Heat and Mass Transfer
ME-341

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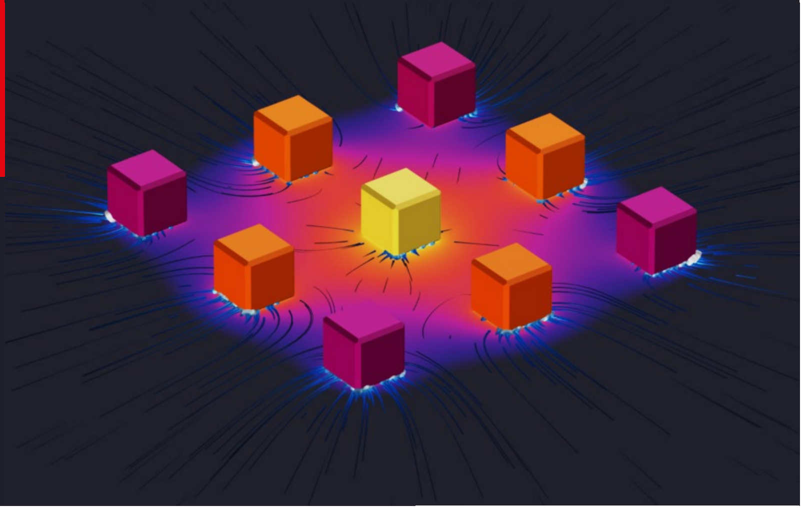

École
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Spring Semester

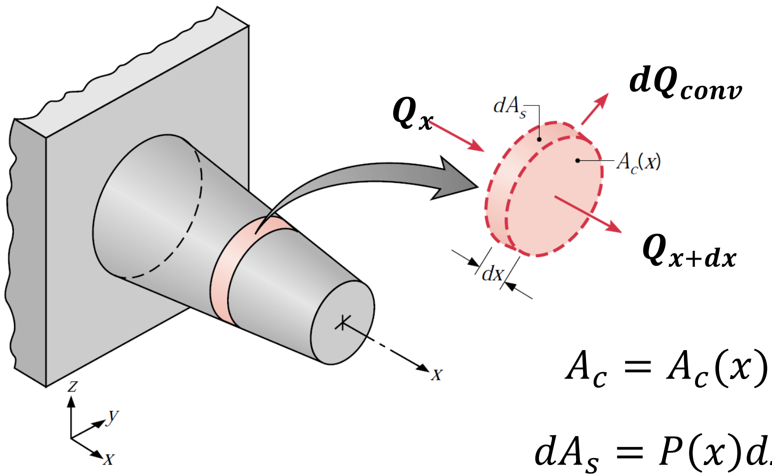
Previously

- ❑ Heat transfer from extended surfaces (Fins)
  - ☑ Boundary Conditions and Temperature Profiles

Learning Objectives:

- ☑ Calculate the temperature profile in fins with constant cross-section

Heat Transfer from Fin



1st Law of Thermodynamics (Energy Balance):

$$0 = Q_{x+dx} - Q_x + Q_{conv}$$

$$A_c = A_c(x)$$

$$dA_s = P(x)dx$$

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{h}{A_c k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

Assumption 5: constant cross-section

Assumption 1: isotropic material
Assumption 2: k independent of T
Assumption 3: steady state
Assumption 4: no heat sources

