

$$\theta_0 = A_1 \exp(-\lambda_1^2 Fo)$$

θ_0= temperature evolution at the center of the cylinder

$$Fo = \frac{\alpha t}{r_0^2} \qquad Bi = \frac{hr_0}{k}$$

$$\frac{Q}{Q_0} = 1 - \frac{2}{\lambda_1}\theta_0 J_1(\lambda_1)$$

Sphere

$$\theta^* = A_1 \exp(-\lambda_1^2 Fo) \frac{1}{\lambda_1 \, r/r_0} sin\left(\frac{\lambda_1 r}{r_0}\right)$$

$$\theta^* = \theta_0 \frac{1}{\lambda_1 \, r/r_0} sin\left(\frac{\lambda_1 r}{r_0}\right)$$

$$\theta_0 = A_1 \exp(-\lambda_1^2 Fo)$$

θ_0= temperature evolution at the center of the sphere

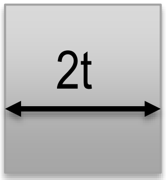
$$Fo = \frac{\alpha t}{r_0^2} \qquad Bi = \frac{hr_0}{k}$$

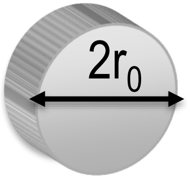
$$\frac{Q}{Q_0} = 1 - \frac{3}{\lambda_1^3}\theta_0[\sin(\lambda_1) - \lambda_1 \cos(\lambda_1)]$$

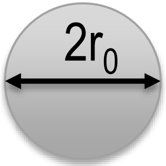
$$\frac{Q}{Q_0} = 1 - D_1 \exp(-\lambda_1^2 Fo)$$

Transient Heat Transfer

Critical dimension L and Bi number







Lumped Capacitance Model

$$\boldsymbol{L} = \boldsymbol{V}/\boldsymbol{A_s}$$

$$Bi_{Lumped} = \frac{h2t}{k}$$

$$Bi_{Lumped} = \frac{hr_0}{2k}$$

$$Bi_{Lumped} = \frac{hr_0}{3k}$$

Bi >0.1

$\boldsymbol{L}$ depends on
non-dimensional variable

$$Bi = \frac{ht}{k}$$

$$Bi = \frac{hr_0}{k}$$

$$Bi = \frac{hr_0}{k}$$

This Lecture

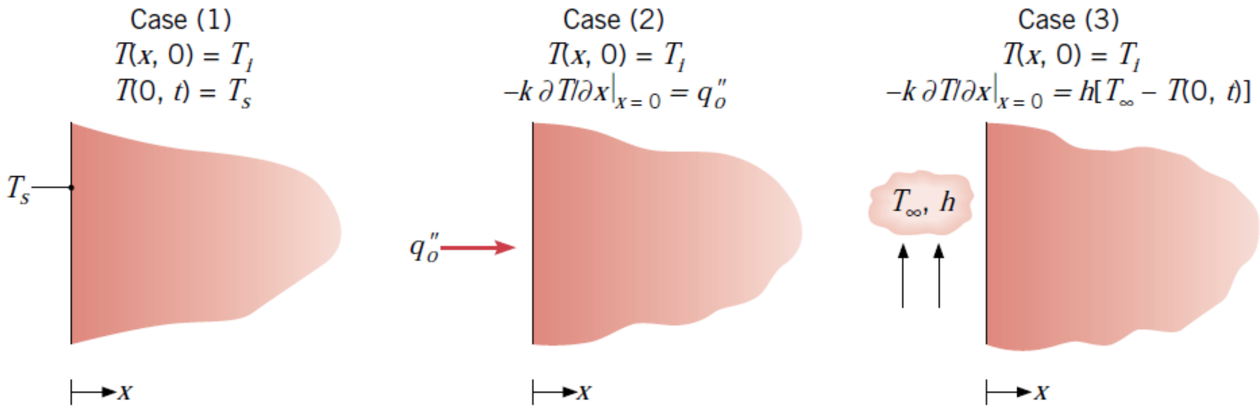
- ❑ Transient Heat Diffusion
  - ❑ Infinite solid
  - ❑ Periodic Heating

Learning Objectives:

- ❑ Solve transient heat conduction under convective cooling

Transient Heat Transfer – Semi-infinite Wall (1D)

$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x \to \infty, t) = T_i$$



Transient Heat Transfer – Semi-infinite Wall (1D)

