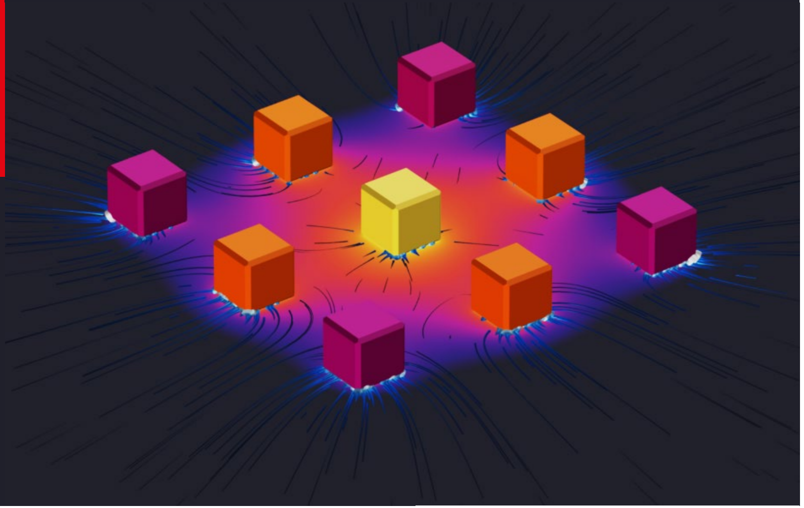
Heat and Mass Transfer
ME-341



Instructor: Giulia Tagliabue

École
polytechnique
fédérale
de Lausanne

Spring Semester

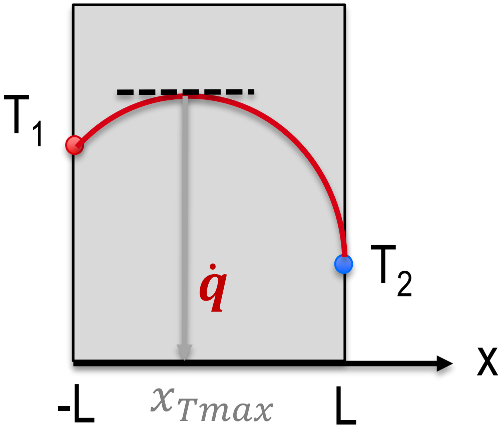
Previously

- [x] 1D steady-state conduction with heat sources

Learning Objectives:

- [ ] Solve 1D steady state heat conduction problems in different geometries, with heat sources

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k}L^2\left(1 - \frac{\boldsymbol{x^2}}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

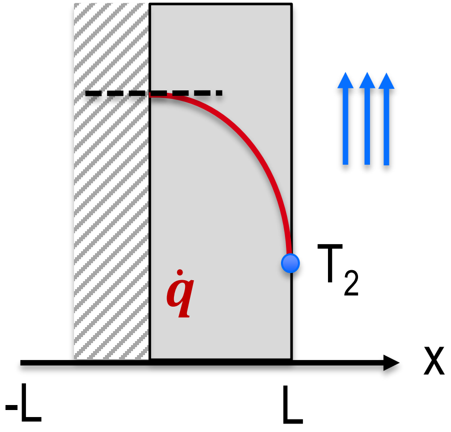
$$Q = -kA\frac{\partial T}{\partial x} = -kA\left(-\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L}\right) = Q(x)$$

The heat transfer rate is not constant!
It depends on the position along x.

$$Q(x) \neq \frac{\Delta T}{R_{th}}$$

The electrical analogy fails!
We cannot use the thermal resistance concept in layers with heat sources.

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k}L^2\left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

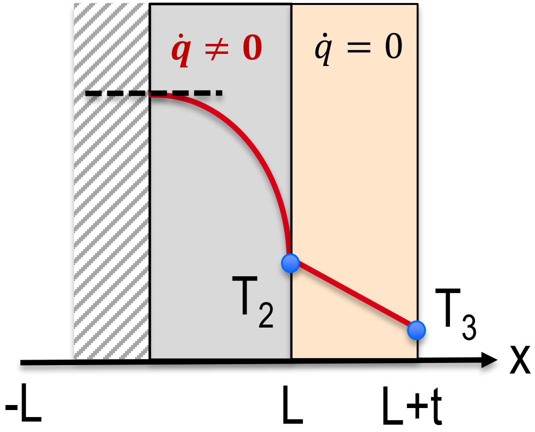
If $T_1 = T_2$ then the T profile must be symmetric and $x_{Tmax} = 0$

$$T(x) = \frac{\dot{q}}{2k}L^2\left(1 - \frac{x^2}{L^2}\right) + T_2 \qquad T_{max} = \frac{\dot{q}}{2k}L^2 + T_2$$

$$q''_{x=0} = -k\left(\frac{\partial T}{\partial x}\right)_{x=0} = -k\left(-\frac{\dot{q}}{k}x\right)_{x=0} = 0$$

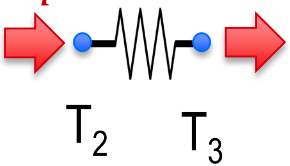
At the centerline of the wall effectively the heat flux is zero, satisfying the symmetry of the problem.
This is equivalent to having a perfectly insulated boundary at x=0.

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

$Q = \dot{q}L$



We can draw an equivalent electrical circuit only for the layers WITHOUT heat sources. Layers WITH heat sources inject into the equivalent circuit a certain Q

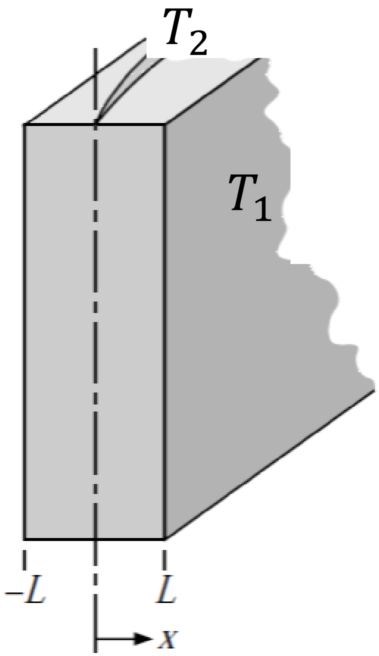
This Lecture

- ❑ 2D Conduction & Shape Factor
- ❑ Exercises

Learning Objectives:

- ❑ Approach simple 2D conduction problems

2D Conduction – Method of Separation of Variables



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \theta \equiv \frac{T - T_1}{T_2 - T_1} \quad \Rightarrow \quad \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

$$\theta(x, y) = X(x) \cdot Y(y) \quad \Rightarrow \quad -\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

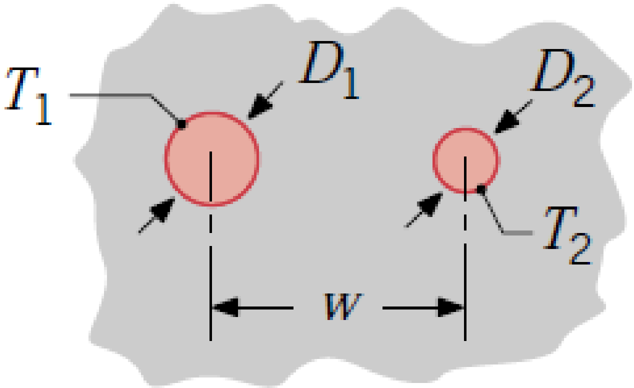
$$\begin{cases} \dfrac{d^2 X}{dx^2} + \lambda^2 X = 0 \\ \dfrac{d^2 Y}{dy^2} - \lambda^2 Y = 0 \end{cases} \quad \Rightarrow \quad \begin{aligned} X &= C_1 \cos \lambda x + C_2 \sin \lambda x \\ Y &= C_3 e^{-\lambda y} + C_4 e^{+\lambda y} \end{aligned}$$

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

Shape Factor for known 2D and 3D Heat Diffusion Problems

For numerous geometries an analytical solution has already been found and can be easily retrieved using a *Shape Factor, S*

$L \gg D_1, D_2$
$L \gg w$



$$Q = k \frac{2\pi L}{\cosh^{-1}\left(\frac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}(T_1 - T_2)$$

$$Q = kS(T_1 - T_2) \qquad R_{cond} = \frac{1}{kS}$$

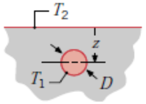
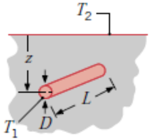
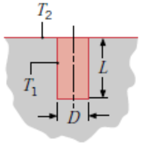
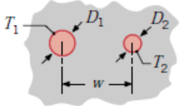
$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$$

Shape Factor for known 2D and 3D Heat Diffusion Problems

For numerous geometries an analytical solution has already been found and can be easily retrieved using a *Shape Factor, S*

TABLE 4.1 Conduction shape factors and dimensionless conduction heat rates for selected systems.

(*a*) Shape factors [$q = Sk(T_1 - T_2)$]

System	Schematic	Restrictions	Shape Factor
Case 1 Isothermal sphere buried in a semi-infinite medium		$z > D/2$	$\frac{2\pi D}{1 - D/4z}$
Case 2 Horizontal isothermal cylinder of length L buried in a semi-infinite medium		$L \gg D$ $L \gg D$ $z > 3D/2$	$\frac{2\pi L}{\cosh^{-1}(2z/D)}$ $\frac{2\pi L}{\ln(4z/D)}$
Case 3 Vertical cylinder in a semi-infinite medium		$L \gg D$	$\frac{2\pi L}{\ln(4L/D)}$
Case 4 Conduction between two cylinders of length L in infinite medium		$L \gg D_1, D_2$ $L \gg w$	$\frac{2\pi L}{\cosh^{-1}\left(\frac{4w^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$

This Lecture

- [x] 2D Conduction & Shape Factor
- [ ] Exercises

Learning Objectives:

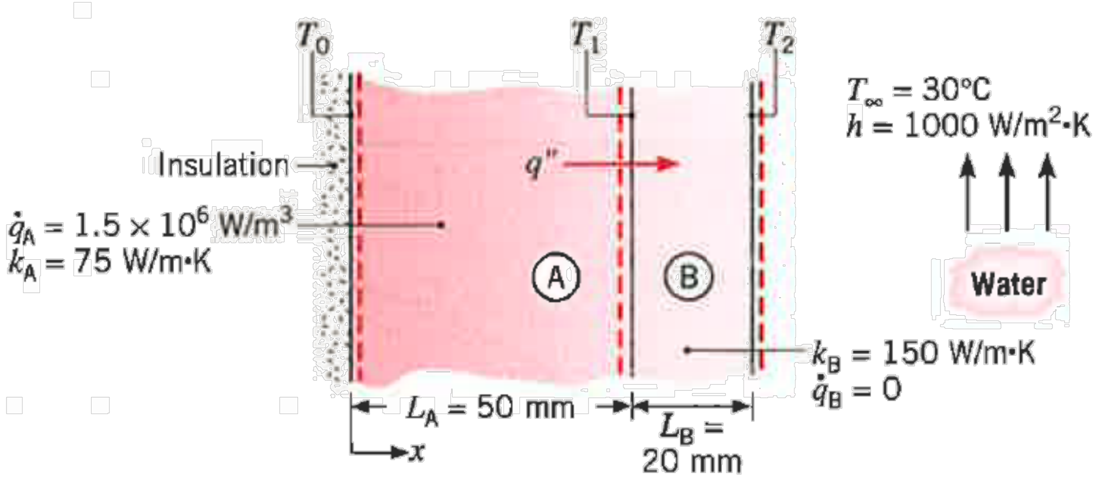
- [x] Approach simple 2D conduction problems

Heat Diffusion Equation – The Heated Slab Case (Example)

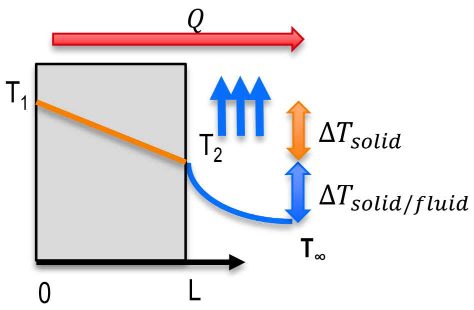
A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m^3, $k_A = 75$ W/m · K, and thickness L_A = 50 mm. The wall material B has no generation with $k_B = 150$ W/m · K and thickness L_B = 20 mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°C$ and $h = 1000$ W/m^2 · K.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