$$A_c = \text{constant}$$

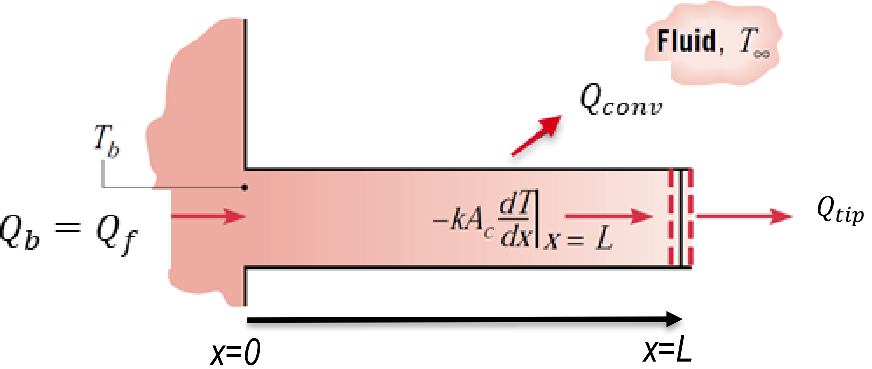
$$dA_s = P dx$$

$$\boldsymbol{\frac{d^2\theta}{dx^2} - m^2\theta = 0}$$

$$\theta = T - T_\infty$$

$$m^2 = {hP}/{kA_c}$$

Heat Transfer from Fin of Uniform Cross-Section



$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \quad \Rightarrow \quad \boldsymbol{\theta = C_1 e^{mx} + C_2 e^{-mx}}$$

Boundary conditions:

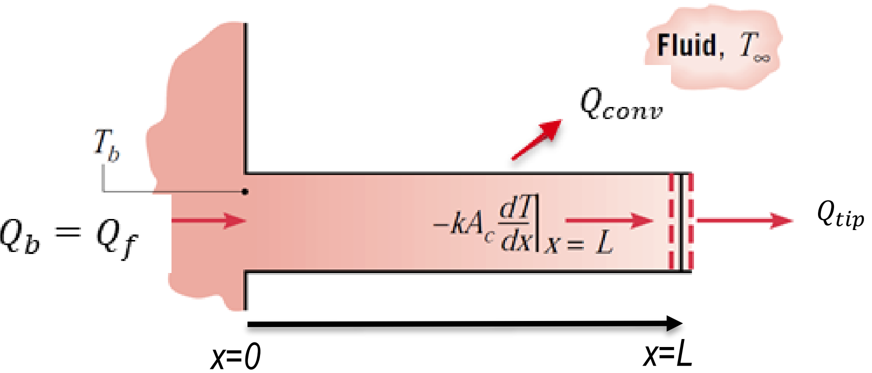
- Fin base

$$T = T_b \quad \Rightarrow \quad \theta_{x=0} = \theta_b$$

- Fin tip

1. Adiabatic $\left.\frac{d\theta}{dx}\right|_{x=L} = 0$
2. Convection $-kA_c \left.\frac{d\theta}{dx}\right|_{x=L} = hA_c(T(L) - T_\infty)$
3. Temperature $\theta_{x=L} = \theta_L$
4. Infinite fin $\theta_L \to 0$

Heat Transfer from Fin of Uniform Cross-Section



The fin was added to enable better heat transfer. So we want to know how much heat is being removed by the fin. We observe that convection on the fin can remove only as much heat as it initially enters the fin by conduction.

Therefore the total heat removed by the fin is equal to:

$$Q_{fin} = Q_{b,cond} = -kA\frac{d\theta}{dx}\Big|_{x=0}$$

Knowing the temperature profile we can calculate the total heat removed by a single fin.

Heat Transfer from Fin of Uniform Cross-Section - Recap

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate Q_f
A	Convection heat transfer: $h\theta(L) = -k\,d\theta/dx\rvert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.70)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ (3.72)
B	Adiabatic $d\theta/dx\rvert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$ (3.75)	$M\tanh mL$ (3.76)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.77)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ (3.78)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx} (3.79)	M (3.80)

$\theta \equiv T - T_\infty$ $m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $M \equiv \sqrt{hPkA_c}\,\theta_b$

This Lecture

- ❑ Fins Performance
- ❑ Array of Fins

Learning Objectives:

