# Heat and Mass Transfer ME-341

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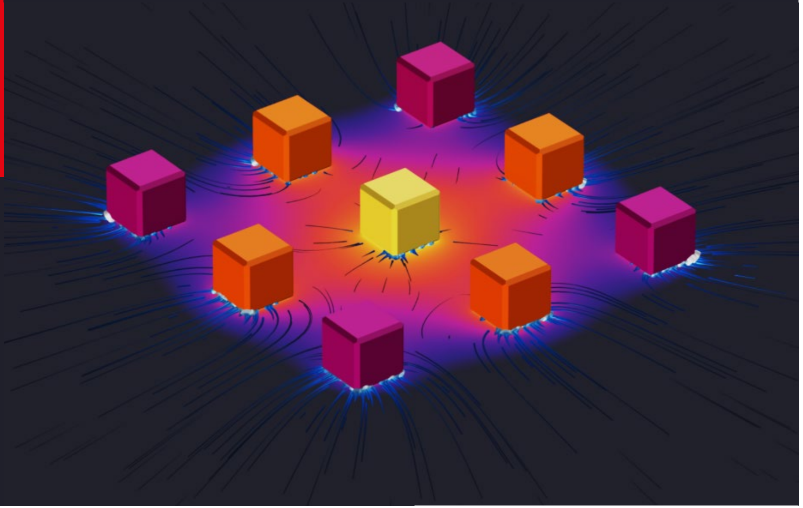


École polytechnique fédérale de Lausanne

Spring Semester

# Until Now

- ☑ Heat Diffusion and Boundary Conditions (W1L2-3)
  - ☑ Steady State Heat Diffusion Equation
    - ☑ Without Heat sources (W1L3-4; W2L1)
      - ☑ Thermal Resistance & Overall Heat Transfer Coefficient
      - ☑ Bi number
      - ☑ Thermal Circuits
    - ☑ WITH Heat Sources (W2L2-3)
    - ☑ Fins and Fin Arrays (W3L1-3)

# This Week

- Transient Heat Diffusion
  - Bi number and Spatial Effects
  - Lumped Capacitance Model
  - Generalized solution for planar/radial/spherical geometries
  - Infinite solid
  - Periodic Heating

Learning Objectives:

- Solve transient heat conduction under convective cooling

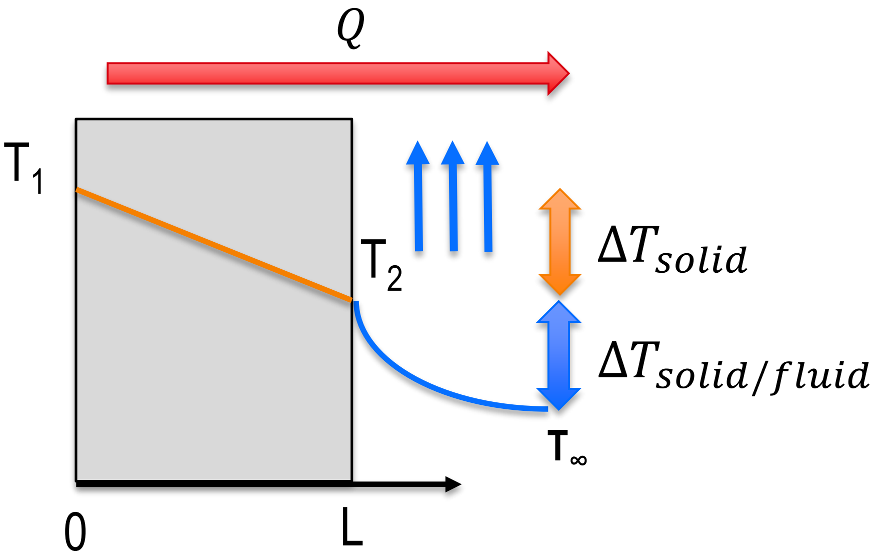
# This Lecture

- ❑ Transient Heat Diffusion
  - ❑ Bi number and Spatial Effects
  - ❑ Lumped Capacitance Model

Learning Objectives:

- ❑ Solve transient heat conduction under convective cooling

# Biot Number




$$Q = \frac{(T_1 - T_2)}{R_{th,cond}} = \frac{(T_2 - T_\infty)}{R_{th,conv}}$$

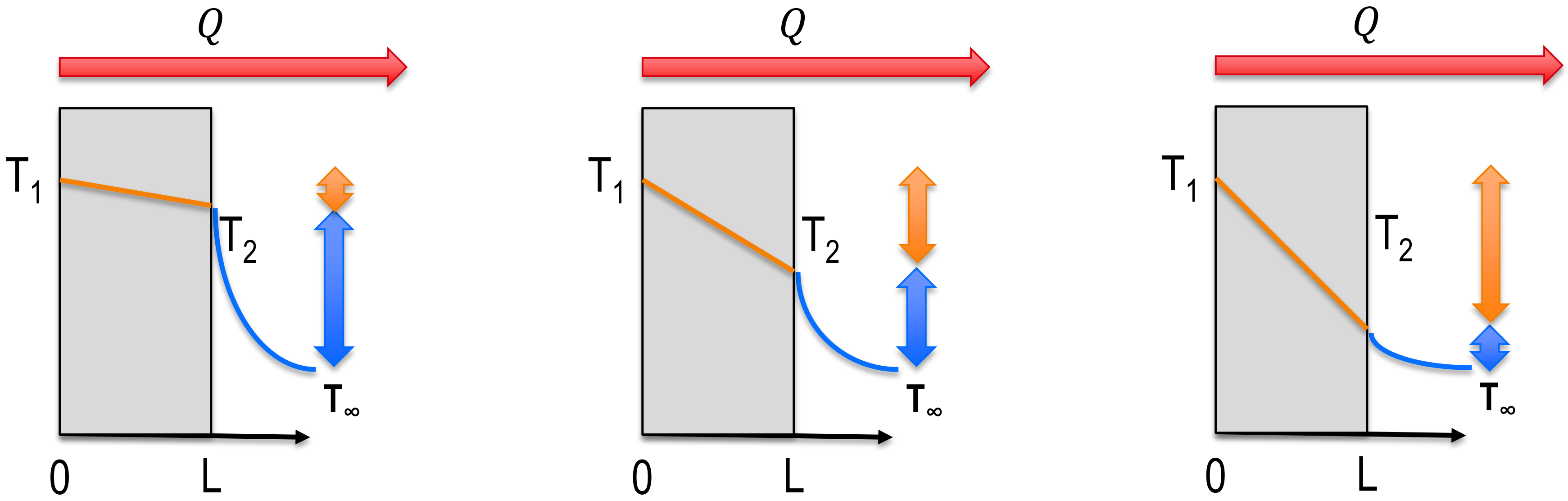
$$\Rightarrow \frac{(T_1 - T_2)}{(T_2 - T_\infty)} = \frac{R_{th,cond}}{R_{th,conv}} \equiv Bi$$

$$\Rightarrow Bi \equiv \frac{hL}{k}$$

*Note*: L can be generalized to be a characteristic dimension of a body (e.g. diameter of a sphere)

$$L = \frac{V}{A_s} = characteristic\ dimension$$

# Biot Number and Spatial Effects




$\Delta T_{solid} \ll \Delta T_{solid/fluid}$

$\Delta T_{solid} \sim \Delta T_{solid/fluid}$

$\Delta T_{solid} \gg \Delta T_{solid/fluid}$

$\boldsymbol{Bi \equiv \frac{hL}{k} \ll 1}$

$\boldsymbol{Bi \equiv \frac{hL}{k} \sim 1}$

$\boldsymbol{Bi \equiv \frac{hL}{k} \gg 1}$

***Solid has a spatially constant T***

***Solid has a spatially dependent T***

# This Lecture

- ❑ Transient Heat Diffusion
  - ✓ Bi number and Spatial Effects
  - ❑ Lumped Capacitance Model

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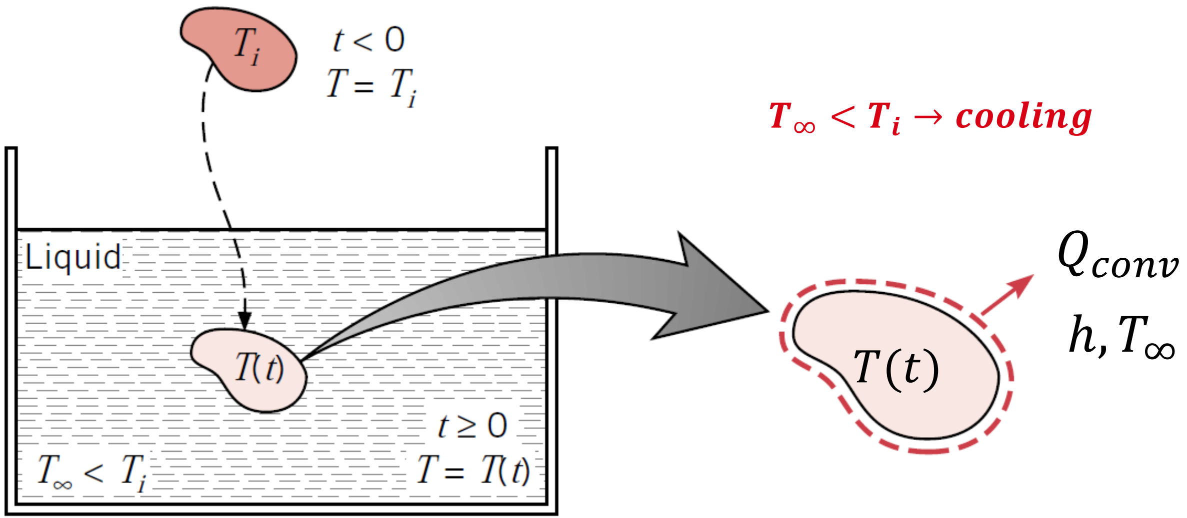
# Introduction to Transient Heat Conduction




**Assumption 1: k is high ensuring a uniform temperature $T(t)$ in the solid (Bi <0.1)**

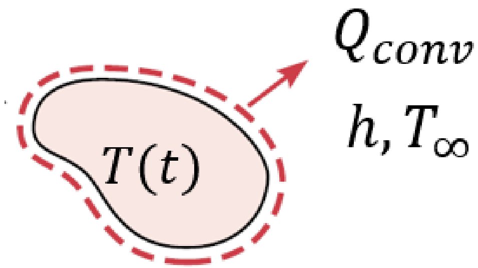
Assumption 2: the volume of liquid is so large that the liquid temperature remains constant at all times