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

We need to find a similarity variable η that transform the function of two variables, $T(x,t)$, into a single variable function $T(\eta)$

$$\eta = \frac{\zeta}{2} = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{(4\alpha t)^{1/2}}\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left[\frac{\partial T}{\partial x}\right]\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{x}{2t(4\alpha t)^{1/2}}\frac{dT}{d\eta}$$

$$\frac{1}{4\alpha t}\frac{\partial^2 T}{\partial \eta^2} = -\frac{1}{\alpha}\frac{x}{2t\sqrt{4\alpha t}}\frac{\partial T}{\partial \eta}$$

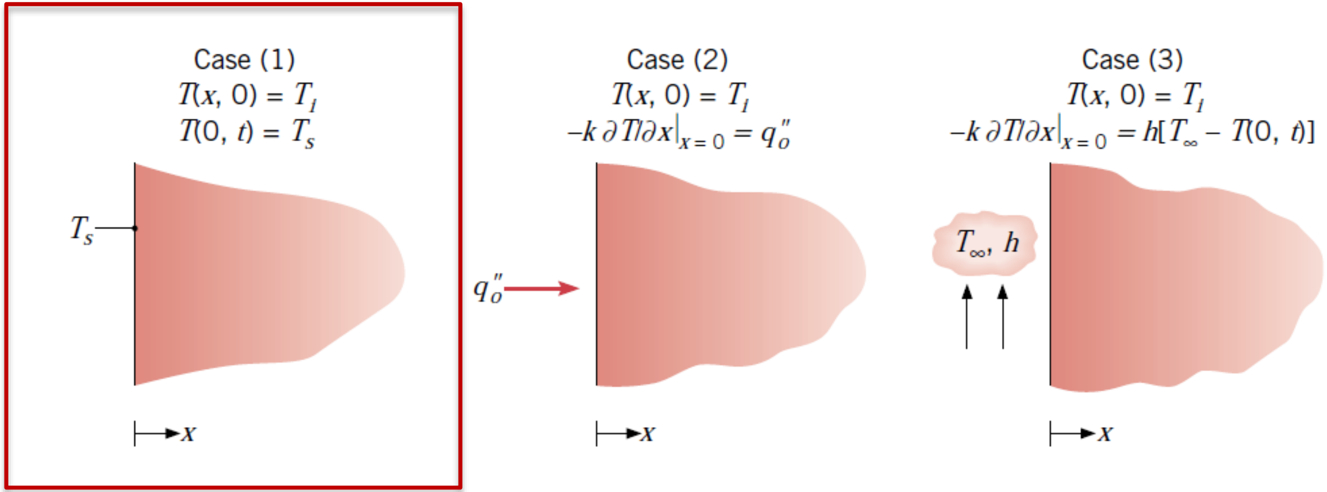
$$\boldsymbol{\frac{\partial^2 T}{\partial \eta^2} = -2\eta\frac{\partial T}{\partial \eta}}$$

$$\alpha = \frac{k}{\rho c} = thermal\ diffusivity\ [\frac{m^2}{s}]$$

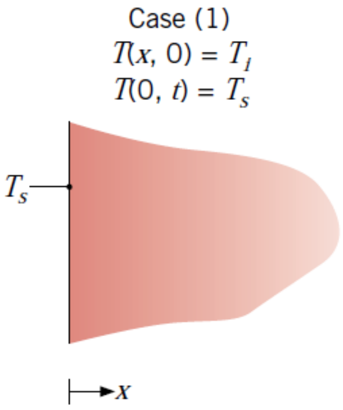
Transient Heat Transfer – Semi-infinite Wall (1D)

$$\frac{\partial^2 T}{\partial \eta^2} = -2\eta \frac{\partial T}{\partial \eta} \Rightarrow \frac{d(dT/d\eta)}{(dT/d\eta)} = -2\eta\, d\eta \Rightarrow \ln(dT/d\eta) = -\eta^2 + C_1' \Rightarrow \frac{dT}{d\eta} = C_1 \exp(-\eta^2)$$

$$\Rightarrow \quad T = C_1 \int_0^{\eta} \exp(-u^2)\, du + C_2 \qquad T(\eta \rightarrow \infty) = T_i$$



Transient Heat Transfer – Semi-infinite Wall (1D)