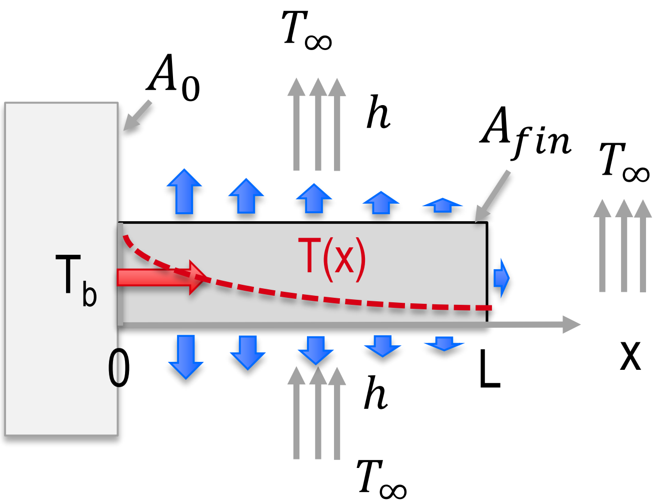
- ❑ Calculate the performance of a fin-based system

Fin Performance – Fin Efficiency

Increased heat transfer proportional to A

$$Q \sim hA\,(T_s - T_\infty) \propto A = A_0 + A_{fin}$$

Increased amount of material / increased conduction resistance /
Decreased temperature along the fin



$$\eta_f \equiv \frac{Q_f}{Q_{f,max}} = \frac{Q_f}{hA_f(T_b - T_\infty)}$$

← Heat transfer rate with fin

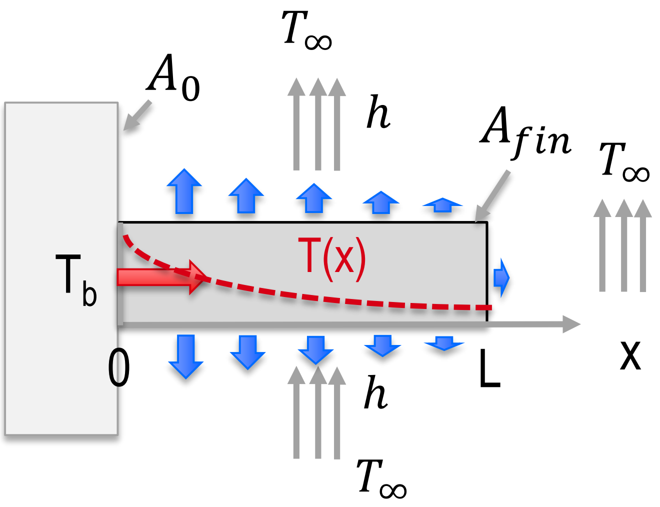
← Convection on all the fin at T_0

$$Q_f = h(\eta_f A_f)(T_b - T_\infty) = h\boldsymbol{A_f^{red}}(T_b - T_\infty)$$

Due to conduction resistance, the additional heat removed is equivalent to an ideal fin of smaller surface area $A_f^{red} < A_f$.

Fin Performance – Fin Efficiency

$$\eta_f \equiv \frac{Q_f}{Q_{f,max}} = \frac{Q_f}{hA_f(T_b - T_\infty)}$$



$$Q_f = h(\eta_f A_f)(T_b - T_\infty) = hA_f^{red}(T_b - T_\infty)$$

For an adiabatic tip:

$$\theta = \frac{\cosh m(L-x)}{\cosh mL}$$

$$Q_f = M \tanh mL$$

$$\eta_f \equiv \frac{Q_f}{hA_f\theta_b} = \frac{M tanh(mL)}{hPL\theta_b}$$

$$L \to 0 \qquad M tanh(mL) \to mL$$

$$\eta_f \to 1$$

Efficiency is not a good metric to evaluate the performance of a fin but is useful to estimate Q_f for complex fins

