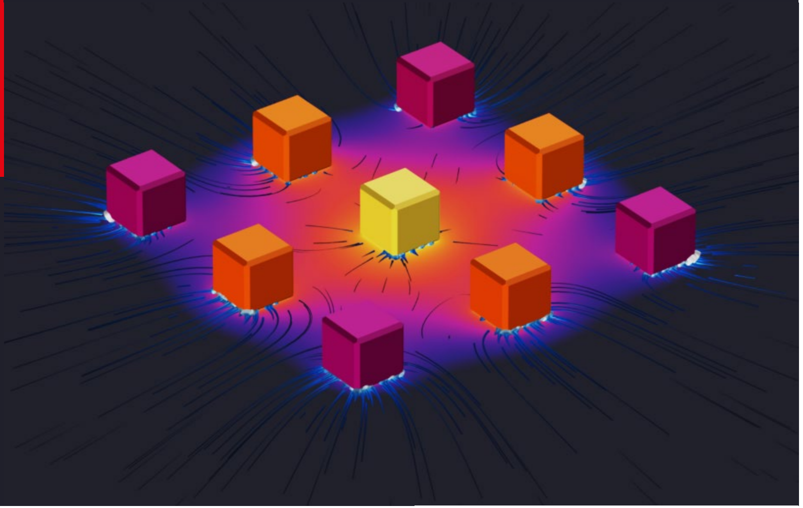
Heat and Mass Transfer
ME-341

Instructor: Giulia Tagliabue



École
polytechnique
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de Lausanne

Spring Semester

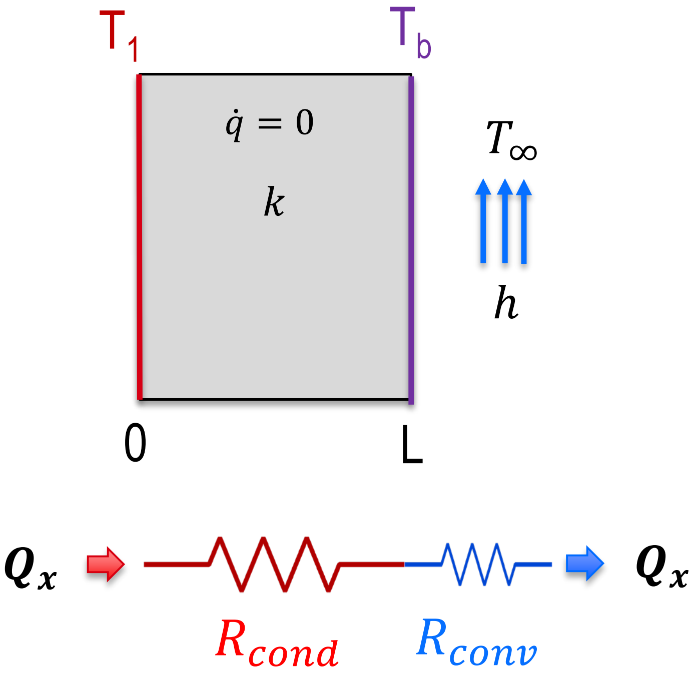
Previously

☑ Heat transfer from extended surfaces (Fins)

Learning Objectives:

☑ Understand the concept of fins

Heat Transfer from Extended Surfaces



$$Q_x = \frac{T_1 - T_\infty}{R_{cond} + R_{conv}}$$

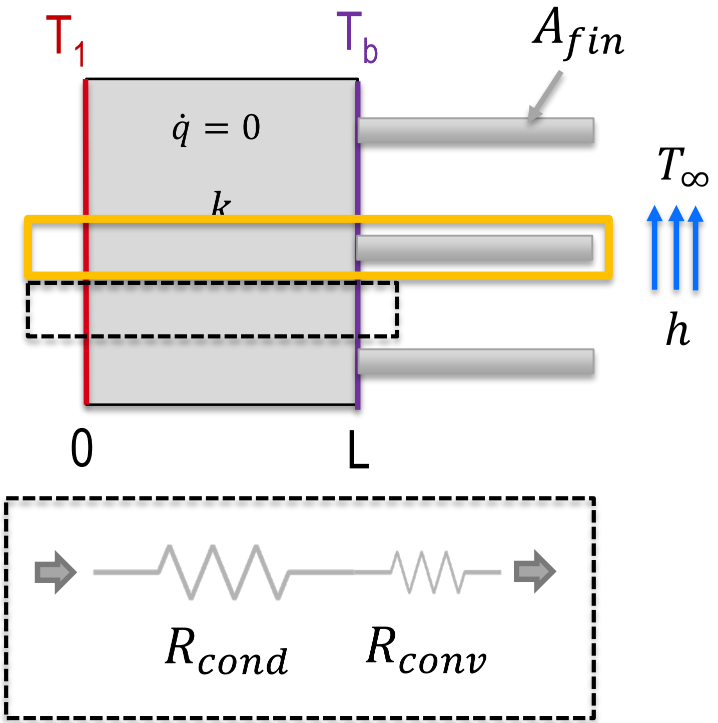
In many technical problems, T_1 and R_{cond} are fixed by the operating conditions and the mechanical requirements of a structure. Then the heat transfer rate is primarily controlled by convection. If we need to increase Q_x then we can:

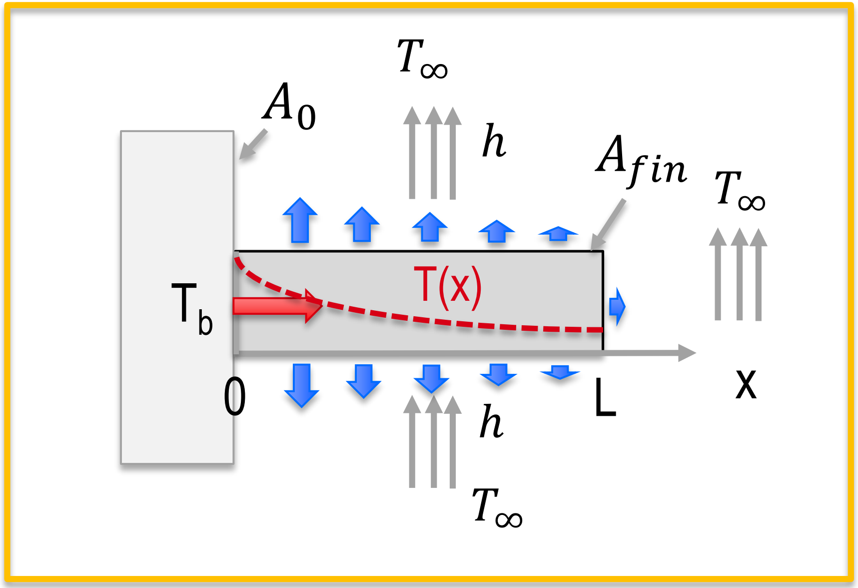
(i) decrease T_∞;

(ii) decrease $R_{conv} = {}^{1}/_{hA}$ by increasing h or A.