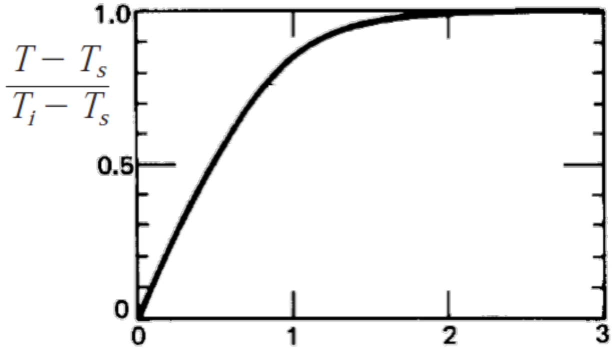


$$T = C_1 \int_0^{\eta} \exp(-u^2)\, du + C_2$$

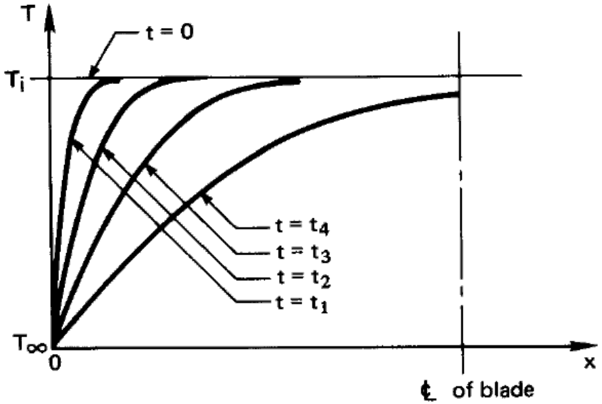
$$T(\eta = 0) = T_s \quad \Rightarrow \quad C_2 = T_s$$

$$T(\eta \to \infty) = T_i \quad \Rightarrow \quad C_1 = \frac{2(T_i - T_s)}{\pi^{1/2}}$$

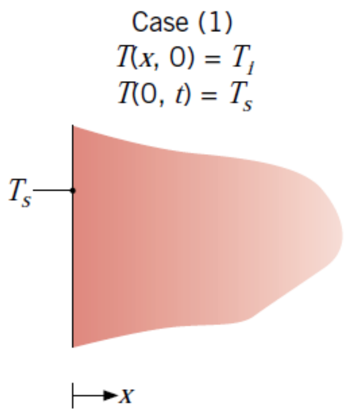
$$\frac{T - T_s}{T_i - T_s} = (2/\pi^{1/2}) \int_0^{\eta} \exp(-u^2)\, du \equiv \text{erf}\, \eta$$



$\eta = x/\sqrt{4\alpha t}$



Transient Heat Transfer – Semi-infinite Wall (1D)

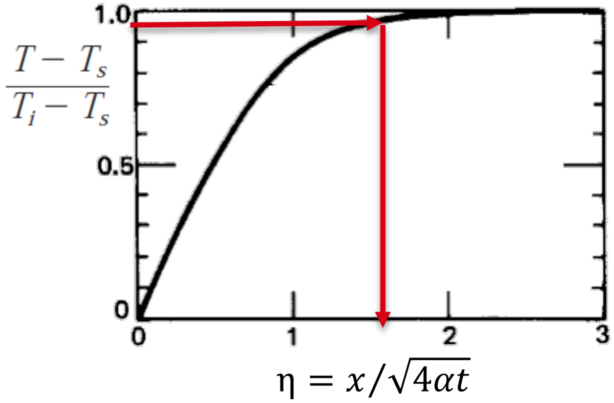


$$T = C_1 \int_0^{\eta} \exp(-u^2)\, du + C_2$$

$$T(\eta = 0) = T_s \;\Rightarrow\; C_2 = T_s$$

$$T(\eta \to \infty) = T_i \;\Rightarrow\; C_1 = \frac{2(T_i - T_s)}{\pi^{1/2}}$$

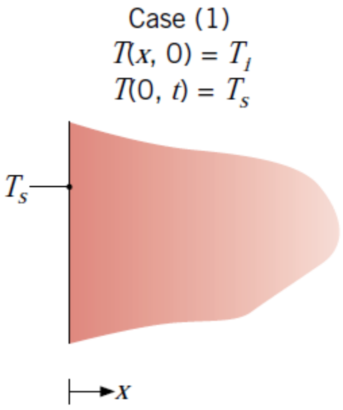
$$\Rightarrow\; \frac{T - T_s}{T_i - T_s} = (2/\pi^{1/2}) \int_0^{\eta} \exp(-u^2)\, du \equiv \operatorname{erf} \eta$$