Fin Performance – Fin Efficiency for Non-uniform Cross-section

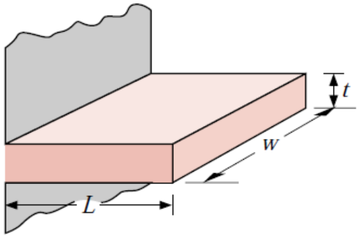
Straight Fins

Rectangular[a]

$A_f = 2wL_c$

$L_c = L + (t/2)$

$A_p = tL$

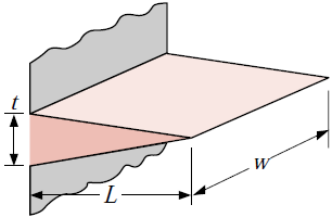


$$\eta_f = \frac{\tanh mL_c}{mL_c} \qquad (3.89)$$

Triangular[a]

$A_f = 2w[L^2 + (t/2)^2]^{1/2}$

$A_p = (t/2)L$



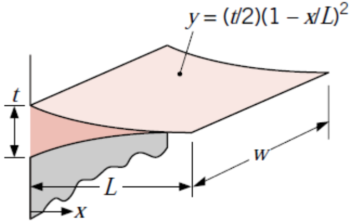
$$\eta_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)} \qquad (3.93)$$

Parabolic[a]

$A_f = w[C_1 L + (L^2/t)\ln(t/L + C_1)]$

$C_1 = [1 + (t/L)^2]^{1/2}$

$A_p = (t/3)L$



$$\eta_f = \frac{2}{[4(mL)^2 + 1]^{1/2} + 1} \qquad (3.94)$$

Fin Performance – Fin Efficiency for Non-uniform Cross-section

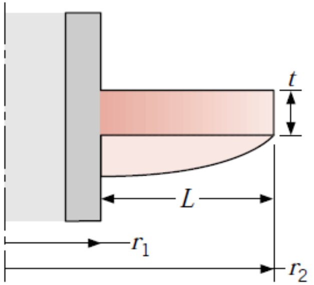
Circular Fin

Rectangular[a]

$A_f = 2\pi\,(r_{2c}^2 - r_1^2)$

$r_{2c} = r_2 + (t/2)$

$V = \pi\,(r_2^2 - r_1^2)t$



$$\eta_f = C_2 \frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})} \qquad (3.91)$$

$$C_2 = \frac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$$

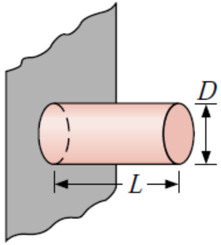
Pin Fins

Rectangular[b]

$A_f = \pi D L_c$

$L_c = L + (D/4)$

$V = (\pi D^2/4)L$



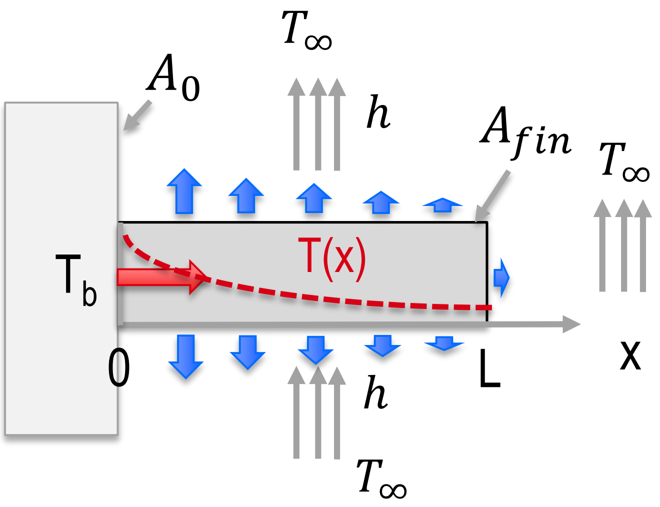
$$\eta_f = \frac{\tanh mL_c}{mL_c} \qquad (3.95)$$