Decreasing T_∞ and increasing h can have a high energy cost so we increase A by adding a fin.

Heat Transfer from Extended Surfaces



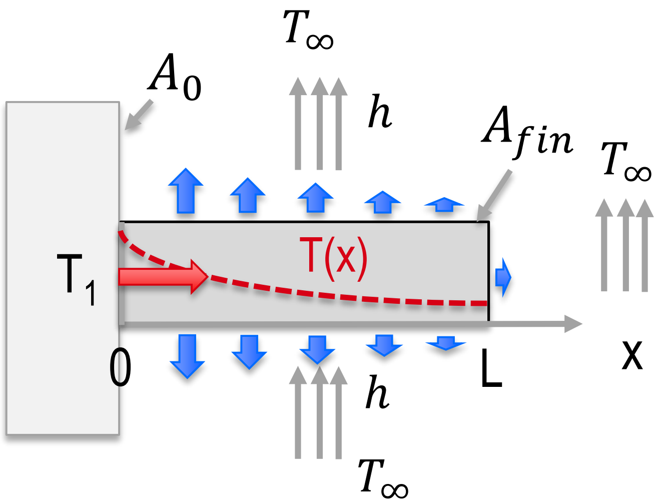


The fin has a more complex behavior than the simple planar system because heat transfer by conduction and convection occur along different directions and are interdependent. Hence a simple equivalent electrical circuit cannot be immediately identified.

Heat Transfer from Extended Surfaces

$$q'' = \bar{h}\,(T_s(x) - T_\infty) \propto T_s(x) - T_\infty$$

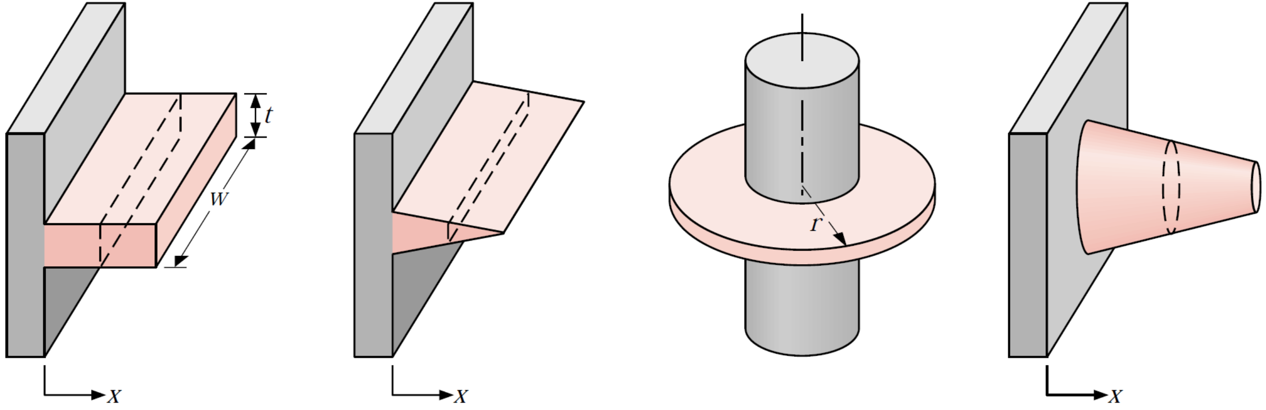
$$Q = \bar{h}A\,(T_s - T_\infty) \propto A = A_0 + A_{fin}$$



It is necessary to **optimize the fin** in order to maximize the advantage of adding extra surface area (A_{fin}) without wasting material (beyond a certain length L, the temperature difference will be so small that the heat transfer becomes negligible)

Heat Transfer from Extended Surfaces

Uniform Cross-Section | **Non-uniform Cross-Section**



What is the temperature profile along the fin, $T = T(x)$?

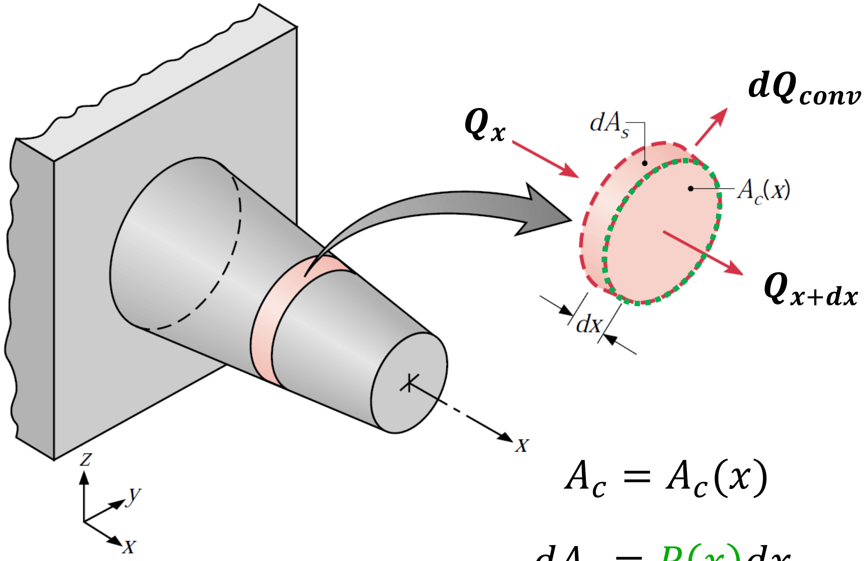
This Lecture

- ❑ Heat transfer from extended surfaces (Fins)
  - ❑ Boundary Conditions and Temperature Profiles

Learning Objectives:

- ❑ Calculate the temperature profile in fins with constant cross-section

Heat Transfer from Generalized Fin



$$A_c = A_c(x)$$

$$dA_s = P(x)dx$$

$$A_c \neq A_s$$

Assumption 1: isotropic material
Assumption 2: k independent of T
Assumption 3: steady state
Assumption 4: no heat sources