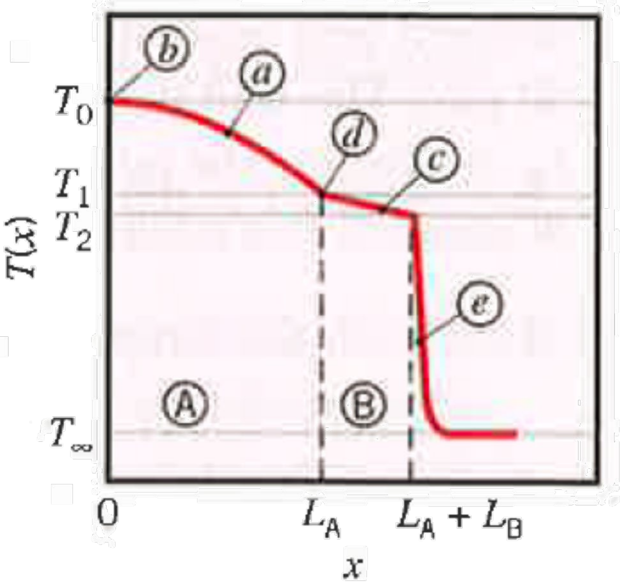
Assumption 1: incompressible medium (solid)
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material
Assumption 4: *k* independent of *T*
Assumption 5: steady state



(a) Parabolic in material A.
(b) Zero slope at insulated boundary.
(c) Linear in material B.
(d) Slope change $= k_B/k_A = 2$ at interface.
The temperature distribution in the water is characterized by
(e) Large gradients near the surface.



$$\frac{(T_1 - T_2)}{(T_2 - T_\infty)} = \frac{R_{th,cond}}{R_{th,conv}} \equiv Bi$$



$$R''_{\text{cond, B}} = \frac{L_B}{k_B} \qquad R''_{\text{conv}} = \frac{1}{h}$$

$$\frac{0.02\ \text{m}}{150\ \text{W/m}\cdot\text{K}} < \frac{1}{1000\ \text{W/m}^2\cdot\text{K}}$$

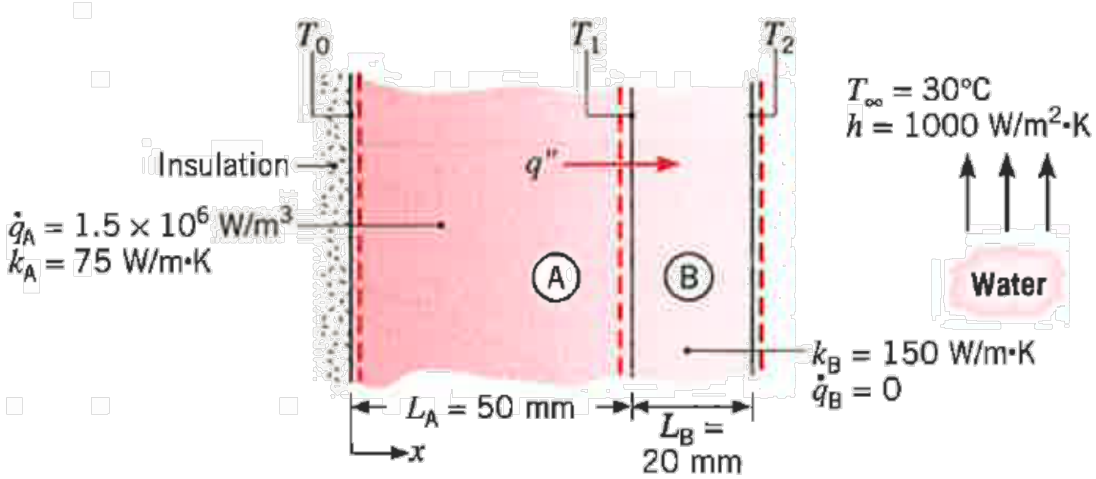
$$R''_{cond,B}/R''_{conv} = \text{Bi} = 0.13$$

Heat Diffusion Equation – The Heated Slab Case (Example)

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6\ \mathrm{W/m^3}$, $k_A = 75\ \mathrm{W/m \cdot K}$, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150\ \mathrm{W/m \cdot K}$ and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°\mathrm{C}$ and $h = 1000\ \mathrm{W/m^2 \cdot K}$.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

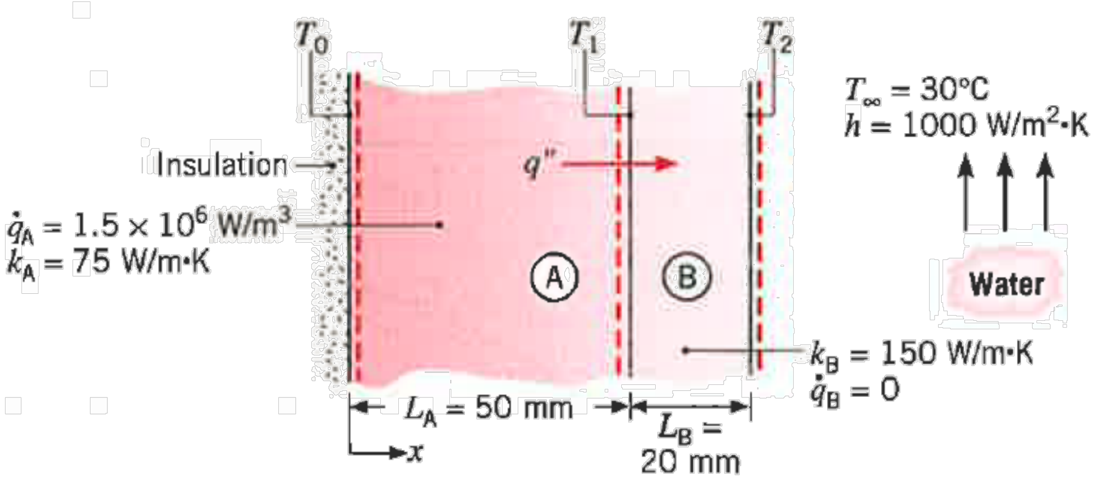
<u>Assumption 1</u>: incompressible medium (solid)
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* independent of *T*
<u>Assumption 5</u>: steady state



Heat Diffusion Equation – The Heated Slab Case (Example)

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6\ \text{W/m}^3$, $k_A = 75\ \text{W/m} \cdot \text{K}$, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150\ \text{W/m} \cdot \text{K}$ and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°\text{C}$ and $h = 1000\ \text{W/m}^2 \cdot \text{K}$.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.