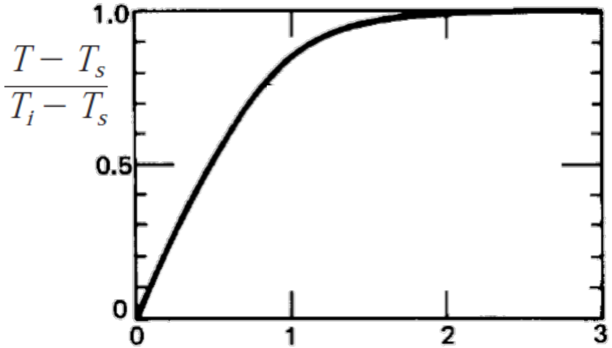
$$\frac{T - T_s}{T_i - T_s} \geq 0.99 \;\Rightarrow\; \eta = \frac{x}{\sqrt{4\alpha t}} \geq 1.8214 \;\Rightarrow\; x \geq 3.64\sqrt{\alpha t}$$

- For position in the slabs farther from the surface than $3.64\sqrt{\alpha t}$ the local temperature has changed less than 1% compared to T_s
- If an object of interest is larger than $3.64\sqrt{\alpha t}$ then up to time *t* it can be analyzed using the semi-infinite body approximation.

Transient Heat Transfer – Semi-infinite Wall (1D)



$$\frac{T - T_s}{T_i - T_s} = (2/\pi^{1/2}) \int_0^{\eta} \exp(-u^2)\, du \equiv \operatorname{erf} \eta$$



$\eta = x/\sqrt{4\alpha t}$

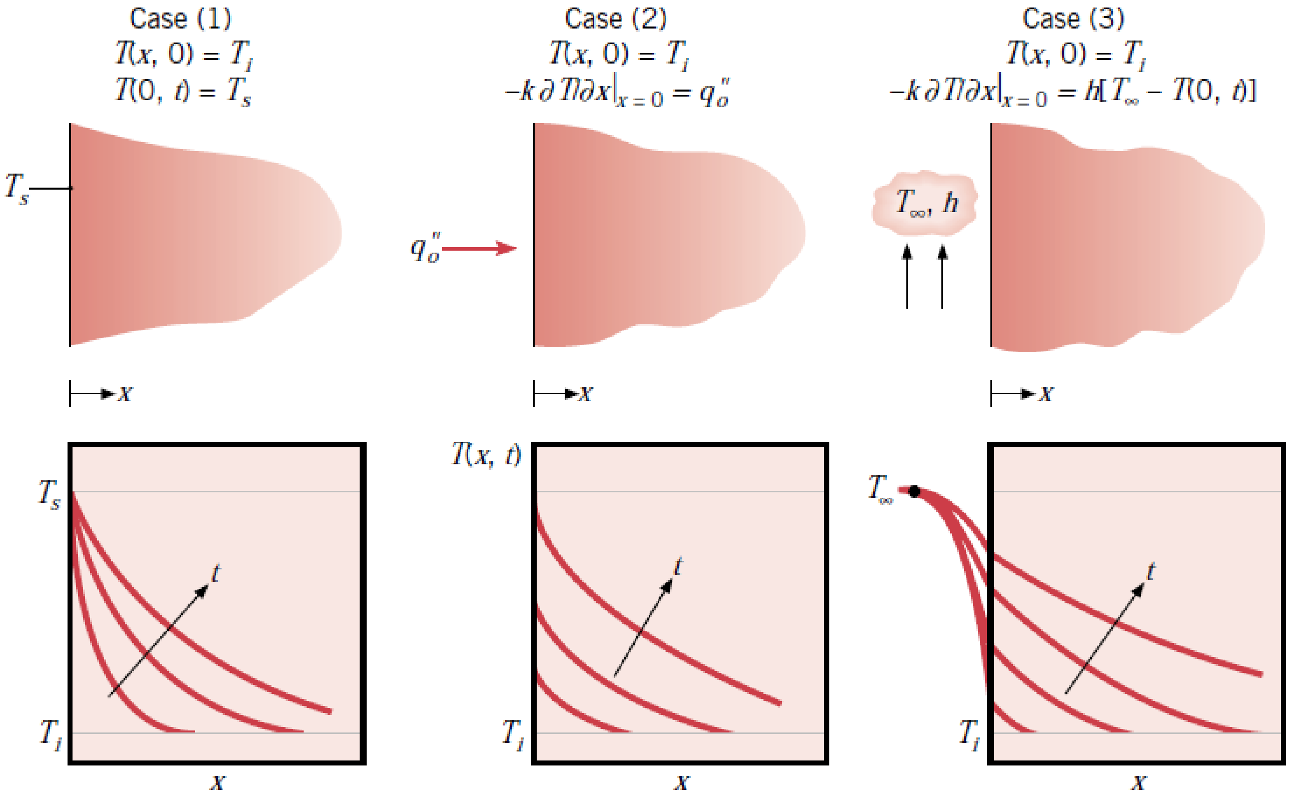
$$q_s'' = -k \frac{\partial T}{\partial x}\bigg|_{x=0} = -k(T_i - T_s) \frac{d(\operatorname{erf} \eta)}{d\eta} \frac{\partial \eta}{\partial x}\bigg|_{\eta=0}$$

