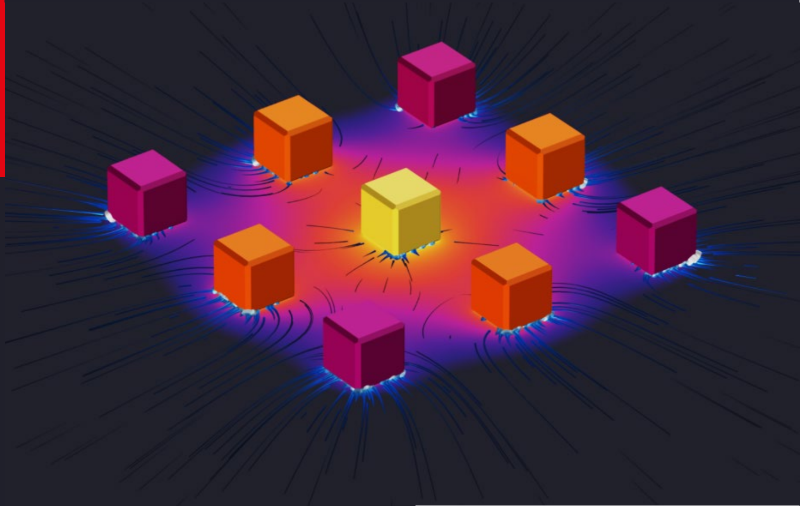
Heat and Mass Transfer
ME-341



Instructor: Giulia Tagliabue

École
polytechnique
fédérale
de Lausanne

Spring Semester

Previously

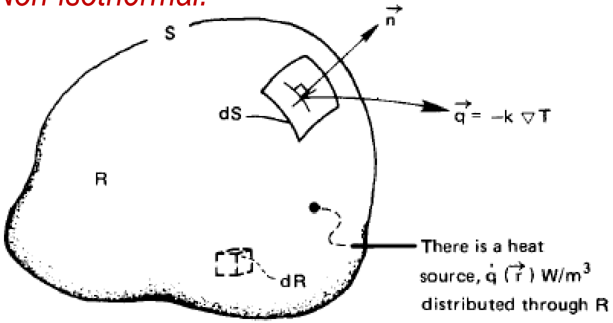
- ☑ Heat Diffusion Equation without Heat sources
  - ☑ Thermal Resistance & Overall Heat Transfer Coefficient
  - ☑ Bi number
  - ☑ Thermal Circuits

Learning Objectives:

- ☑ Solve 1D steady state heat transfer problems without heat sources

Heat Diffusion Equation – 3D

Non-isothermal!



$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

To solve the equation we need:

- Initial condition: $\mathrm{T(t = 0)} = T_i(x, y, z)$
- Boundary conditions

<u>Assumption 1</u>: incompressible medium
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* is independent of *T*

<u>Assumption 5</u>: steady-state ($\boldsymbol{\partial/\partial t = 0}$)

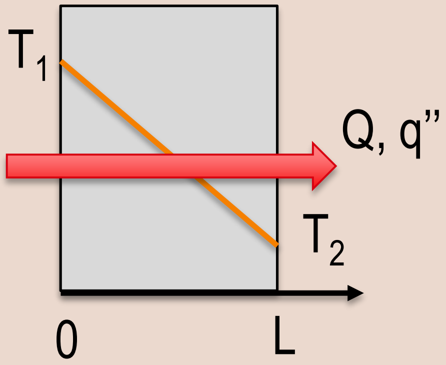
<u>Assumption 6</u>: no heat sources ($\boldsymbol{\dot{q} = 0}$)

$$\boldsymbol{\nabla^2 T = 0}$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources, Dirichlet's BC

Planar Wall

$$T(x) = \frac{T_2 - T_1}{L} x + T_1$$

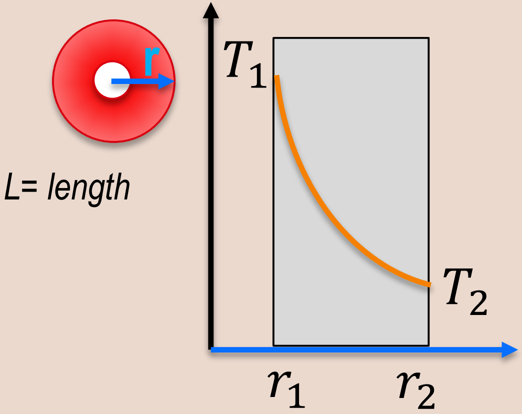


$$Q = -kA\frac{dT}{dx} = -\frac{kA}{L}(T_2 - T_1) = const$$

$$q'' = Q/A = const$$

Radial System

$$T(r) = \frac{T_1 - T_2}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_2$$