<u>Assumption 1</u>: incompressible medium (solid)
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* independent of *T*
<u>Assumption 5</u>: steady state

(a) Parabolic in material A.
(b) Zero slope at insulated boundary.
(c) Linear in material B.
(d) Slope change $= k_B/k_A = 2$ at interface.
The temperature distribution in the water is characterized by
(e) Large gradients near the surface.

$$R''_{\text{cond, B}} = \frac{L_B}{k_B} \qquad R''_{\text{conv}} = \frac{1}{h}$$

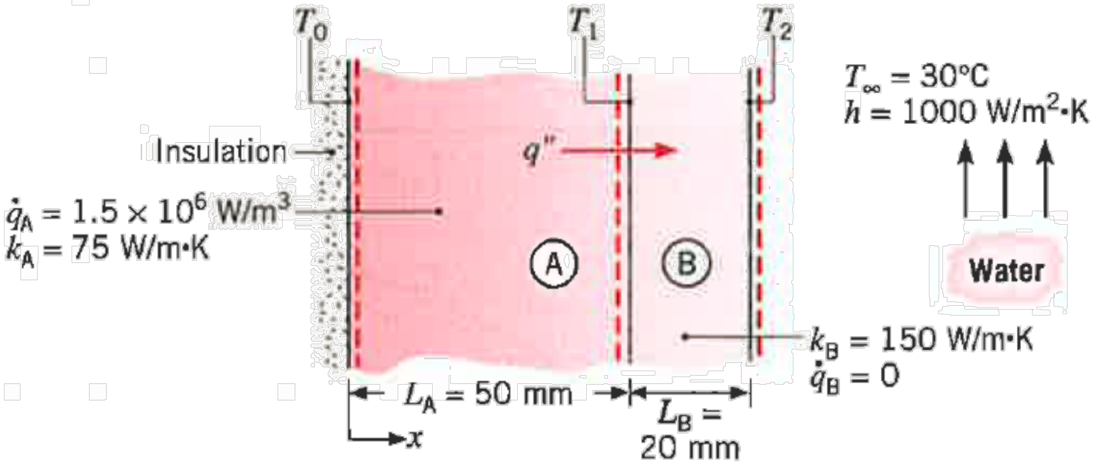
$$\frac{0.02\ \text{m}}{150\ \text{W/m} \cdot \text{K}} < \frac{1}{1000\ \text{W/m}^2 \cdot \text{K}}$$

$$R''_{cond,B}/R''_{conv} = \text{Bi} = 0.13$$

Heat Diffusion Equation – The Heated Slab Case (Example)

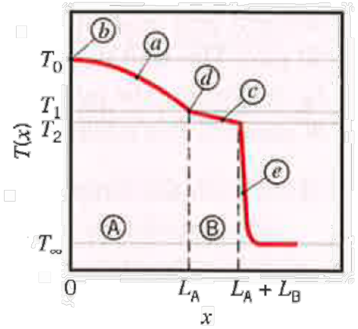
A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m^3, $k_A = 75$ W/m · K, and thickness $L_A =$ 50 mm. The wall material B has no generation with $k_B = 150$ W/m · K and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°C$ and $h = 1000$ W/m^2 · K.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.



<u>Assumption 1</u>: incompressible medium (solid)
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* independent of *T*
<u>Assumption 5</u>: steady state

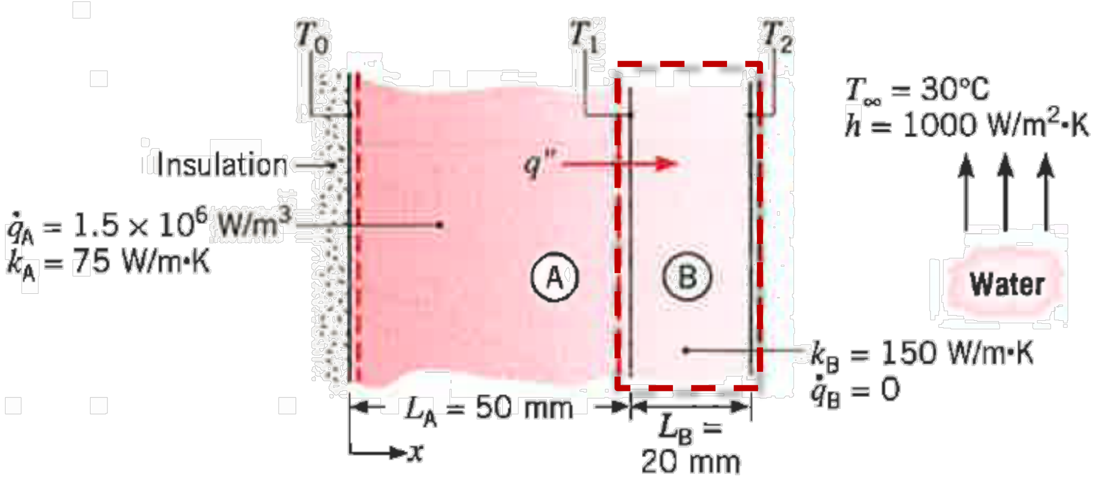
(a) Parabolic in material A.
(b) Zero slope at insulated boundary.
(c) Linear in material B.
(d) Slope change $= k_B/k_A = 2$ at interface.
The temperature distribution in the water is characterized by
(e) Large gradients near the surface.