$$q_s'' = k(T_s - T_i)(2/\pi^{1/2}) \exp(-\eta^2)(4\alpha t)^{-1/2}\big|_{\eta=0}$$

$$q_s'' = \frac{k(T_s - T_i)}{(\pi \alpha t)^{1/2}}$$

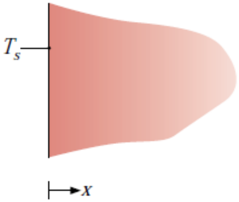
The initial heat flux is very large $\propto 1/\sqrt{t}$ but then it decreases rapidly

Transient Heat Transfer – Semi-infinite Wall (1D)



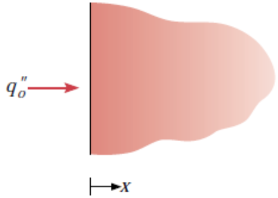
Transient Heat Transfer – Semi-infinite Wall (1D)

Case (1)
$T(x, 0) = T_i$
$T(0, t) = T_s$



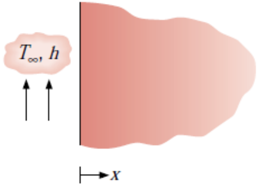
$$\frac{T(x, t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Case (2)
$T(x, 0) = T_i$
$-k\,\partial T/\partial x|_{x=0} = q_o''$



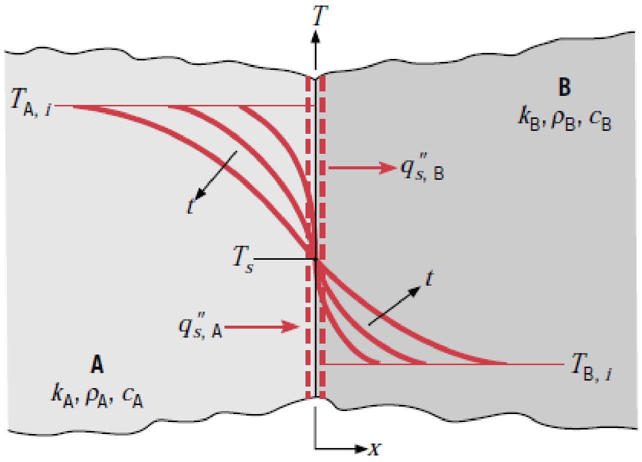
$$T(x, t) - T_i = \frac{2q_o''(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q_o'' x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Case (3)
$T(x, 0) = T_i$
$-k\,\partial T/\partial x|_{x=0} = h[T_\infty - T(0, t)]$



$$\frac{T(x, t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

Transient Heat Transfer – Semi-infinite Wall (1D)



$$q''_{s,A} = q''_{s,B}$$

$$\frac{-k_A(T_s - T_{A,i})}{(\pi\alpha_A t)^{1/2}} = \frac{k_B(T_s - T_{B,i})}{(\pi\alpha_B t)^{1/2}}$$

$$\frac{T(x,t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

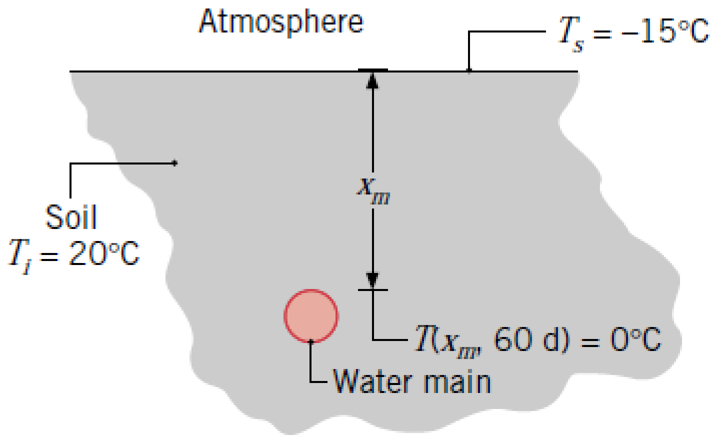
$$\Rightarrow \quad T_s = \frac{(k\rho c)_A^{1/2} T_{A,i} + (k\rho c)_B^{1/2} T_{B,i}}{(k\rho c)_A^{1/2} + (k\rho c)_B^{1/2}}$$