Write the 1st law of thermodynamics for a control volume equal to the solid

# Transient Heat Transfer

Assumption 1: k is high ensuring a uniform temperature $T(t)$ in the solid **(Bi<0.1)**
Assumption 2: the volume of liquid is so large that the liquid temperature remains constant at all times

$T_\infty < T_i \rightarrow cooling$




$$Q_{conv} = hA_s(T(t) - T_\infty)$$

1[st] law: $$0 = -Q_{conv} - \cancel{W} + \cancel{E}_{gen} - mc\frac{dT}{dt}$$

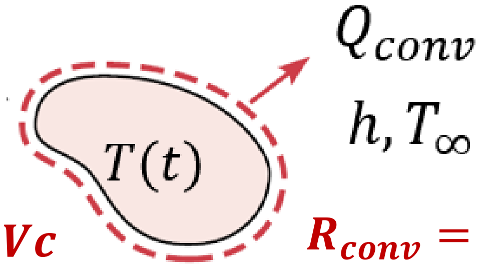
$$\rho V c\frac{dT}{dt} = -hA_s(T(t) - T_\infty)$$

$$\theta(t) = T(t) - T_\infty \qquad \frac{d\theta}{\theta} = -\frac{hA_s}{\rho V c}$$

$$\frac{\boldsymbol{\theta}}{\boldsymbol{\theta_i}} = e^{-\frac{hA_s}{\rho V c}t} \qquad T(t) = (T_i - T_\infty)e^{-\frac{hA_s}{\rho V c}t} + T_\infty$$

# Electrical Analogy

$$T_\infty < T_i \rightarrow cooling$$

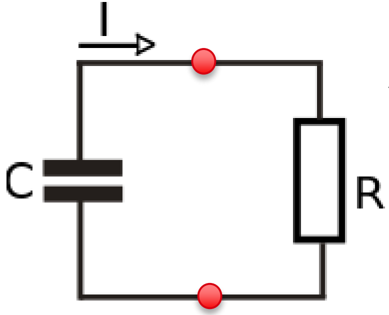



$$C_{th} = \rho V c \qquad R_{conv} = \frac{1}{hA_s}$$

$$\frac{\theta}{\theta_i} = e^{-\frac{hA_s}{\rho V c}t} = e^{-\frac{1}{R_{conv}C_{th}}t}$$

$$I(t) = C\frac{dV}{dt}$$

$V_C(t)$




$V_R(t) = RI(t)$

$$V_C(t) + V_R(t) = 0$$

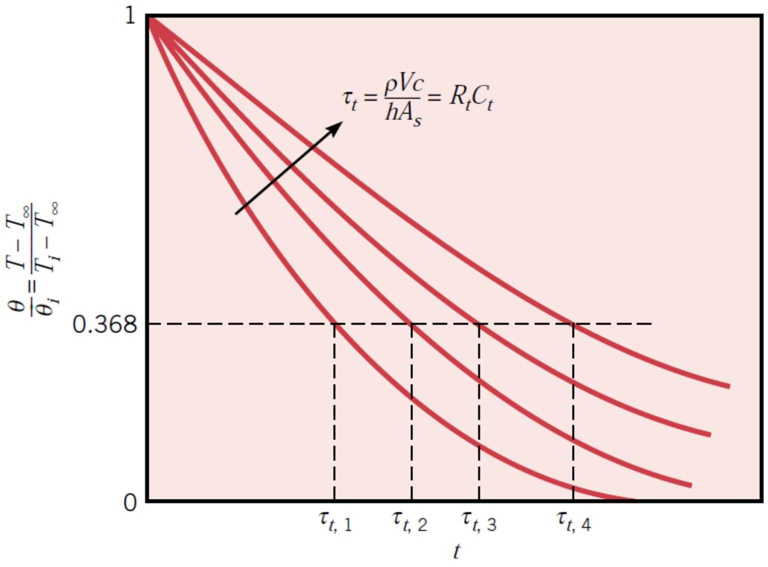
$$\frac{V}{V_i} = e^{-\frac{1}{RC}t}$$

Cooling is equivalent to the discharge of a capacitor over a resistor

**Lumped Capacitance Model**

# Lumped Capacitance Model

Assumption 1: k is high ensuring a uniform temperature $T(t)$ in the solid **(Bi<0.1)**
Assumption 2: the volume of liquid is so large that the liquid temperature remains constant at all times




$$\frac{\theta}{\theta_i} = e^{-\frac{hA_s}{\rho V c}t} = e^{-\frac{1}{R_{conv}C_{th}}t} = e^{-\frac{t}{\tau_{th}}}$$

*Total heat removed* ***[J]****:*

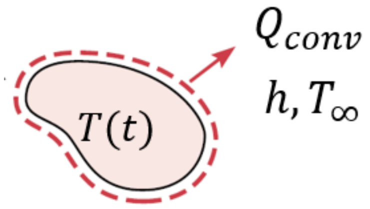
$$Q[J] = \int_0^t Q_{conv}\,dt = hA_s \int_0^t \theta\,dt$$

$$Q[J] = \rho V c \theta_i \left[1 - e^{-\frac{t}{\tau_{th}}}\right]$$

# Lumped Capacitance Model

<u>Assumption 1</u>: k is high ensuring a uniform temperature $T(t)$ in the solid **(Bi<0.1)**