Heat Diffusion Equation – The Heated Slab Case (Example)

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m³, $k_A = 75$ W/m · K, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150$ W/m · K and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°C$ and $h = 1000$ W/m² · K.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

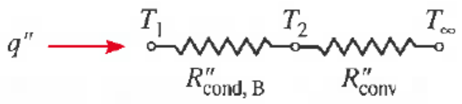


<u>Assumption 1</u>: incompressible medium (solid)
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* independent of *T*
<u>Assumption 5</u>: steady state

$$q'' = h(T_2 - T_\infty) \qquad \dot{q}L_A = q''$$

$$T_2 = T_\infty + \frac{\dot{q}L_A}{h}$$

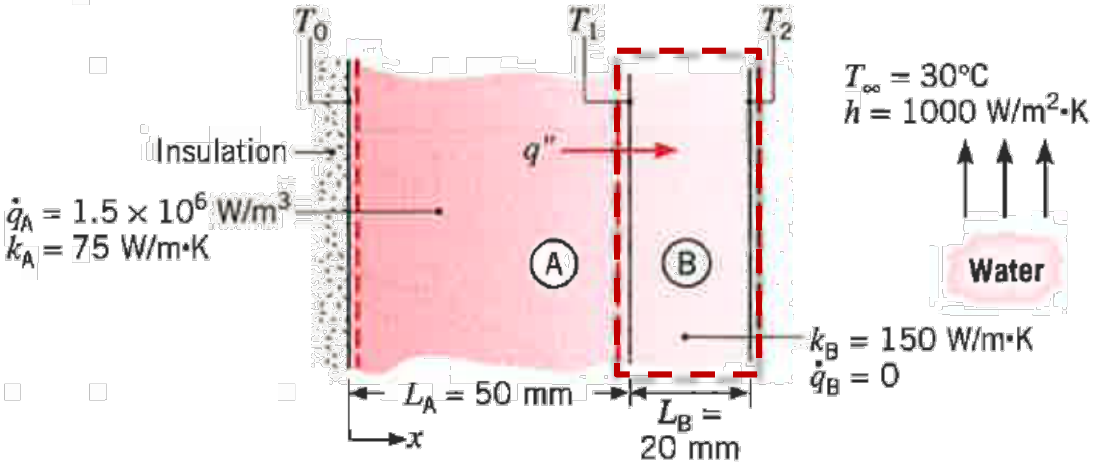
$$T_2 = 30°C + \frac{1.5 \times 10^6 \text{ W/m}^3 \times 0.05 \text{ m}}{1000 \text{ W/m}^2 \cdot \text{K}} = 105°C$$



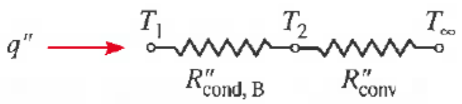
Heat Diffusion Equation – The Heated Slab Case (Example)

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m^3, $k_A = 75$ W/m · K, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150$ W/m · K and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°$C and $h = 1000$ W/m^2 · K.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.



<u>Assumption 1</u>: incompressible medium (solid)
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* independent of *T*
<u>Assumption 5</u>: steady state



$$T_1 = T_\infty + (R''_{cond,B} + R''_{conv})\, q''$$

$$R''_{cond,B} = \frac{L_B}{k_B} \qquad R''_{conv} = \frac{1}{h}$$

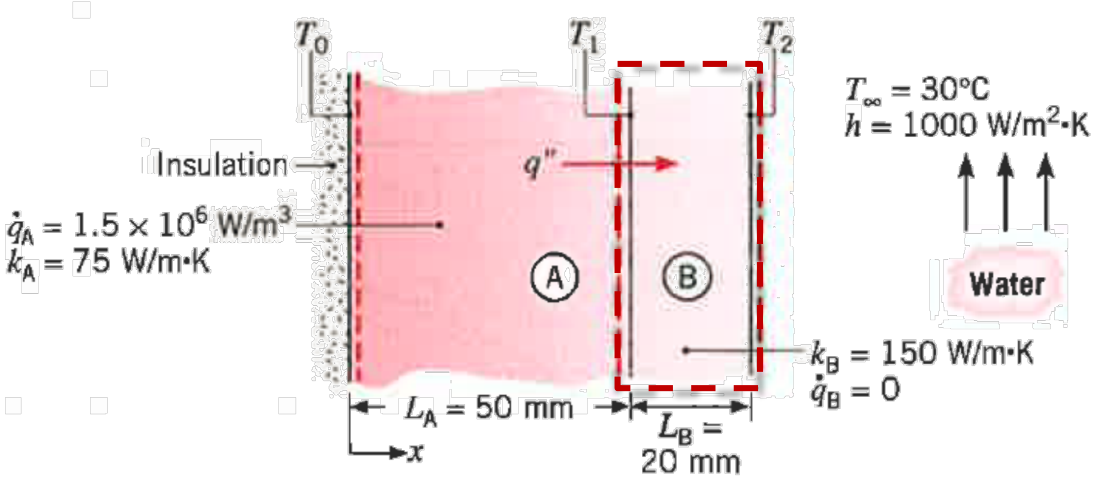
$$\frac{0.02\ \text{m}}{150\ \text{W/m}\cdot\text{K}} \qquad \frac{1}{1000\ \text{W/m}^2\cdot\text{K}}$$

$$R''_{cond,B} = 7.5R''_{conv}$$

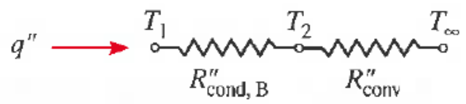
Heat Diffusion Equation – The Heated Slab Case (Example)

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6\ \mathrm{W/m^3}$, $k_A = 75\ \mathrm{W/m \cdot K}$, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150\ \mathrm{W/m \cdot K}$ and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°\mathrm{C}$ and $h = 1000\ \mathrm{W/m^2 \cdot K}$.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.



Assumption 1: incompressible medium (solid)
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material
Assumption 4: *k* independent of *T*
Assumption 5: steady state