$$q''_s = \frac{k(T_s - T_i)}{(\pi\alpha t)^{1/2}}$$

Hence the quantity $m \equiv (k\rho c)^{1/2}$ is a weighting factor that determines whether T_s will more closely approach $T_{A,i}$ $(m_A > m_B)$ or $T_{B,i}$ $(m_B > m_A)$.

Transient Heat Transfer – Semi-infinite Wall (1D)

In laying water mains, utilities must be concerned with the possibility of freezing during cold periods. Although the problem of determining the temperature in soil as a function of time is complicated by changing surface conditions, reasonable estimates can be based on the assumption of a constant surface temperature over a prolonged period of cold weather. What minimum burial depth x_m would you recommend to avoid freezing under conditions for which soil, initially at a uniform temperature of 20°C, is subjected to a constant surface temperature of −15°C for 60 days?



Transient Heat Transfer – Semi-infinite Wall (1D)

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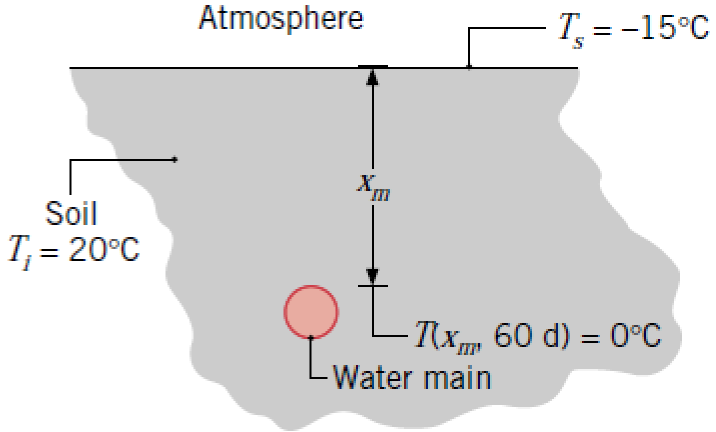
$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$\eta = \frac{\zeta}{2} = \frac{x}{\sqrt{4\alpha t}}$$



ζ/2	erf(ζ/2)	erfc(ζ/2)	ζ/2	erf(ζ/2)	erfc(ζ/2)
0.00	0.00000	1.00000	1.10	0.88021	0.11980
0.05	0.05637	0.94363	1.20	0.91031	0.08969
0.10	0.11246	0.88754	1.30	0.93401	0.06599
0.15	0.16800	0.83200	1.40	0.95229	0.04771
0.20	0.22270	0.77730	1.50	0.96611	0.03389
0.30	0.32863	0.67137	1.60	0.97635	0.02365
0.40	0.42839	0.57161	1.70	0.98379	0.01621
0.50	0.52050	0.47950	1.80	0.98909	0.01091
0.60	0.60386	0.39614	1.8214	*0.99000*	*0.01000*
0.70	0.67780	0.32220	1.90	0.99279	0.00721
0.80	0.74210	0.25790	2.00	0.99532	0.00468
0.90	0.79691	0.20309	2.50	0.99959	0.00041
1.00	0.84270	0.15730	3.00	0.99998	0.00002

$$\frac{0-(-15)}{20-(-15)} = 0.429 = \text{erf}\left(\frac{x_m}{2\sqrt{\alpha t}}\right)$$

$$\frac{x_m}{2\sqrt{\alpha t}} = 0.40$$

$$x_m = 0.80(\alpha t)^{1/2} = 0.80(0.138 \times 10^{-6}\ \text{m}^2/\text{s} \times 60\ \text{days} \times 24\ \text{h/day} \times 3600\ \text{s/h})^{1/2} = 0.68\ \text{m}$$