<u>Assumption 2</u>: the volume of liquid is so large that the liquid temperature remains constant at all times




$$\frac{\theta}{\theta_i} = e^{-\frac{hA_s}{\rho V c}t} = e^{-\frac{1}{R_{conv}C_{th}}t} = e^{-\frac{t}{\tau_{th}}}$$

$$\frac{hA_s}{\rho V c}t = \frac{ht}{\rho c}\frac{A_s}{V} = \frac{ht}{\rho c}\frac{\mathbf{1}}{\mathbf{L}}$$

$$L = \frac{V}{A_s} = characteristic\ dimension$$

$$Bi \equiv \frac{hL}{k}$$

$$\frac{ht}{\rho c}\frac{1}{L} = \frac{hL}{k}\frac{kt}{\rho c L^2} = Bi \cdot \frac{\alpha t}{L^2} = Bi \cdot Fo$$

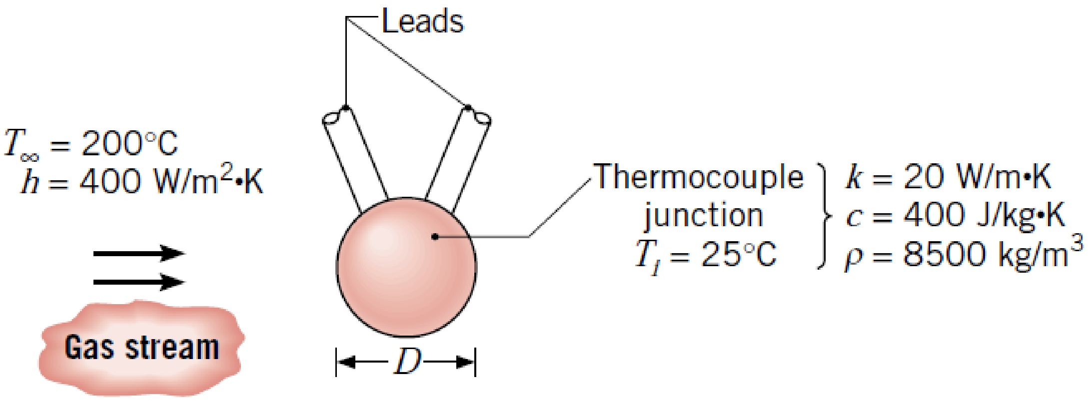
$$\boldsymbol{Fo = \frac{\alpha t}{L^2} = Fourier\ number}$$

*Dimensionless Time*

$$\frac{\theta}{\theta_i} = e^{-Bi \cdot Fo}$$

# Lumped Capacitance Model

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400\ \mathrm{W/m^2 \cdot K}$, and the junction thermophysical properties are $k = 20\ \mathrm{W/m \cdot K}$, $c = 400\ \mathrm{J/kg \cdot K}$, and $\rho = 8500\ \mathrm{kg/m^3}$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

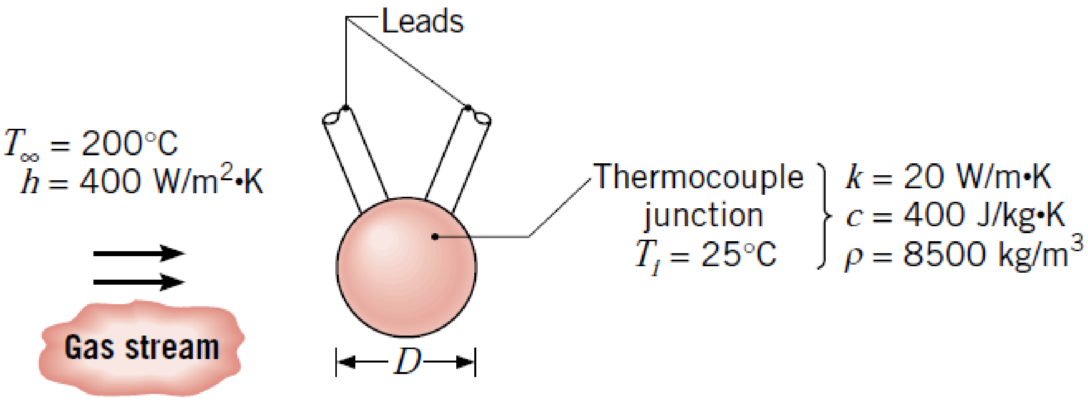



Rule #1: Calculate the Bi Number

In this case we cannot calculate Bi because we do not know the sphere diameter -> we first determine the sphere diameter under the assumption of Lumped Capacitance Model and then we verify whether the assumption is true !!!

# Lumped Capacitance Model

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400\ \mathrm{W/m^2 \cdot K}$, and the junction thermophysical properties are $k = 20\ \mathrm{W/m \cdot K}$, $c = 400\ \mathrm{J/kg \cdot K}$, and $\rho = 8500\ \mathrm{kg/m^3}$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?