$$T_1 = T_\infty + (R''_{cond,B} + R''_{conv})\, q''$$

$$T_1 = 30°\mathrm{C} + 85°\mathrm{C} = 115°\mathrm{C}$$

$$T_0 = \frac{\dot{q} L_A^2}{2 k_A} + T_1$$

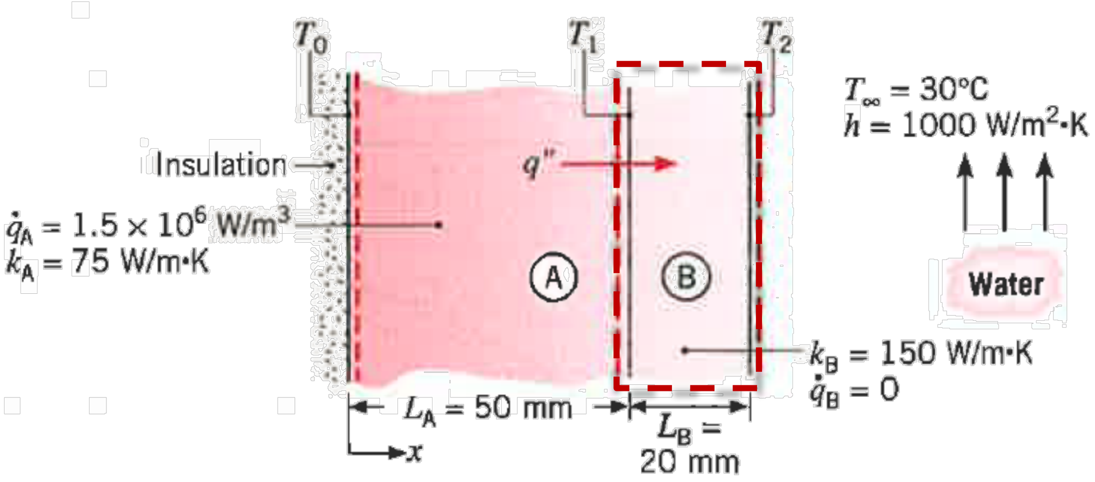
$$T_0 = 25°\mathrm{C} + 115°\mathrm{C} = 140°\mathrm{C}$$

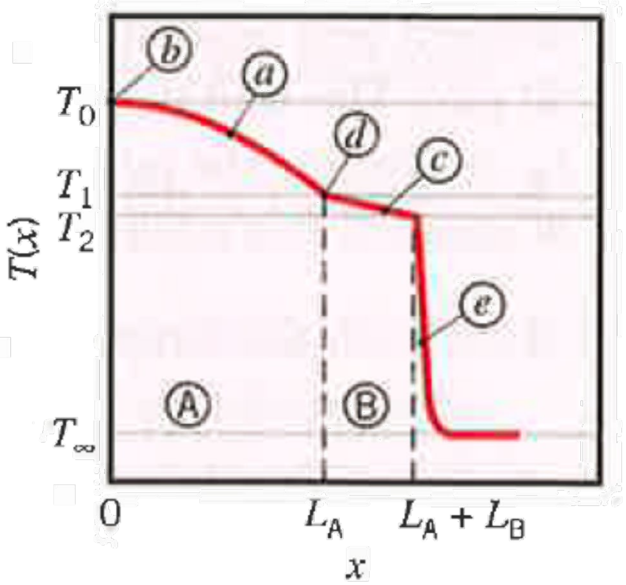
Heat Diffusion Equation – The Heated Slab Case (Example)

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m³, $k_A = 75$ W/m · K, and thickness $L_A =$ 50 mm. The wall material B has no generation with $k_B = 150$ W/m · K and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°C$ and $h = 1000$ W/m² · K.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

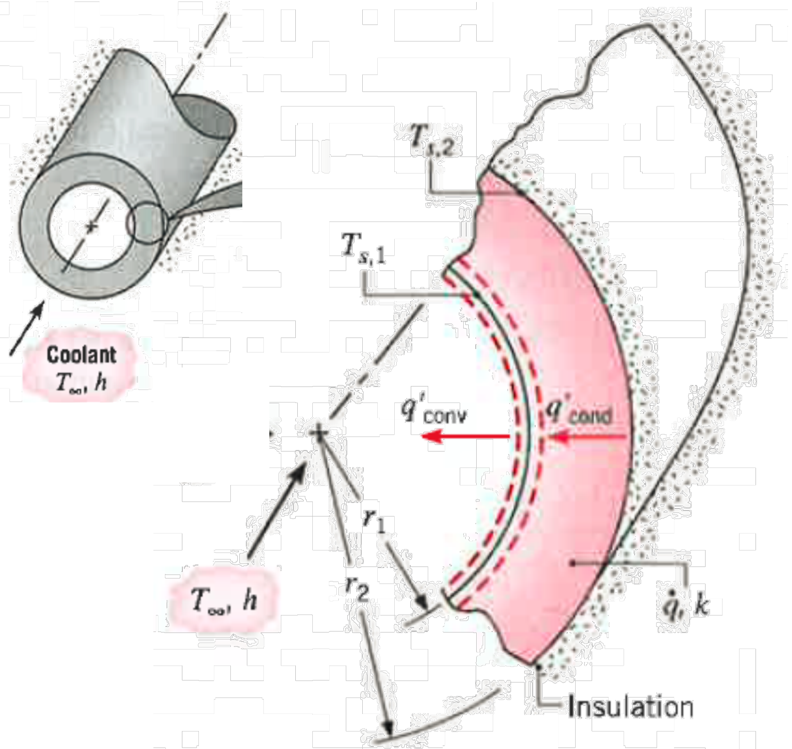
<u>Assumption 1</u>: incompressible medium (solid)
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* independent of *T*
<u>Assumption 5</u>: steady state





Heat Diffusion Equation – The Heated Cylinder (Example)

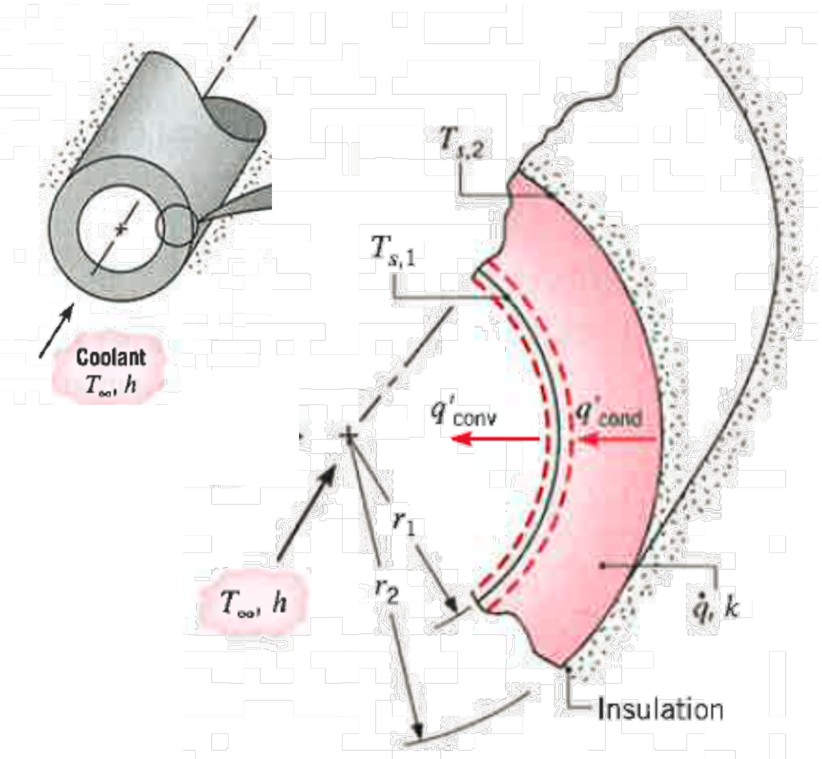
Consider a long solid tube, insulated at the outer radius r_2 and cooled at the inner radius r_1, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.