1st Law of Thermodynamics (Energy Balance):

$$Q_x = Q_{x+dx} + dQ_{conv}$$

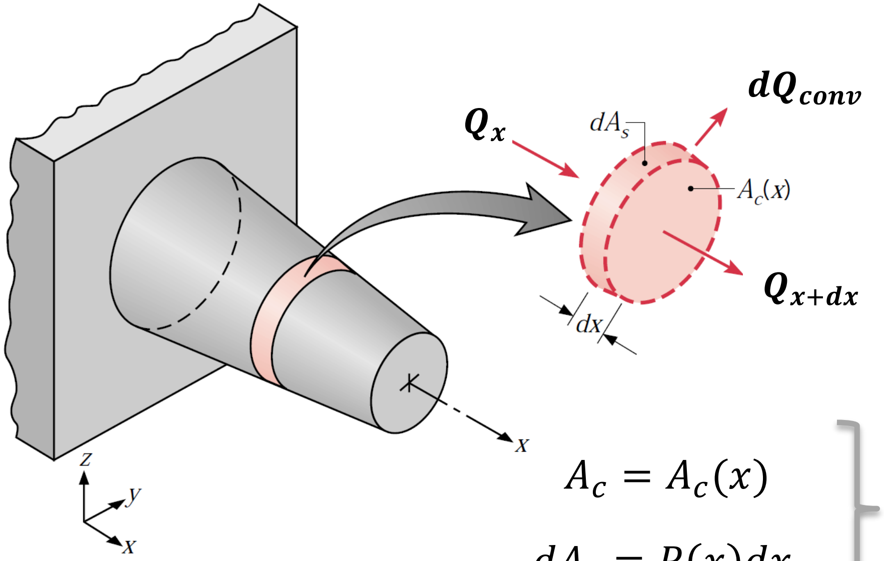
Transport Laws:

$$dQ_{conv} = h dA_s (T(x) - T_\infty)$$

$$Q_x = -kA_c \frac{dT}{dx}$$

$$Q_{x+dx} = Q_x + \frac{dQ_x}{dx}dx = -kA_c\frac{dT}{dx} - \mathrm{k}\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)dx$$

Heat Transfer from Generalized Fin



1st Law of Thermodynamics (Energy Balance):

$$0 = Q_{x+dx} - Q_x + dQ_{conv}$$

$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) dx - \frac{h}{k} dA_s (T(x) - T_\infty) = 0$$

$$A_c = A_c(x)$$

$$dA_s = P(x)dx$$

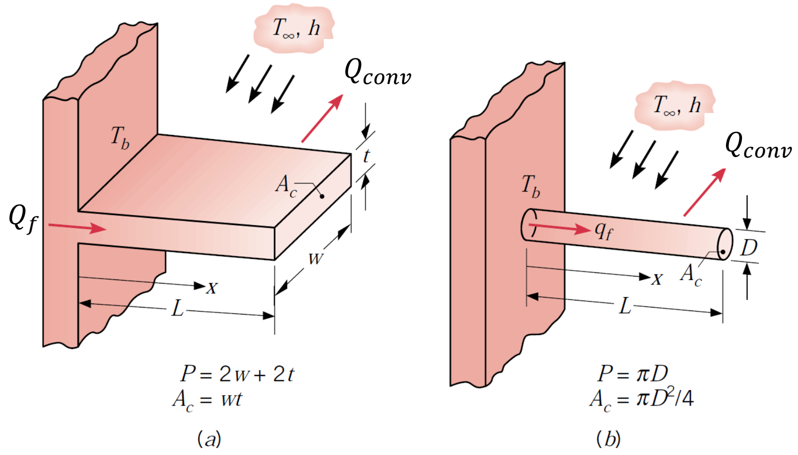
$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{h}{A_c k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

Assumption 1: isotropic material
Assumption 2: k independent of T
Assumption 3: steady state
Assumption 4: no heat sources

Assumption 5: constant cross-section

$$A_c = \text{constant} \qquad dA_s = Pdx$$

Heat Transfer from Generalized Fin



1st Law of Thermodynamics (Energy Balance):

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{h}{A_c k}P\right)(T - T_\infty) = 0$$

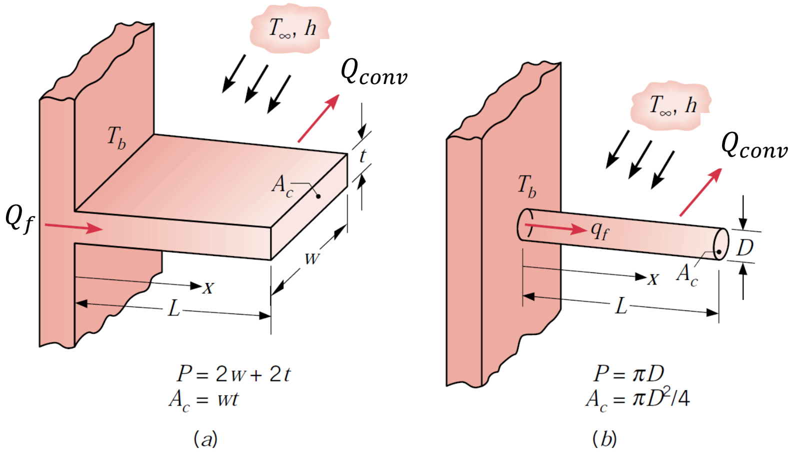
$$\boldsymbol{\frac{d^2T}{dx^2} - \frac{hP}{A_c k}(T - T_\infty) = 0}$$

Non-dimensional variables: $\Theta = \frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty}$ $\quad \xi = x/L \quad m^2 = {hP}/{kA_c}$

Assumption 1: isotropic material
Assumption 2: k independent of T
Assumption 3: steady state
Assumption 4: no heat sources
Assumption 5: constant cross-section

$$\Rightarrow \frac{d^2\theta}{dx^2} - m^2\theta = 0 \quad \Rightarrow \quad \boldsymbol{\frac{d^2\Theta}{d\xi^2} - (mL)^2\Theta = 0}$$

Heat Transfer from Generalized Fin



$$\frac{d^2\Theta}{d\xi^2} - (mL)^2\Theta = 0$$

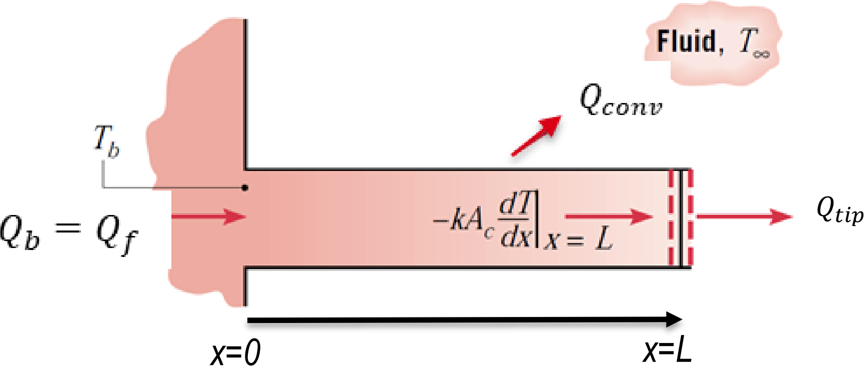
$$(mL)^2 = \frac{hPL^2}{kA_c} = \frac{\left(L/_{kA_c}\right)}{\left(1/_{hPL}\right)} \sim \frac{R_{cond}}{R_{conv}}$$

If $(mL)^2 \gg 1$ the fin has added a large thermal resistance;
If $(mL)^2 \ll 1$ the fin has added little surface area;

Assumption 1: isotropic material
Assumption 2: k independent of T
Assumption 3: steady state
Assumption 4: no heat sources
Assumption 5: constant cross-section