Fin Performance – Fin Effectiveness

Heat transfer rate with fin

$$\varepsilon_f \equiv \frac{Q_f}{hA_{c,b}(T_b - T_\infty)}$$

Heat transfer rate without fin (base surface)



***For an infinitely long fin*:**

$$\theta = e^{-mx}$$

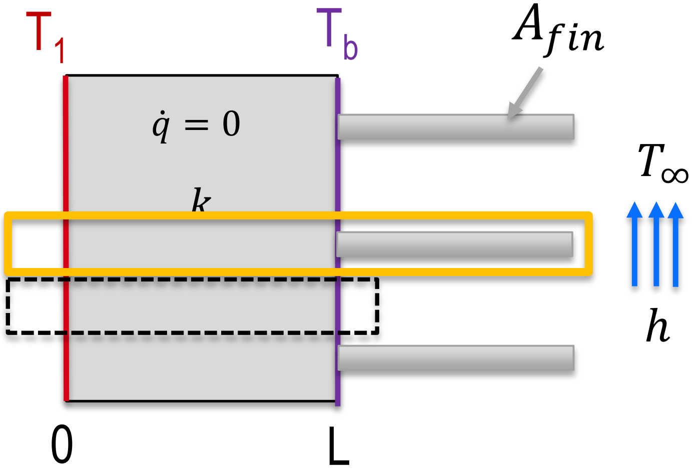
$$Q_f = M$$

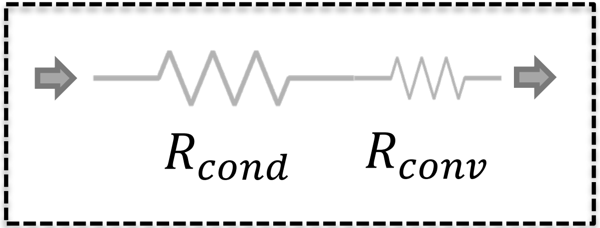
$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}$$

Effectiveness is increased if:

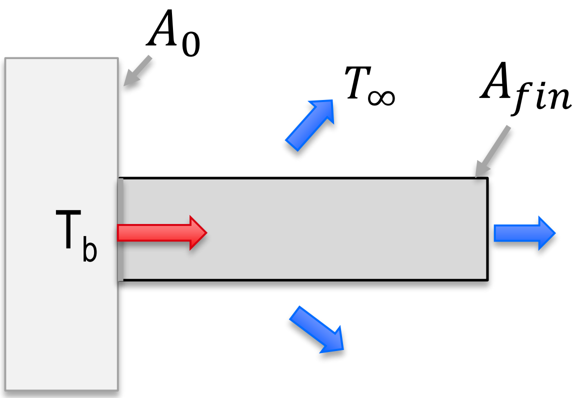
- k increases (high conductivity material)
- h decreases (used for cases with poor convection, i.e. gas flow, natural convection)

Fin Performance – Fin Resistance





$$Q_f = h(\eta_f A_f)(T_b - T_\infty)$$



Temperature difference at the base

$$R_f \equiv \frac{(T_b - T_\infty)}{Q_f} = \frac{1}{h(\eta_f A_f)}$$

Heat transfer rate with fin

Fin Performance

$$R_f \equiv \frac{(T_b - T_\infty)}{Q_f} = \frac{1}{h(\eta_f A_f)}$$

$$\varepsilon_f \equiv \frac{Q_f}{hA_{c,b}(T_b - T_\infty)}$$

$$\varepsilon_f = \frac{1}{R_f h A_{c,b}} = \frac{1/hA_{c,b}}{R_f} = \frac{\boldsymbol{R_{conv,b}}}{\boldsymbol{R_f}} = \frac{1/hA_{c,b}}{1/h(\eta_f A_f)}$$

To be useful the fin must have a combined thermal resistance R_f that is lower than the convection resistance of the base, $R_{conv,b}$.