1. Obtain the general solution for the temperature distribution in the tube.
2. In a practical application a limit would be placed on the maximum temperature that is permissible at the insulated surface ($r = r_2$). Specifying this limit as $T_{s,2}$, identify appropriate boundary conditions that could be used to determine the arbitrary constants appearing in the general solution. Determine these constants and the corresponding form of the temperature distribution.
3. Determine the heat removal rate per unit length of tube.
4. If the coolant is available at a temperature T_∞, obtain an expression for the convection coefficient that would have to be maintained at the inner surface to allow for operation at prescribed values of $T_{s,2}$ and $\dot{q}$.

Heat Diffusion Equation – The Heated Cylinder (Example)

Consider a long solid tube, insulated at the outer radius r_2 and cooled at the inner radius r_1, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.



Assumption 1: incompressible medium (solid)
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material
Assumption 4: *k* independent of *T*
Assumption 5: steady state

Assumption 6: uniform volumetric heat generation
Assumption 7: outer surface is adiabatic (no heat flux)

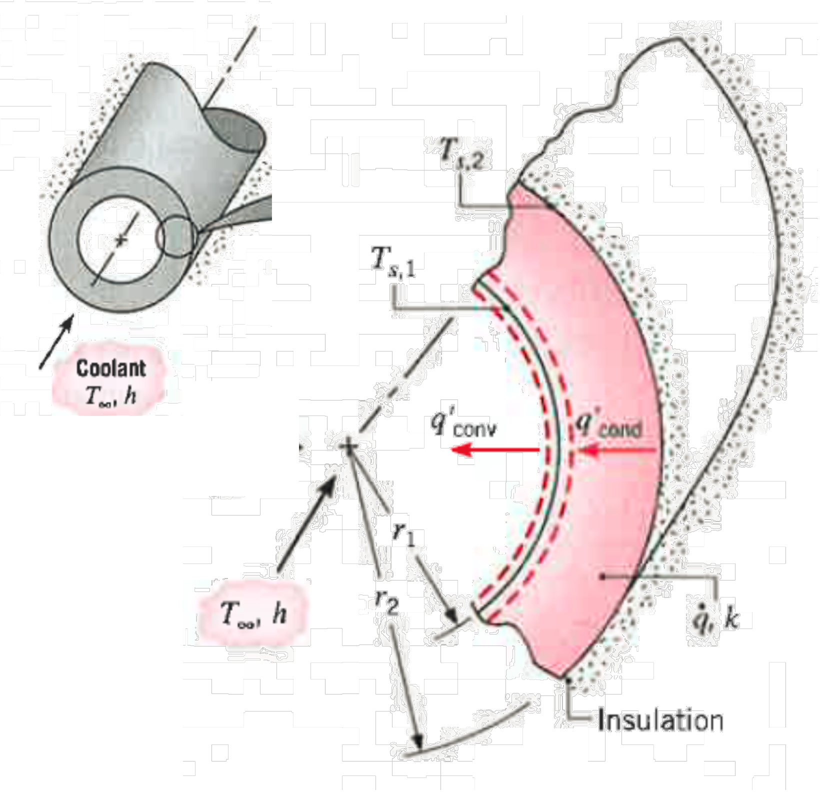
1. Obtain the general solution for the temperature distribution in the tube.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = -\frac{\dot{q}}{k} \qquad T(r) = -\frac{\dot{q}}{4k}r^2 + C_1\ln(r) + C_2$$

Boundary Conditions: $T(r_2) = T_{s,2}$ $\qquad \left.\frac{dT}{dr}\right|_{r_2} = 0$

Heat Diffusion Equation – The Heated Cylinder (Example)

Consider a long solid tube, insulated at the outer radius r_2 and cooled at the inner radius r_1, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.



2. In a practical application a limit would be placed on the maximum temperature that is permissible at the insulated surface ($r = r_2$). Specifying this limit as $T_{s,2}$, identify appropriate boundary conditions that could be used to determine the arbitrary constants appearing in the general solution. Determine these constants and the corresponding form of the temperature distribution.

$$T(r) = -\frac{\dot{q}}{4k} r^2 + C_1 \ln(r) + C_2$$

Boundary Conditions: $T(r_2) = T_{s,2}$ $\quad \left.\frac{dT}{dr}\right|_{r_2} = 0$

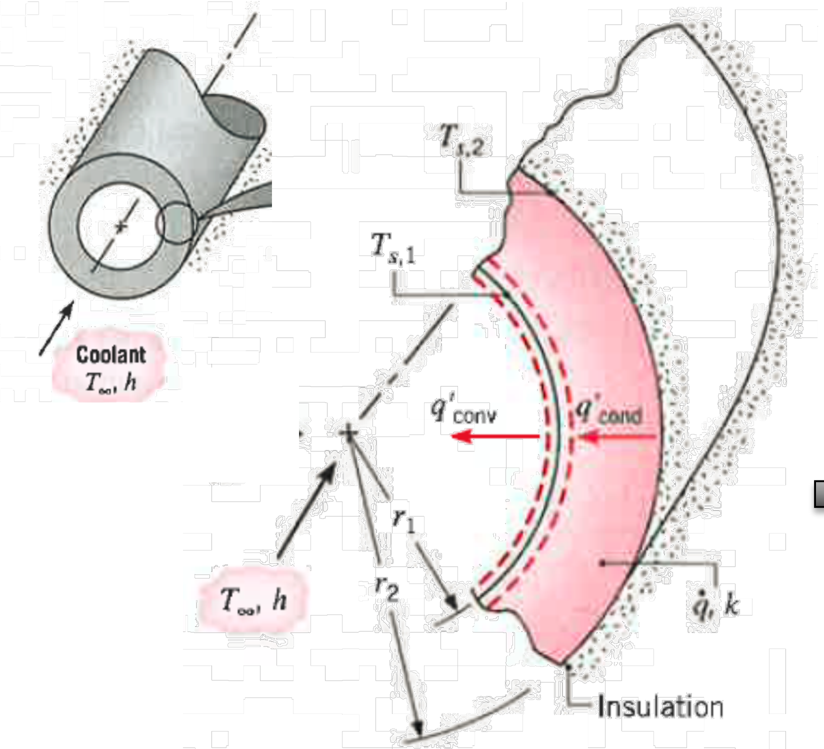
<u>Assumption 7</u>: outer surface is adiabatic (no heat flux)