A good fin will have a value of $(mL)^2$ of the order of 10^0 to balance internal and external resistance

Heat Transfer from Generalized Fin



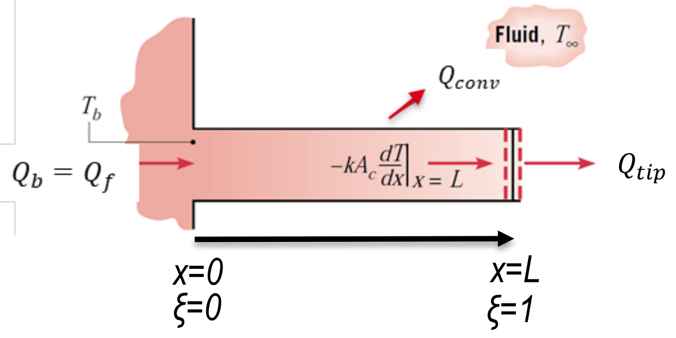
The fin was added to enable better heat transfer. So we want to know how much heat is being removed by the fin. We observe that convection on the fin can remove only as much heat as it initially enters the fin by conduction.

Therefore the total heat removed by the fin is equal to:

$$Q_{fin} = Q_{b,cond} = -kA\frac{d\theta}{dx}\bigg|_{x=0} = -\frac{kA}{L}(T_b - T_\infty)\frac{d\Theta}{d\xi}\bigg|_{\xi=0}$$

Knowing the temperature profile we can calculate the total heat removed by a single fin.

Heat Transfer from Fin of Uniform Cross-Section



$$\frac{d^2\Theta}{d\xi^2} - (mL)^2\Theta = 0 \quad \Rightarrow \quad \boldsymbol{\Theta = C_1 e^{mL\xi} + C_2 e^{-mL\xi}}$$

$$\Theta = \frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty}$$

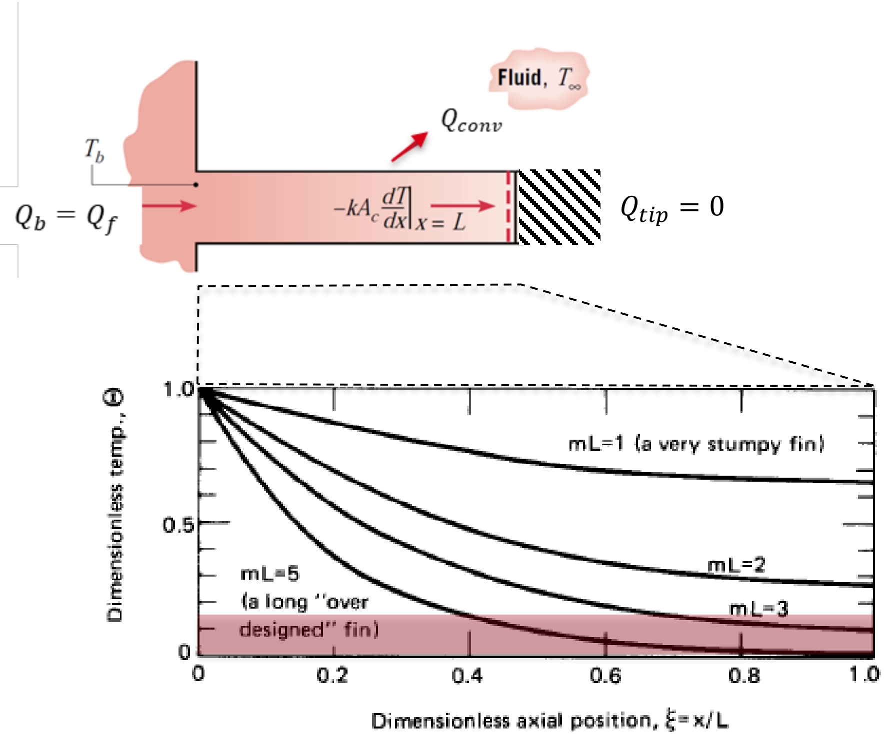
$$\xi = x/L$$

$$m^2 = {hP}/{kA_c}$$

Boundary conditions:

- Fin base $\quad T = T_b \Rightarrow \Theta_{\xi=0} = 1$
- Fin tip
  1. Adiabatic $\quad \left.\frac{d\Theta}{d\xi}\right|_{\xi=1} = 0$
  2. Convection $\quad \left.\frac{d\Theta}{d\xi}\right|_{\xi=1} = hA_c(T(L) - T_\infty)$
  3. Temperature $\quad \Theta_{\xi=1} = \Theta_L$
  4. Infinite fin $\quad \Theta_L \rightarrow 0$

Heat Transfer from Fin of Uniform Cross-Section – Adiabatic Tip



If mL>3 at the tip very small temperature difference

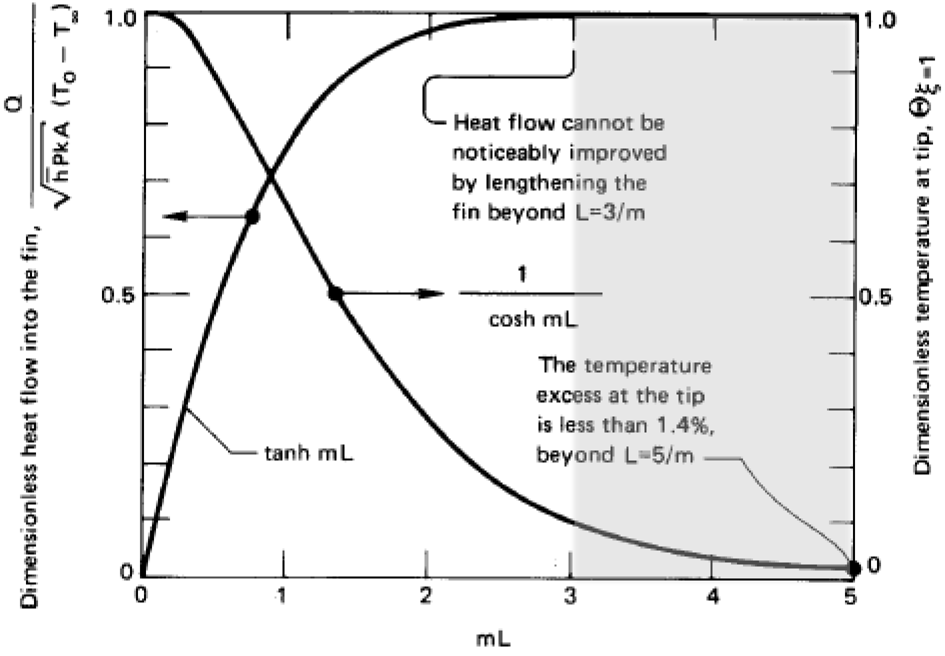
$$\frac{d^2\Theta}{d\xi^2} - (mL)^2\Theta = 0 \quad \Rightarrow \quad \boldsymbol{\Theta = C_1 e^{mL\xi} + C_2 e^{-mL\xi}}$$