This Lecture

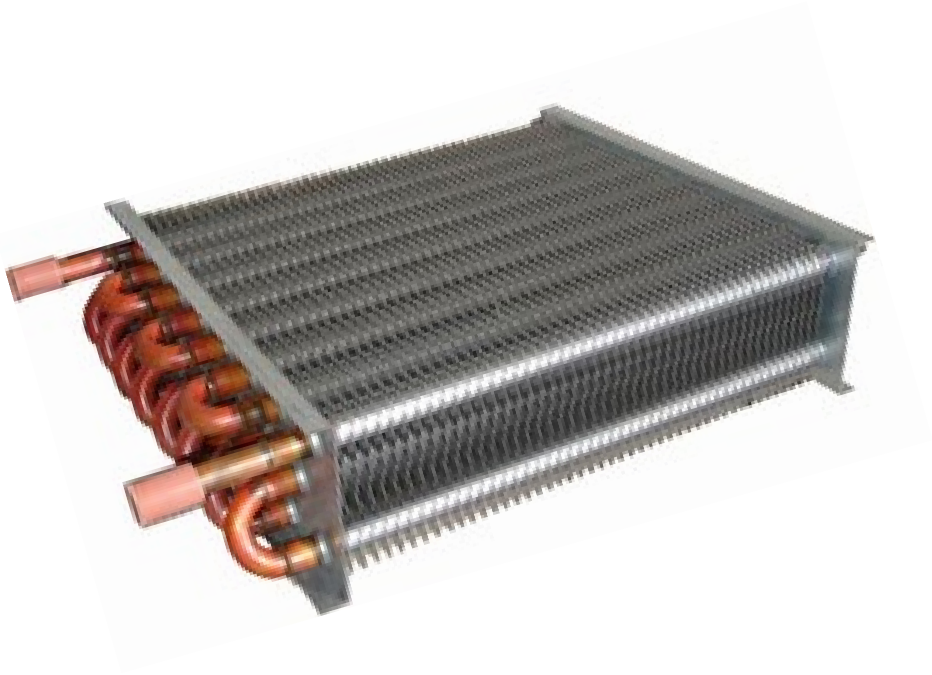
- [x] Fins Performance
- [ ] Array of Fins

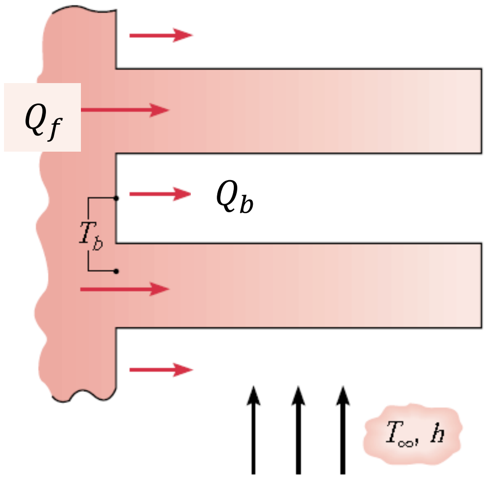
Learning Objectives:

- [ ] Calculate the performance of a fin-based system

Fin Arrays

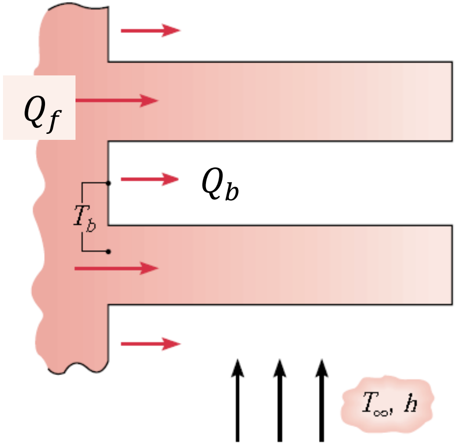
In most practical cases we utilize an array of fins rather than a single fin.