$$C_1 = \frac{\dot{q}}{2k} r_2^2 \qquad C_2 = T_{s,2} + \frac{\dot{q}}{4k} r_2^2 - \frac{\dot{q}}{2k} r_2^2 \ln r_2$$

$$T(r) = T_{s,2} + \frac{\dot{q}}{4k}(r_2^2 - r^2) - \frac{\dot{q}}{2k} r_2^2 \ln \frac{r_2}{r}$$

Heat Diffusion Equation – The Heated Cylinder (Example)

Consider a long solid tube, insulated at the outer radius r_2 and cooled at the inner radius r_1, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.



3. Determine the heat removal rate per unit length of tube.

$$T(r) = T_{s,2} + \frac{\dot{q}}{4k}(r_2^2 - r^2) - \frac{\dot{q}}{2k} r_2^2 \ln\frac{r_2}{r}$$

$$Q_r = -kA(r)\frac{dT}{dr} = -k(\mathbf{2\pi r L})\frac{dT}{dr}$$

$$q'_r(r) = \frac{Q_r}{L} = -k(\mathbf{2\pi r})\frac{dT}{dr}$$

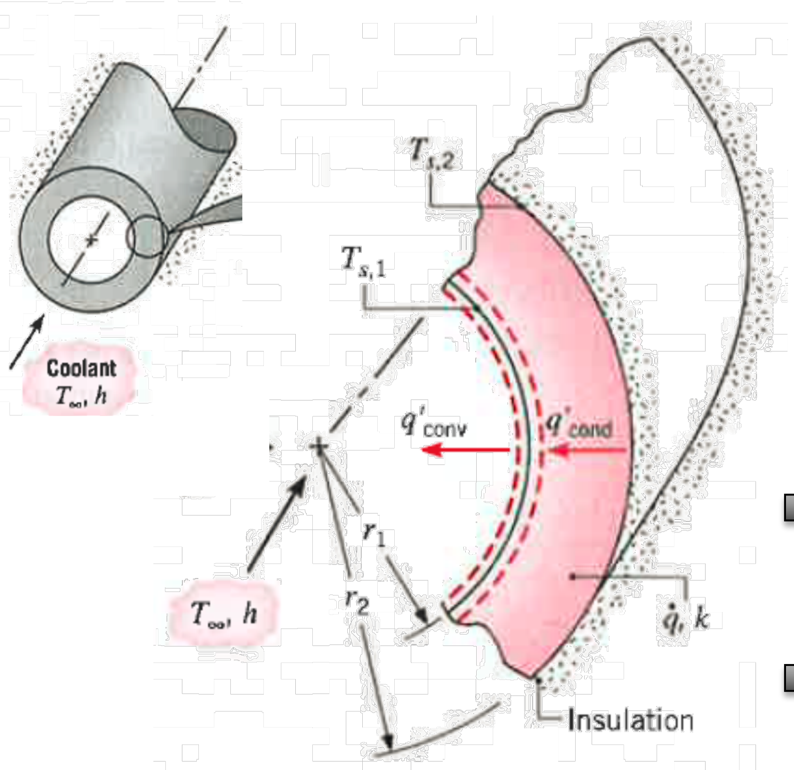
$$\Rightarrow \quad q'_r(r_1) = -k2\pi r_1\left(-\frac{\dot{q}}{2k}r_1 + \frac{\dot{q}}{2k}\frac{r_2^2}{r_1}\right) = -\pi\dot{q}(r_2^2 - r_1^2)$$

Sanity check: we know that $\left.\frac{dT}{dr}\right|_{r_2} = 0 \Rightarrow q'_r(r_2) = 0$

$\dot{E}_g = \dot{q}\pi(r_2^2 - r_1^2)L$ *heat generated equal the heat removed OK*

Heat Diffusion Equation – The Heated Cylinder (Example)

Consider a long solid tube, insulated at the outer radius r_2 and cooled at the inner radius r_1, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.



4. If the coolant is available at a temperature T_∞, obtain an expression for the convection coefficient that would have to be maintained at the inner surface to allow for operation at prescribed values of $T_{s,2}$ and $\dot{q}$.

$$q'_r(r_1) = -k2\pi r_1\left(-\frac{\dot{q}}{2k}r_1 + \frac{\dot{q}}{2k}\frac{r_2^2}{r_1}\right) = -\pi\dot{q}(r_2^2 - r_1^2)$$

$$q'_{conv} = h(2\pi r_1)(T_{s,1} - T_\infty) = |q'_{cond}(r_1)|$$

$$h = \frac{\dot{q}(r_2^2 - r_1^2)}{2r_1(T_{s,1} - T_\infty)}$$

We will see later that h is related to the velocity of the fluid so this would be our free parameter. However, pressure drops can become too large and hence a cooler fluid might be needed

Until Now

- ☑ Heat Diffusion and Boundary Conditions (W1L2-3)
- ☑ Heat Diffusion Equation without Heat sources (W1L3-4; W2L1)
  - ☑ Thermal Resistance & Overall Heat Transfer Coefficient
  - ☑ Bi number
  - ☑ Thermal Circuits
- ☑ Heat Diffusion WITH Heat Sources (W2L2-3)

Learning Objectives:

- ☑ Solve 1D&2D steady state heat transfer problems with/without heat sources

Next Lectures

- ❑ Heat transfer from extended surfaces (Fins)
- ❑ Fins of uniform cross-section
- ❑ Fins Performance

Learning Objectives:

- ❑ Understand the concept of fins
- ❑ Calculate the heat transfer in fins of different shapes
- ❑ Calculate the performance of a fin-based system