Boundary conditions:

- Fin base $\quad T = T_b \;\Rightarrow\; \Theta_{\xi=0} = 1$
- Fin tip $\quad Q_{tip} = 0 \;\Rightarrow\; \left.\frac{d\Theta}{d\xi}\right|_{\xi=1} = 0$

$$\Theta = \frac{\theta}{\theta_b} = \frac{\cosh(mL(1-\xi))}{\cosh mL}$$

Heat Transfer from Fin of Uniform Cross-Section – Adiabatic Tip



$$\Theta = \frac{\theta}{\theta_b} = \frac{\cosh(mL(1-\xi))}{\cosh mL}$$

$$Q = -kA \left.\frac{d(T-T_\infty)}{dx}\right|_{x=0}$$

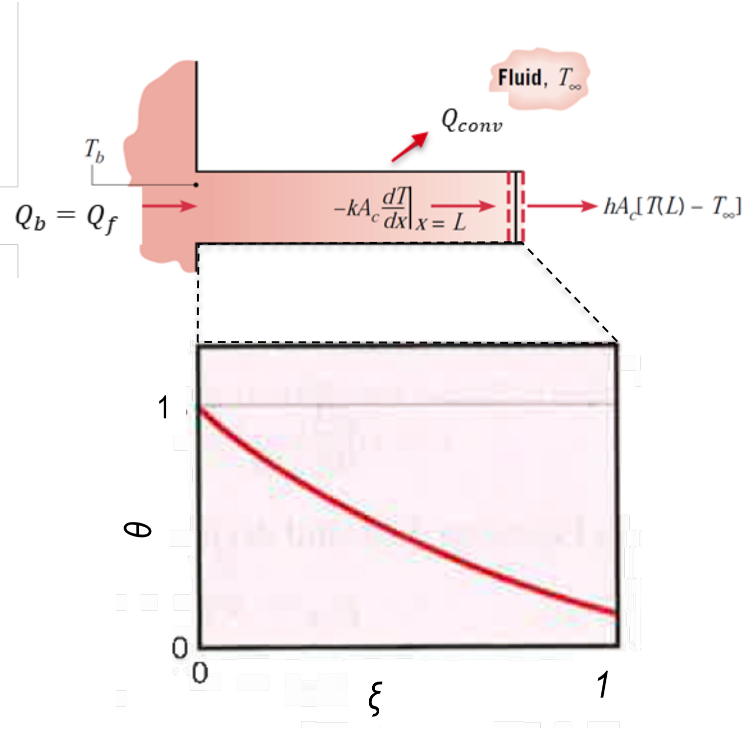
$$Q = -\frac{kA}{L}(T_b - T_\infty)\left.\frac{d\Theta}{d\xi}\right|_{\xi=0}$$

$$\frac{QL}{kA(T_0 - T_\infty)} = mL\frac{\sinh mL}{\cosh mL}$$

$$\frac{Q}{\sqrt{kA\bar{h}P}(T_0 - T_\infty)} = \tanh mL$$

If mL>3 at the tip very small temperature difference
And there is negligible residual heat flow in the fin.

Heat Transfer from Fin of Uniform Cross-Section – Convection



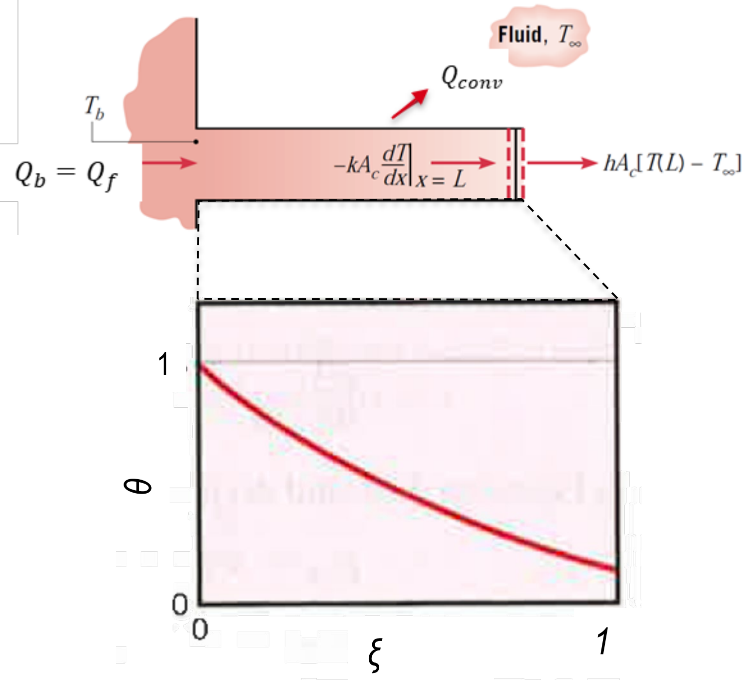
$$\frac{d^2\theta}{d\xi^2} - (mL)^2\theta = 0 \quad \Rightarrow \quad \boldsymbol{\theta = C_1 e^{mL\xi} + C_2 e^{-mL\xi}}$$

Boundary conditions:

- Fin base $\quad T = T_b \Rightarrow \Theta_{\xi=0} = 1$
- Fin tip $\quad \left.\frac{d\Theta}{d\xi}\right|_{\xi=1} = hA_c(T(L) - T_\infty) = -kA_c \left.\frac{dT}{dx}\right|_{x=}$

$$\Rightarrow \quad \Theta = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

Heat Transfer from Fin of Uniform Cross-Section – Convection



$$\Theta = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

$$Q_f = Q_b = -kA_c \frac{dT}{dx}\Big|_{x=0} - kA_c \frac{d\theta}{dx}\Big|_{x=0}$$

$$Q_f = M\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

Sanity Check 1: $Q_f = \int_{A_f} h[T(x) - T_\infty]\, dA_s = \int_{A_f} h\theta(x)\, dA_s$

Sanity Check 2: if $\frac{hL}{mLk} \ll 1$ i.e. $\frac{hL}{k} = \frac{L/k}{1/h} \sim \frac{R_{cond}}{R_{conv}} \ll 1$ then $\frac{dT}{dx}\Big|_{x=L} \sim 0$ equivalent to an adiabatic tip.