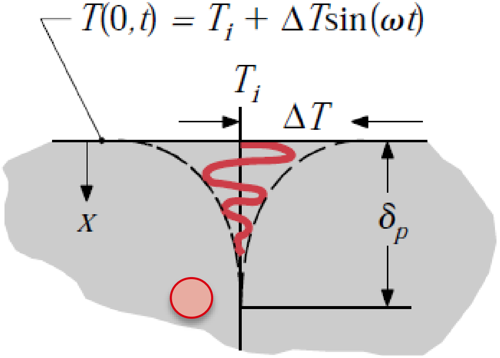
This Lecture

- ❑ Transient Heat Diffusion
  - ✓ Infinite solid
  - ❑ Periodic Heating

Learning Objectives:

- ❑ Solve transient heat conduction under convective cooling

Transient Heat Transfer – Periodic Heating



There are many situations in which the surface temperature or the heat flux change periodically (e.g. seasonal temperature variation of soil temperature, periodic heat dissipation in electronic chips)

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \Rightarrow \frac{T(x,t) - T_i}{\Delta T} = \exp[-x\sqrt{\omega/2\alpha}]\sin[\omega t - x\sqrt{\omega/2\alpha}]$$

$$exp\left(-\frac{x}{\sqrt{\omega/2\alpha}}\right) \geq 0.9 \Rightarrow x \geq 4\sqrt{\alpha/\omega}$$

The amplitude of temperature fluctuations is reduced by more than 90% for depths larger than $4\sqrt{\alpha/\omega}$

$$q_s''(t) = k\Delta T\sqrt{\omega/\alpha}\sin(\omega t + \pi/4)$$

The average heat flux over a period is zero

This Lecture

- ❑ Transient Heat Diffusion
  - ☑ Infinite solid
  - ☑ Periodic Heating

Learning Objectives:

- ☑ Solve transient heat conduction under convective cooling

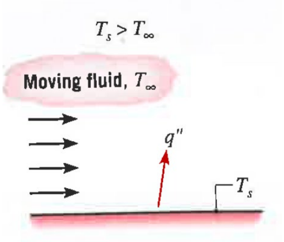
Until Now

- ☑ Heat Diffusion and Boundary Conditions (W1L2-3)
  - ☑ Steady State Heat Diffusion Equation
    - ☑ Without Heat sources (W1L3-4; W2L1)
      - ☑ Thermal Resistance & Overall Heat Transfer Coefficient
      - ☑ Bi number
      - ☑ Thermal Circuits
    - ☑ WITH Heat Sources (W2L2-3)
    - ☑ Fins and Fin Arrays (W3L1-3)
  - ☑ Transient Heat Diffusion (W4L1-3)

What's Next

convection

Part II: Newton's Law and Heat Convection

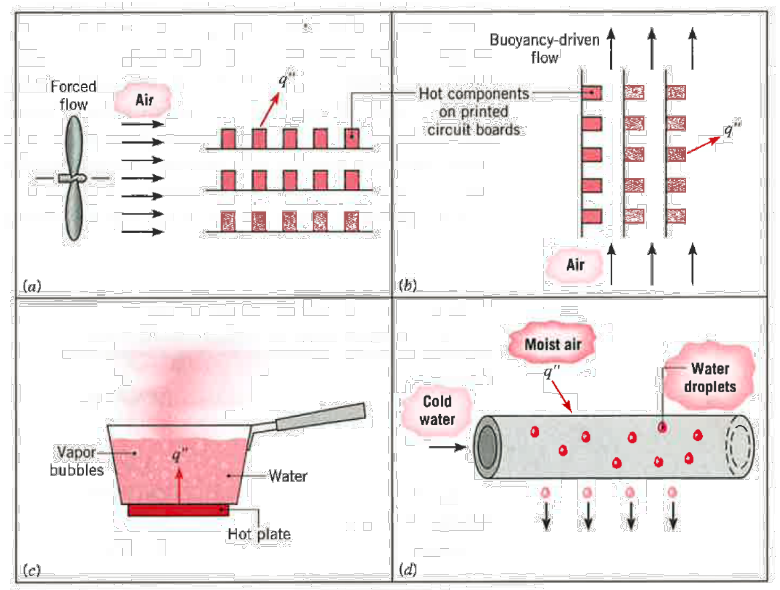


$$q'' = \bar{h}\,(T_s - T_\infty)$$

h = convective heat transfer coefficient, [W/m²K]

Forced Convection

Natural Convection



Boiling

Condensation

Transient Heat Transfer – Spatial Effects

The Semi-infinite Wall (Example)

Most of us have passed our finger through an 800°C candle flame and know that if we limit exposure to about 1/4 s we will not be burned. Why not?