$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho V c)$$

$$A_s = \pi D^2$$

$$V = \pi D^3/6$$

$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} c$$

$$D = \frac{6h\tau_t}{\rho c} = 7.06 \times 10^{-4}\ \mathrm{m}$$

$$Bi = \frac{h\left(D/6\right)}{k} = 2.35 \times 10^{-3}$$

Lumped Capacitance Model valid if:

**Bi < 0.1**

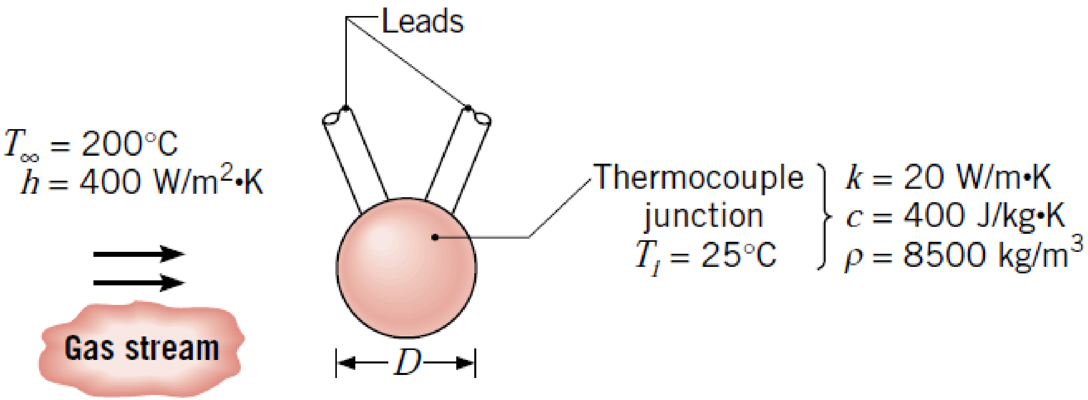
Valid assumption!!

# Lumped Capacitance Model

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400\ \mathrm{W/m^2 \cdot K}$, and the junction thermophysical properties are $k = 20\ \mathrm{W/m \cdot K}$, $c = 400\ \mathrm{J/kg \cdot K}$, and $\rho = 8500\ \mathrm{kg/m^3}$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

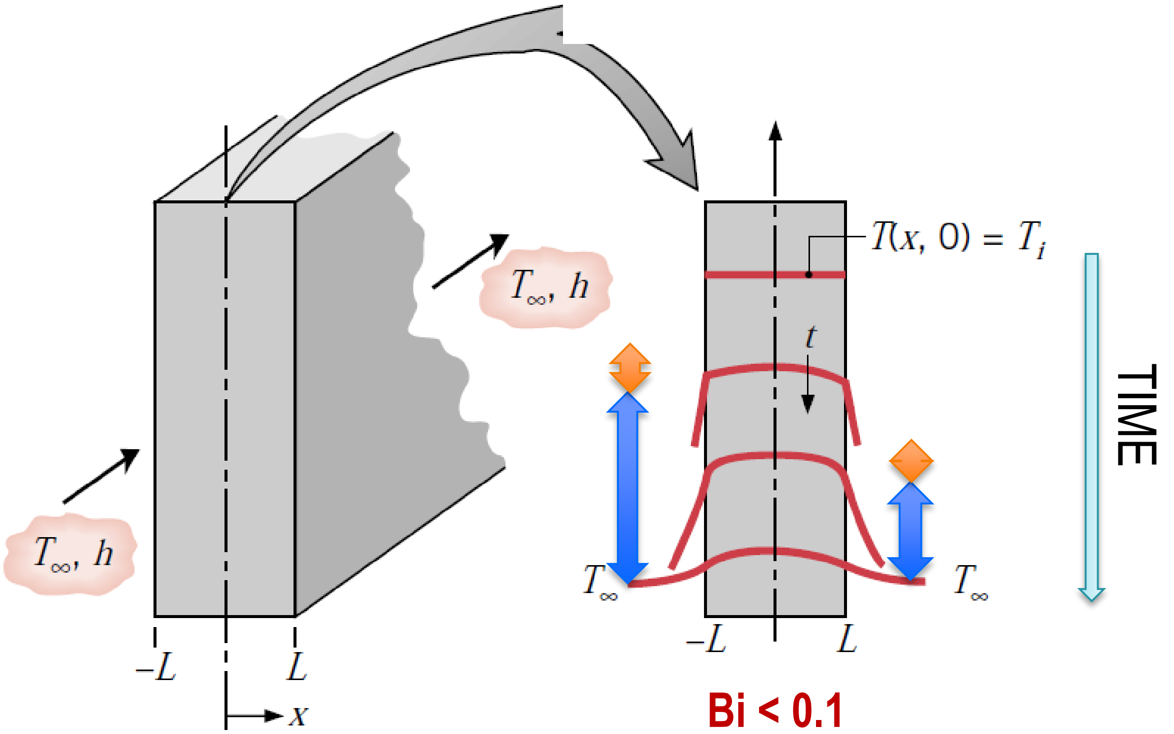
$$\frac{\theta}{\theta_i} = e^{-\frac{hA_s}{\rho V c}t}$$

$$t = \frac{\rho(\pi D^3/6)c}{h(\pi D^2)} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho D c}{6h} \ln \frac{T_i - T_\infty}{T - T_\infty}$$




$$t = 5.2\ \mathrm{s}$$

# Lumped Capacitance Model




Solid is at a ~constant T

# This Lecture

- ❑ Transient Heat Diffusion
  - ✓ Bi number and Spatial Effects
  - ✓ Lumped Capacitance Model

Learning Objectives:

- ❑ Solve transient heat conduction under convective cooling

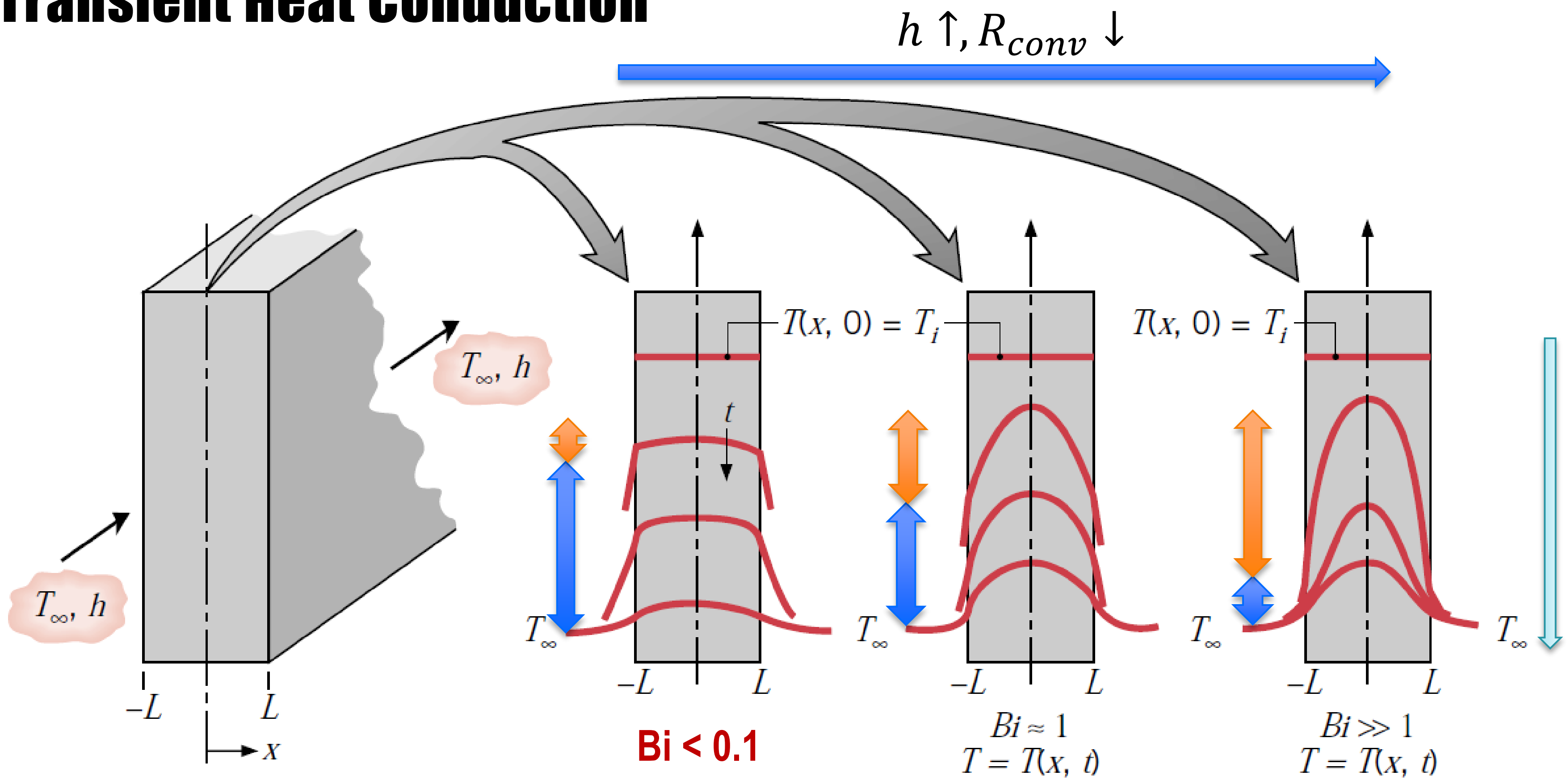
# Next Lecture

- ❑ Transient Heat Diffusion
  - ❑ Generalized solution for planar/radial/spherical geometries
  - ❑ Infinite solid
  - ❑ Periodic Heating

Learning Objectives:

- ❑ Solve transient heat conduction under convective cooling

# Transient Heat Conduction



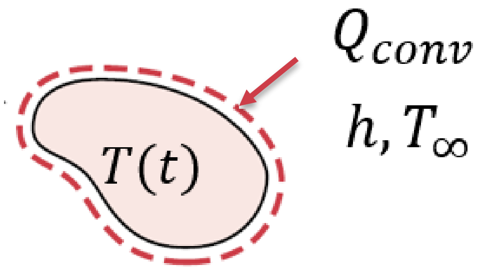

*Note: during the transient the temperature is NOT linear within the wall even if there are no heat sources*

# Supplementary Slides

# Transient Heat Transfer

Assumption 1: the solid is sufficiently small and k sufficiently high to ensure a uniform temperature $T(t)$ in the solid
Assumption 2: the volume of liquid is so large that the liquid temperature remains constant at all times

$$T_\infty > T_i \rightarrow heating$$




$$Q_{conv} = hA_s(T_\infty - T(t))$$

1st law: $$0 = +Q_{conv} - \dot{W} + \dot{E}_{gen} - mc\frac{dT}{dt}$$

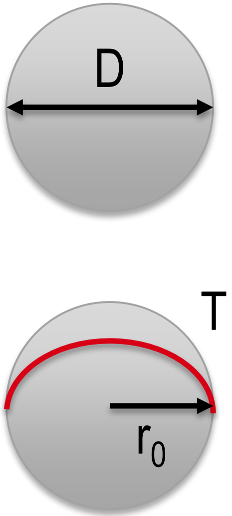
$$\rho V c \frac{dT}{dt} = +hA_s(T_\infty - T(t))$$

$$\theta(t) = T(t) - T_\infty \qquad \frac{d\theta}{\theta} = -\frac{hA_s}{\rho V c}$$

$$\frac{\boldsymbol{\theta}}{\boldsymbol{\theta_i}} = e^{-\frac{hA_s}{\rho V c}t} \qquad T(t) = -(T_\infty - T_i)e^{-\frac{hA_s}{\rho V c}t} + T_\infty$$