Heat Transfer from Fin of Uniform Cross-Section - Recap

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate Q_f
A	Convection heat transfer: $h\theta(L) = -k\,d\theta/dx\vert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.70)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ (3.72)
B	Adiabatic $d\theta/dx\vert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$ (3.75)	$M\tanh mL$ (3.76)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.77)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ (3.78)
D	Infinite fin ($L \to \infty$): $\theta(L) = 0$	e^{-mx} (3.79)	M (3.80)

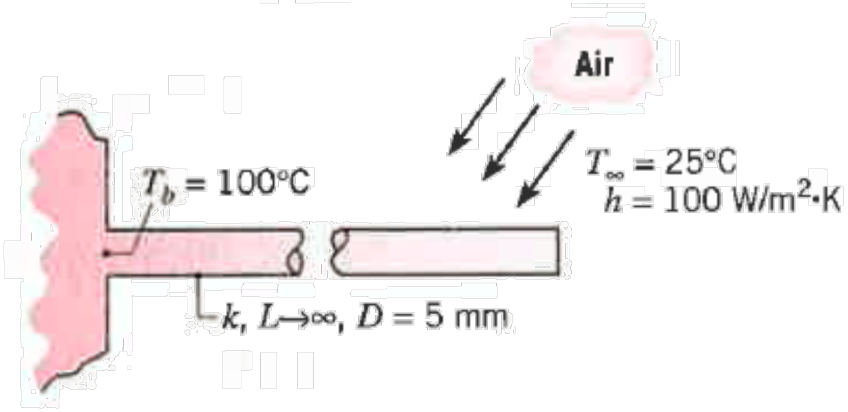
$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

Heat Transfer from Fin of Uniform Cross-Section (Example)

A very long rod 5 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 100 $W/m^2 \cdot K$.

- We first consider a rod made of copper. What is its thermal conductivity?
- What tip boundary condition do we use when describing the rod as a “fin”?



Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction along the rod.
3. Constant properties.
4. Negligible radiation exchange with surroundings.
5. Uniform heat transfer coefficient.
6. Infinitely long rod.

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$$\lim_{mL \to \text{large}} \Theta = e^{-mL\xi}$$

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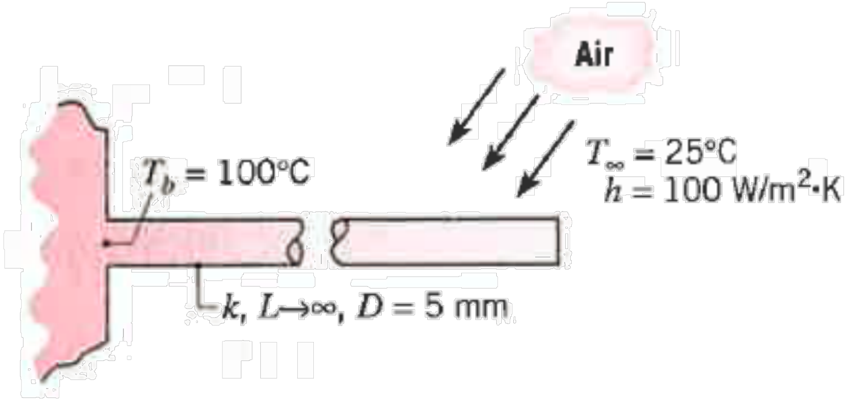

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Properties: Table A.1, copper $[T = (T_b + T_\infty)/2 = 62.5°C \approx 335\ K]$: $k = 398$ $W/m \cdot K$. Table A.1, 2024 aluminum (335 K): $k = 180\ W/m \cdot K$. Table A.1, stainless steel, AISI 316 (335 K): $k = 14\ W/m \cdot K$.

Heat Transfer from Fin of Uniform Cross-Section (Example)

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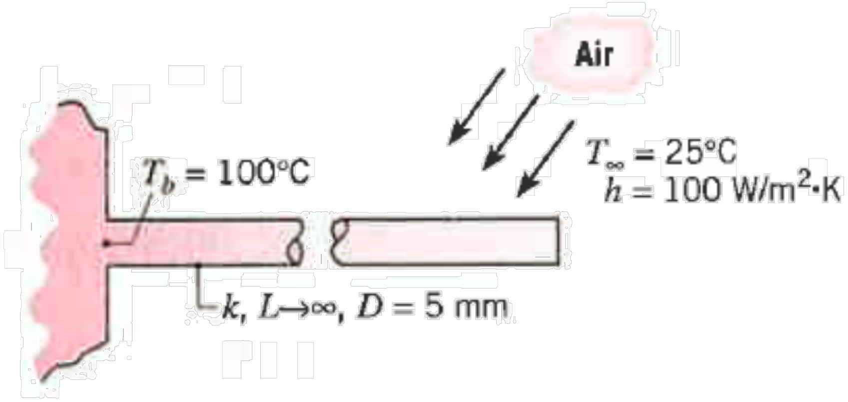
$$\lim_{mL \to \text{large}} \Theta = e^{-mL\xi}$$

1. Determine the temperature distributions along rods constructed from pure copper, 2024 aluminum alloy, and type AISI 316 stainless steel. What are the corresponding heat losses from the rods?



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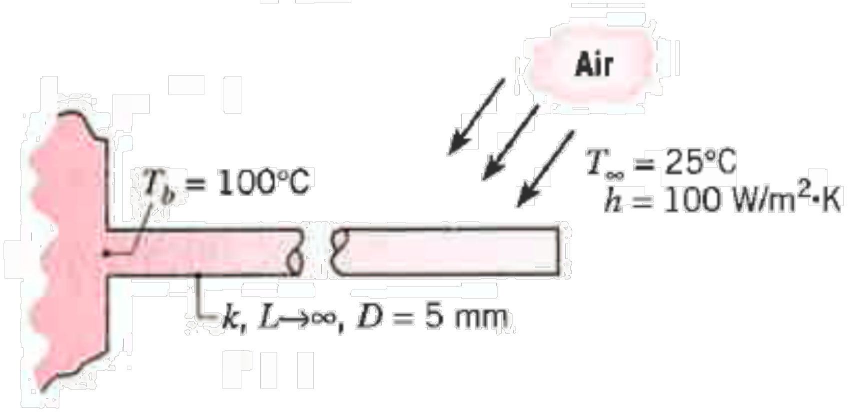
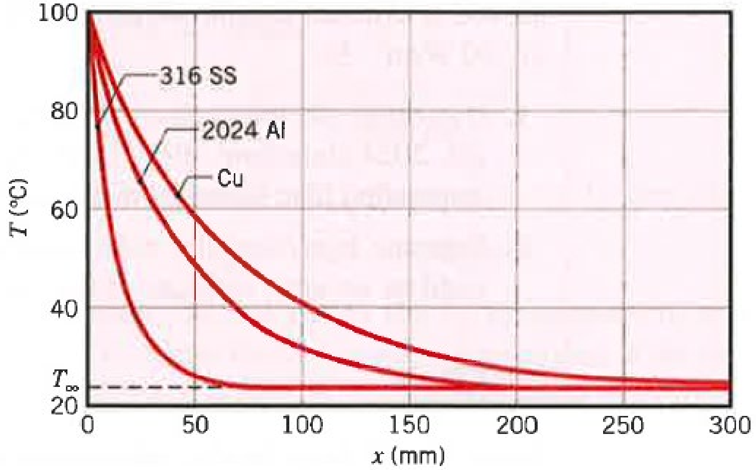
1. Determine the temperature distributions along rods constructed from pure copper, 2024 aluminum alloy, and type AISI 316 stainless steel. What are the corresponding heat losses from the rods?