Solution. The short exposure to the flame causes only a *very* superficial heating, so we consider the finger to be a semi-infinite region and go to eqn. (5.53) to calculate $(T_{burn} - T_{flame})/(T_i - T_{flame})$. It turns out that the burn threshold of human skin, T_{burn}, is about 65°C. (That is why 140°F or 60°C tap water is considered to be "scalding.") Therefore, we shall calculate how long it will take for the surface temperature of the finger to rise from body temperature (37°C) to 65°C, when it is protected by an assumed $\overline{h} \cong 100\ \text{W/m}^2\text{K}$. We shall assume that the thermal conductivity of human flesh equals that of its major component—water—and that the thermal diffusivity is equal to the known value for beef. Then

$$\Theta = \frac{65 - 800}{37 - 800} = 0.963$$

$$\frac{\overline{h}x}{k} = 0 \quad \text{since } x = 0 \text{ at the surface}$$

$$\frac{\overline{h}^2 \alpha t}{k^2} = \frac{100^2(0.135 \times 10^{-6})t}{0.63^2} = 0.0034(t\ \text{s})$$

$$0.963 = \underbrace{\text{erfc}\,0}_{=0} + e^{0.0034t}\left[\text{erfc}\left(0 + \sqrt{0.0034\,t}\right)\right]$$

$$t \cong 0.33\ \text{s}.$$

Thus, it would require about 1/3 s to bring the skin to the burn point. ■

Case 3 Surface Convection: $-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\right]\left[\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

$\zeta/2$	erf$(\zeta/2)$	erfc$(\zeta/2)$	$\zeta/2$	erf$(\zeta/2)$	erfc$(\zeta/2)$
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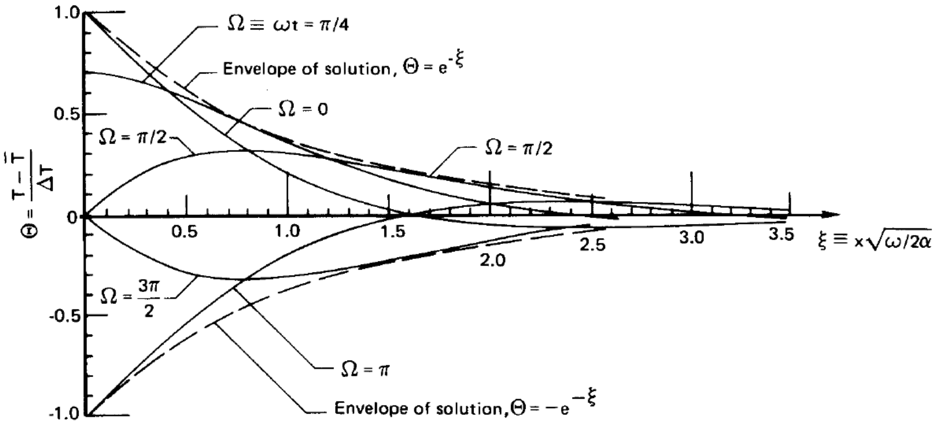
1.00	0.84270	0.15730	3.00	0.99998	0.00002

Transient Heat Transfer – Periodic Heating



How deep in the earth must we dig to find the temperature wave that was launched by the coldest part of the last winter if it is now high summer?

Solution. $\omega = 2\pi$ rad/yr, and $\Omega = \omega t = 0$ at the present. First, we must find the depths at which the $\Omega = 0$ curve reaches its local extrema. (We pick the $\Omega = 0$ curve because it gives the highest temperature at $t = 0$.)

$$\left.\frac{d\Theta}{d\xi}\right|_{\Omega=0} = -e^{-\xi}\cos(0-\xi) + e^{-\xi}\sin(0-\xi) = 0$$

This gives

$$\tan(0-\xi) = 1 \quad \text{so} \quad \xi = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$$

and the first minimum occurs where $\xi = 3\pi/4 = 2.356$, as we can see in Fig. 5.19. Thus,

$$\xi = x\sqrt{\omega/2\alpha} = 2.356$$

or, if we take $\alpha = 0.139 \times 10^{-6}$ m^2/s (given in [5.14] for coarse, gravelly earth),

$$x = 2.356 \Bigg/ \sqrt{\frac{2\pi}{2\,(0.139 \times 10^{-6})}\,\frac{1}{365(24)(3600)}} = 2.783 \text{ m}$$

If we dug in the earth, we would find it growing older and colder until it reached a maximum coldness at a depth of about 2.8 m. Farther down, it would begin to warm up again, but not much. In midwinter ($\Omega = \pi$), the reverse would be true. ■