# Transient Heat Transfer

## *Lumped Capacitance Model – Critical dimension $L_c$*




We defined the critical dimension $L_c$ as the ratio of volume (heat capacity) and surface area (convection). However, with Bi we want to estimate how uniform the T profile is, so it is also reasonable to use as critical length the length over which the temperature profile might change (so for a sphere it would be r0).

In this course we will stick with the first definition and always use the volume/surface ratio.

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho V c) \qquad A_s = \pi D^2 \qquad V = \pi D^3/6$$

$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} c$$

$$D = \frac{6h\tau_t}{\rho c} = 7.06 \times 10^{-4}\ \mathrm{m}$$

$$Bi = \frac{h(L/6)}{k} = 2.35 \times 10^{-3}$$

# Transient Heat Transfer

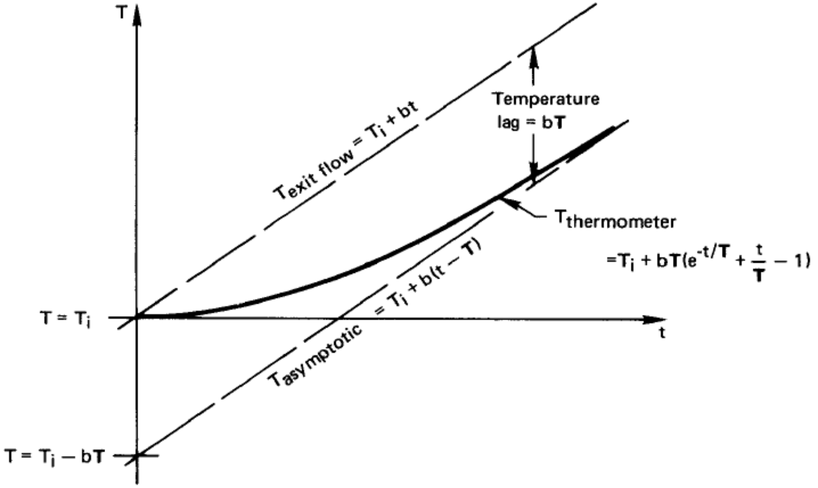
## *Lumped Capacitance Model (Example)* Assumption: $k \rightarrow \infty \equiv R_{th,cond} \rightarrow 0$

The flow rates of hot and cold water are regulated into a mixing chamber. We measure the temperature of the water as it leaves, using a thermometer with a time constant, $T$. On a particular day, the system started with cold water at $T = T_i$ in the mixing chamber. Then hot water is added in such a way that the outflow temperature rises linearly, as shown in Fig. 5.4, with $T_{\text{exit flow}} = T_i + bt$. How will the thermometer report the temperature variation?


$$T - T_i = C_1 e^{-t/T} + b(t - T)$$

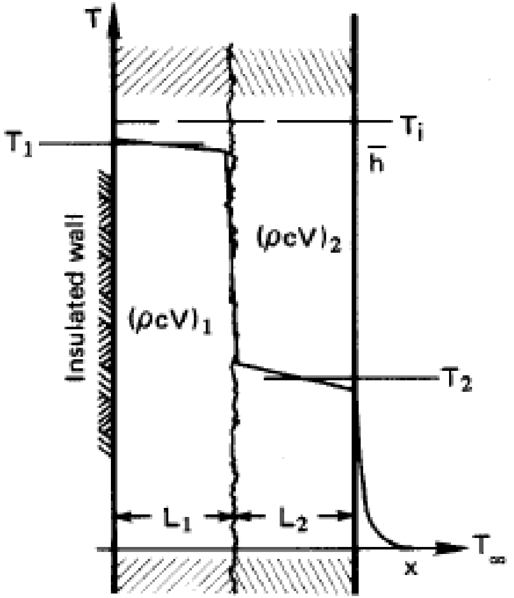
$$T - T_i = 0 \text{ at } t = 0. \qquad C_1 = bT$$