$$\lim_{mL \to \text{large}} \Theta = e^{-mL\xi}$$

$$T = T_\infty + (T_b - T_\infty)e^{-mx}$$

$$m = (hP/kA_c)^{1/2} = (4h/kD)^{1/2}.$$





$$Q = \sqrt{(kA\bar{h}P)}\,(T_0 - T_\infty)$$

This Lecture

- ❑ Heat transfer from extended surfaces (Fins)
  - ☑ Boundary Conditions and Temperature Profiles

Learning Objectives:

- ☑ Calculate the temperature profile in fins with constant cross-section

Next lecture

- ❑ Fins Performance
- ❑ Array of Fins

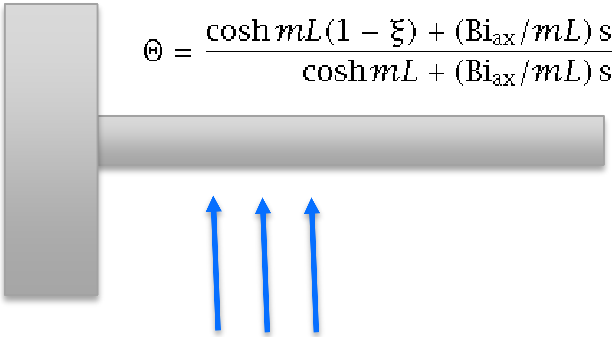
Learning Objectives:

- ❑ Calculate the performance of a fin-based system

Supplementary Slides

Heat Transfer from Fin of Uniform Cross-Section (Example)

A 2 cm diameter aluminum rod with $k = 205$ W/m·K, 8 cm in length, protrudes from a 150°C wall. Air at 26°C flows by it, and $\overline{h} = 120$ W/m^{2}K. Determine whether or not tip conduction is important in this problem. To do this, make the very crude assumption that $\overline{h} \simeq \overline{h}_L$. Then compare the tip temperatures as calculated with and without considering heat transfer from the tip.



$$\Theta = \frac{\cosh mL(1-\xi) + (\mathrm{Bi}_{ax}/mL)\sinh mL(1-\xi)}{\cosh mL + (\mathrm{Bi}_{ax}/mL)\sinh mL}$$

Solution.

$$mL = \sqrt{\frac{\overline{h}PL^2}{kA}} = \sqrt{\frac{120(0.08)^2}{205(0.01/2)}} = 0.8656$$

$$\mathrm{Bi}_{ax} = \frac{\overline{h}L}{k} = \frac{120(0.08)}{205} = 0.0468$$

Therefore, eqn. (4.48) becomes

$$\Theta\,(\xi = 1) = \Theta_{tip} = \frac{\cosh 0 + (0.0468/0.8656)\sinh 0}{\cosh(0.8656) + (0.0468/0.8656)\sinh(0.8656)}$$

$$= \frac{1}{1.3986 + 0.0529} = 0.6886$$

so the exact tip temperature is

$$T_{tip} = T_\infty + 0.6886(T_0 - T_\infty)$$

$$= 26 + 0.6886(150 - 26) = 111.43°\mathrm{C}$$

$$\Theta = \frac{\cosh mL(1-\xi)}{\cosh mL}$$

on the other hand, gives

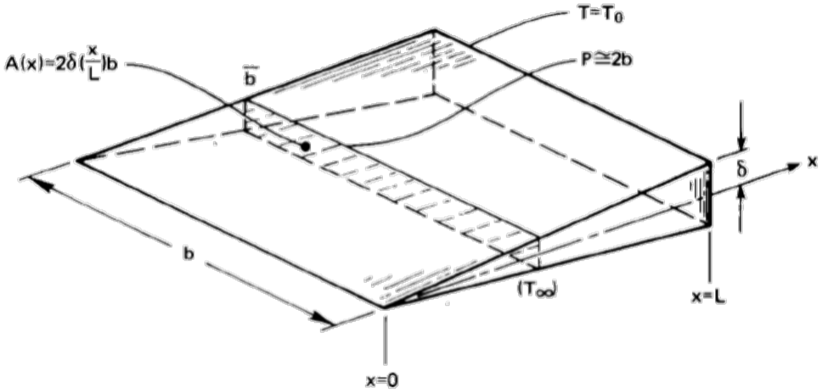
$$\Theta_{tip} = \frac{1}{1.3986} = 0.7150$$

so the approximate tip temperature is

$$T_{tip} = 26 + 0.715(150 - 26) = 114.66°\mathrm{C}$$

Thus the insulated-tip approximation is adequate for the computation in this case. ■

Heat Transfer from Generalized Fin – b>>



Let's consider a fin with non-uniform cross-section for which b>>thickness:

$$A(x) = 2\delta\left({}^{x}/_{L}\right)b$$

$$\frac{d}{dx}\left[A(x)\frac{d(T-T_\infty)}{dx}\right] - \frac{\overline{h}P}{k}(T-T_\infty) = 0$$

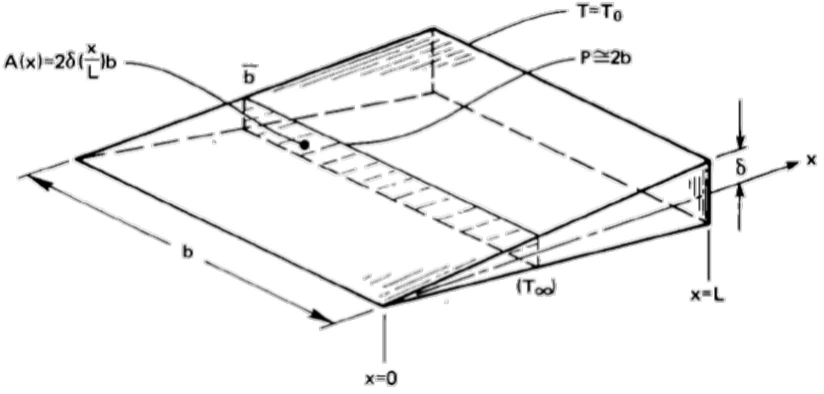
$$\frac{d}{dx}\left[2\delta\left(\frac{x}{L}\right)b\,\frac{d(T-T_\infty)}{dx}\right] - \frac{2\overline{h}b}{k}(T-T_\infty) = 0$$

$$\xi\frac{d^2\Theta}{d\xi^2} + \frac{d\Theta}{d\xi} - \underbrace{\frac{\overline{h}L^2}{k\delta}}_{\substack{\text{a kind}\\ \text{of } (mL)^2}}\Theta = 0$$

This second-order linear differential equation is difficult to solve because it has a variable coefficient. Its solution is expressible in Bessel functions:

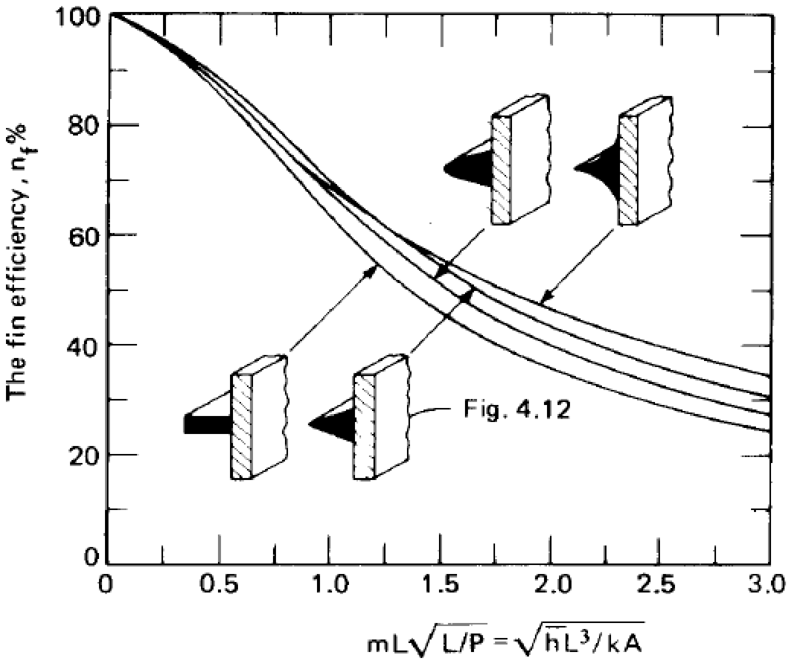
$$\Theta = \frac{I_o\left(2\sqrt{\overline{h}Lx/k\delta}\right)}{I_o\left(2\sqrt{\overline{h}L^2/k\delta}\right)} \qquad (4.62)$$

Heat Transfer from Generalized Fin – b>>



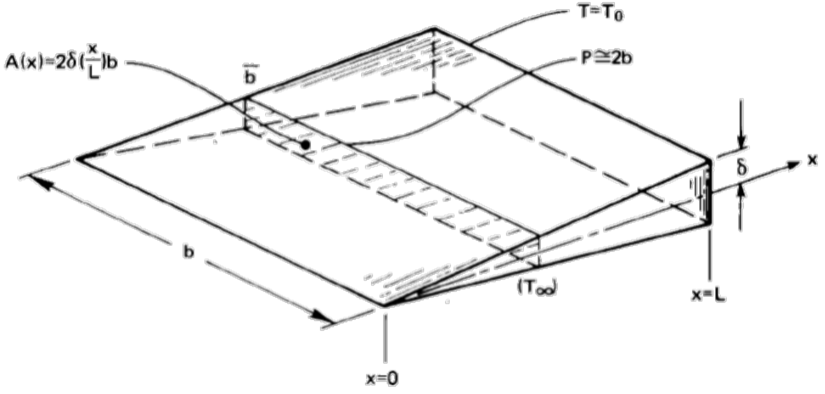
Let's consider a fin with non-uniform cross-section for which b>>thickness:

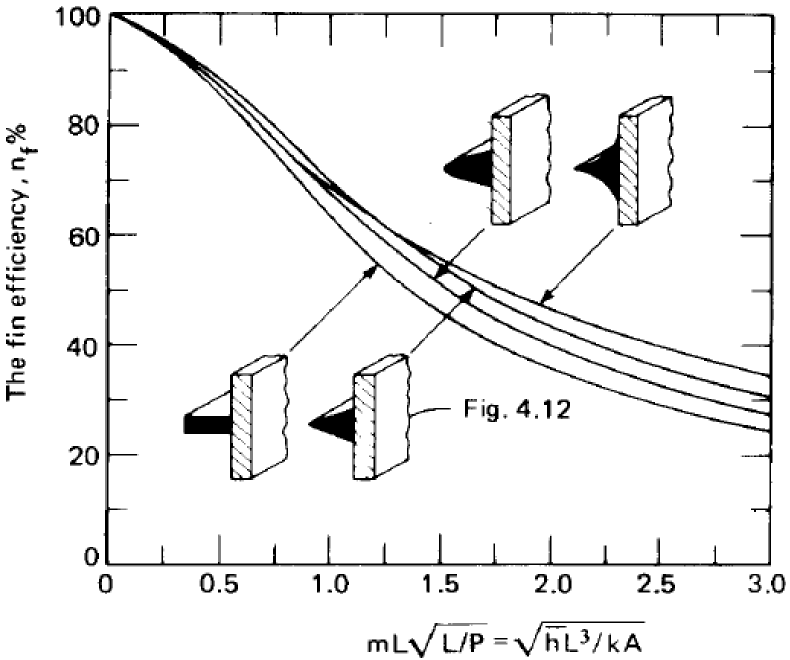
$$A(x) = b2\delta(^{x}/_{L})$$



Comparison of four straight fins: constant thickness, triangular, parabolic, and hyperbolic. (m is based on A shown in black.)

Heat Transfer from Generalized Fin – b>>



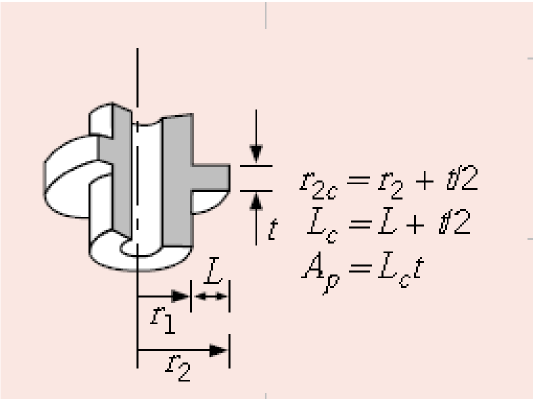


Comparison of four straight fins: constant thickness, triangular, parabolic, and hyperbolic. (m is based on A shown in black.)

$$\eta_f \equiv \frac{Q_f}{Q_{f,max}} = \frac{Q_f}{hA_f(T_0 - T_\infty)}$$

$$R_f \equiv \frac{(T_0 - T_\infty)}{Q_f} = \frac{1}{hA_f\eta_f}$$

Fins of Nonuniform Cross-section – radial fin



Although the fin thickness is uniform (t is independent of r), the cross-sectional area, $A_c = 2\pi rt$, varies with r. Replacing x by r in Equation 3.61 and expressing the surface area as $A_s = 2\pi(r^2 - r_1^2)$, the general form of the fin equation reduces to

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kt}(T - T_\infty) = 0$$

or, with $m^2 \equiv 2h/kt$ and $\theta \equiv T - T_\infty$,

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0$$

The foregoing expression is a *modified Bessel equation* of order zero, and its general solution is of the form

$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr)$$

where I_0 and K_0 are modified, zero-order Bessel functions of the first and second

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