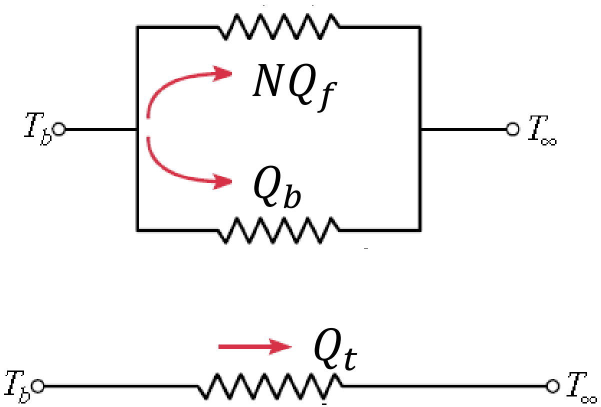




Fin Arrays – Thermal Circuit Representation

In most practical cases we utilize an array of fins rather than a single fin. Assuming that the base of the fin has the same temperature as the exposed surface.





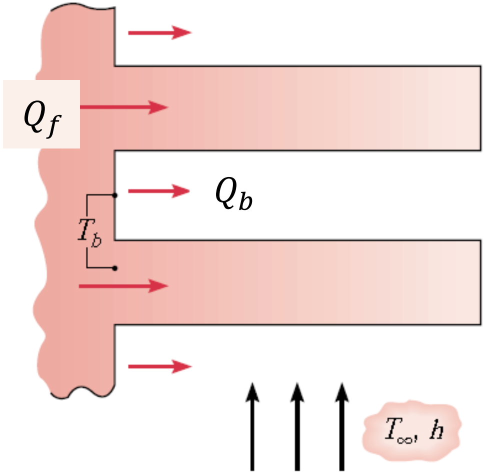
$$Q_b = hA_b(T_b - T_\infty) \qquad A_b = exposed\ surface\ without\ fins$$

$$Q_{Nf} = NQ_f = Nh(\eta_f A_f)(T_b - T_\infty) \qquad R_{Nf} \equiv \frac{(T_b - T_\infty)}{Q_{Nf}} = \frac{1}{Nh(\eta_f A_f)}$$

Fin Arrays - Performance

In most practical cases we utilize an array of fins rather than a single fin.

Both for the heat transfer rate and the areas we have to account for the N fins and the base



$$\eta_t \equiv \frac{Q_t}{Q_{t,max}} = \frac{Q_t}{hA_t(T_b - T_\infty)} = \frac{NQ_f + Q_b}{h(NA_f + A_b)(T_b - T_\infty)}$$

$$\eta_t \equiv \frac{N\eta_f A_f h(T_b - T_\infty) + hA_b(T_b - T_\infty)}{h(NA_f + A_b)(T_b - T_\infty)} = \frac{N\eta_f A_f + A_b}{NA_f + A_b}$$

$$\eta_t \equiv 1 - \frac{NA_f}{A_t}(1 - \eta_f) \quad \Rightarrow \quad R_t \equiv \frac{(T_0 - T_\infty)}{Q_t} = \frac{1}{\eta_t h A_t}$$

This Lecture

- ☑ Fins Performance
- ☑ Array of Fins

Learning Objectives:

- ☑ Calculate the performance of a fin-based system

Until Now

- ☑ Heat Diffusion and Boundary Conditions (W1L2-3)
  - ☑ Steady State Heat Diffusion Equation without Heat sources (W1L3-4; W2L1)
    - ☑ Thermal Resistance & Overall Heat Transfer Coefficient
    - ☑ Bi number
    - ☑ Thermal Circuits
  - ☑ Steady-State Heat Diffusion WITH Heat Sources (W2L2-3)
  - ☑ Fins and Fin Arrays (W3L1-3)

Next Lectures

- Transient Heat Diffusion
  - Bi number and Spatial Effects
  - Lumped Capacitance Model
  - Generalized solution for planar/radial/spherical geometries
  - Infinite solid
  - Periodic Heating