$$T - (T_i + bt) = bT\left(e^{-t/T} - 1\right)$$



# Transient Heat Transfer

***Lumped Capacitance Model*** <u>Assumption</u>: $k \to \infty \equiv R_{th,cond} \to 0$




$$\text{slab 1}: \quad -(\rho c V)_1 \frac{dT_1}{dt} = h_c A (T_1 - T_2)$$

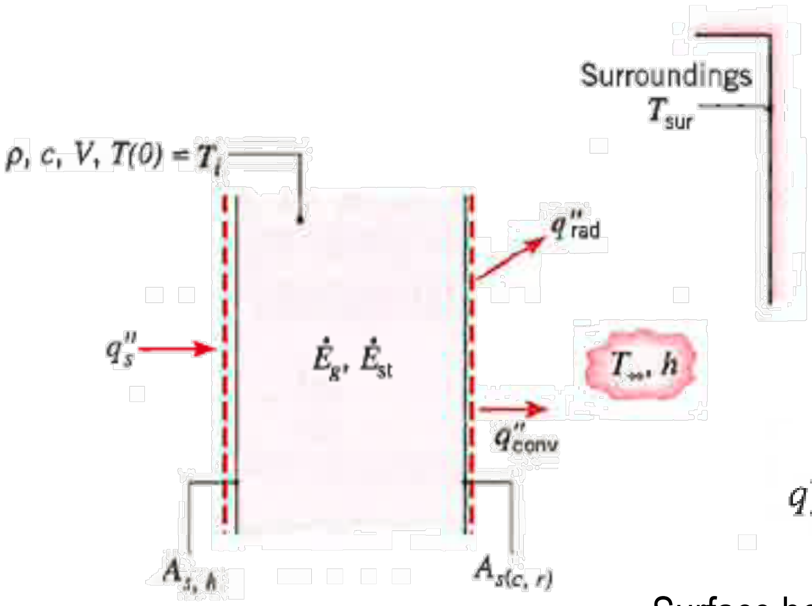
$$\text{slab 2}: \quad -(\rho c V)_2 \frac{dT_2}{dt} = \bar{h} A (T_2 - T_\infty) - h_c A (T_1 - T_2)$$

➡ *The derivation becomes much more involved*

# Transient Heat Transfer

***Lumped Capacitance Model (Bi < 0.1)***

<u>Assumption</u>: $k \to \infty \equiv R_{th,cond} \to 0$




Can you find the transient solution if only radiative heat transfer is present, without heat sources?

Volumetric heating

$$q''_s A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T^4_{sur})]A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

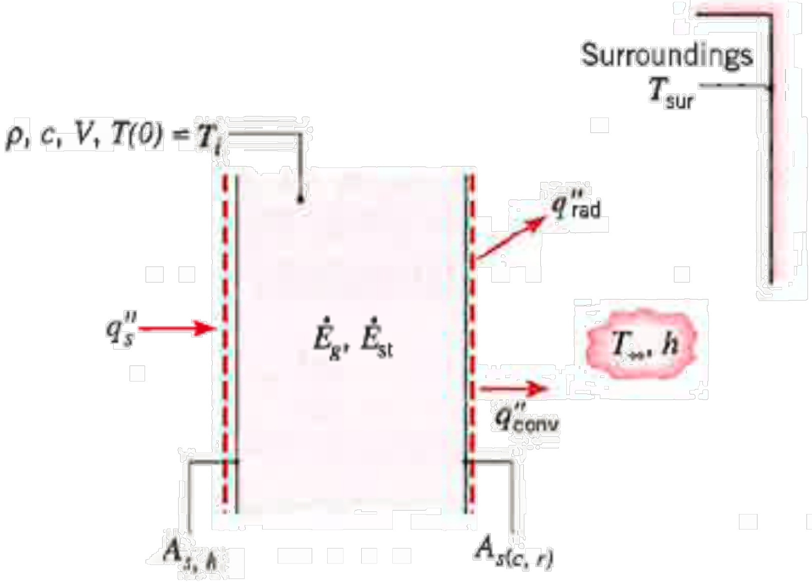
Surface heating

Convection and Radiation

# Transient Heat Transfer

## *Lumped Capacitance Model (Bi < 0.1)*

<u>Assumption</u>: $k \rightarrow \infty \equiv R_{th,cond} \rightarrow 0$




$$q''_s A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T^4_{sur})]A_{s(c,r)} = \rho V c \frac{dT}{dt} \quad (5.15)$$

Equation 5.15 is a nonlinear, first-order, nonhomogeneous, ordinary differential equation that cannot be integrated to obtain an exact solution.[1] However, exact solutions may be obtained for simplified versions of the equation. For example, if there is no imposed heat flux or generation and convection is either nonexistent (a vacuum) or negligible relative to radiation, Equation 5.15 reduces to

$$\rho V c \frac{dT}{dt} = -\varepsilon A_{s,r}\sigma(T^4 - T^4_{sur}) \quad (5.16)$$

Separating variables and integrating from the initial condition to any time $t$, it follows that

$$\frac{\varepsilon A_{s,r}\sigma}{\rho V c}\int_0^t dt = \int_{T_i}^{T} \frac{dT}{T^4_{sur} - T^4} \quad (5.17)$$

Evaluating both integrals and rearranging, the time required to reach the temperature $T$ becomes

$$t = \frac{\rho V c}{4\varepsilon A_{s,r}\sigma T^3_{sur}}\left\{\ln\left|\frac{T_{sur} + T}{T_{sur} - T}\right| - \ln\left|\frac{T_{sur} + T_i}{T_{sur} - T_i}\right| + 2\left[\tan^{-1}\left(\frac{T}{T_{sur}}\right) - \tan^{-1}\left(\frac{T_i}{T_{sur}}\right)\right]\right\} \quad (5.18)$$

This expression cannot be used to evaluate $T$ explicitly in terms of $t$, $T_i$, and $T_{sur}$, nor does it readily reduce to the limiting result for $T_{sur} = 0$ (radiation to deep space). However, returning to Equation 5.17, its solution for $T_{sur} = 0$ yields

$$t = \frac{\rho V c}{3\varepsilon A_{s,r}\sigma}\left(\frac{1}{T^3} - \frac{1}{T_i^3}\right) \quad (5.19)$$