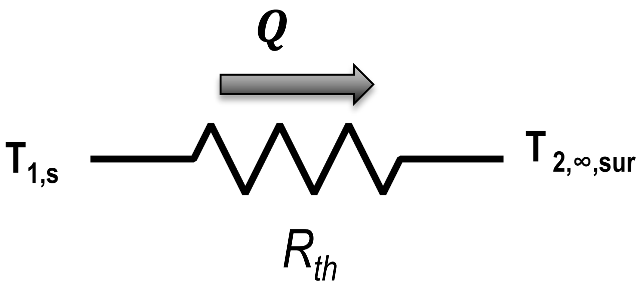


$$Q = -kA\frac{dT}{dr} = -\frac{k(2\pi \cancel{r} l)}{\ln(r_1/r_2)} \frac{(T_2 - T_1)}{\cancel{r}} = const$$

$$q = Q/(2\pi r l) = q''(r) \neq const$$

Thermal Resistance

$$Q = \frac{\left(T_{1,s} - T_{2,\infty sur}\right)}{R_{th}}$$

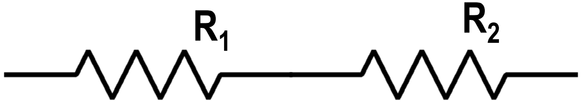


	Planar Wall	Radial System
Conduction:	$\frac{L}{kA}$	$\frac{ln(r_2/r_1)}{2\pi Lk}$
Convection:	$\frac{1}{hA}$	$\frac{1}{h2\pi rL}$
Radiation:	$\frac{1}{h_{rad}A}$	$\frac{1}{h_{rad}2\pi rL}$

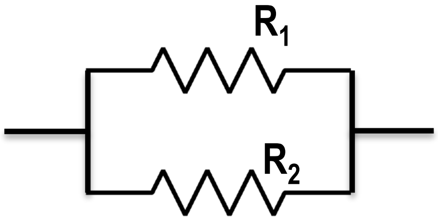
$$q" = \frac{Q}{A} = \frac{\left(T_{1,s} - T_{2,\infty sur}\right)}{AR_{th}} = \frac{\left(T_{1,s} - T_{2,\infty sur}\right)}{R"_{th}} \longrightarrow R^{"}_{th} = R_{th}A$$

$$q' = \frac{Q}{L} = \frac{\left(T_{1,s} - T_{2,\infty sur}\right)}{lR_{th,cyl}} = \frac{\left(T_{1,s} - T_{2,\infty sur}\right)}{R'_{th,cyl}} \longrightarrow R'_{th,cyl} = R_{th,cyl}L$$

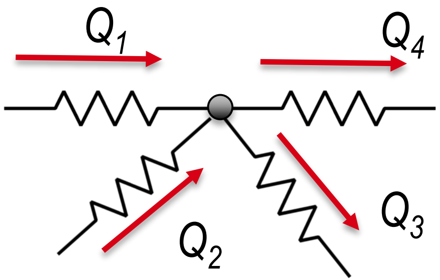
Thermal Circuits



$$R_{series} = R_1 + R_2 + .. = \sum R_i$$



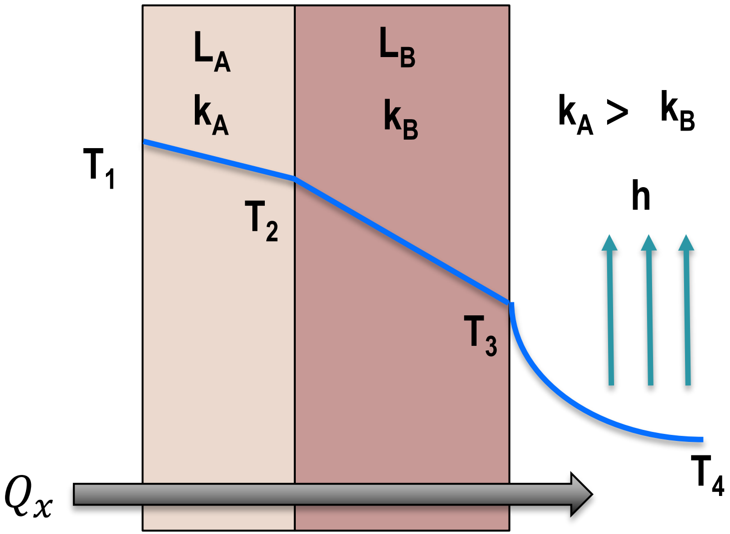
$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + .. = \sum \frac{1}{R_i}$$

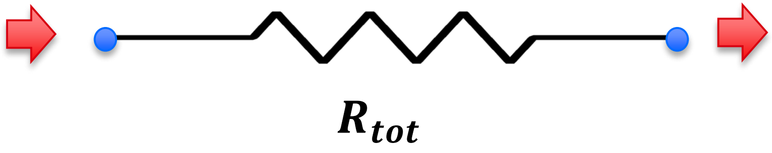


Kirchoff's 1st Law:

$$\sum Q_i = 0 \qquad \begin{array}{c} Q_i > 0\ enters \\ Q_i < 0\ exits \end{array}$$

Overall Heat Transfer Coefficient





$A = 4m^2$

$T_1 = 900C = 1173K$
$T_4 = 10C = 283K$

$L_A = 10\ cm = 0.1m$
$L_B = 50\ \text{cm} = 0.5\text{m}$

$k_A = 100\ {}^{W}/_{mK}$
$k_B = 0.1\ {}^{W}/_{mK}$

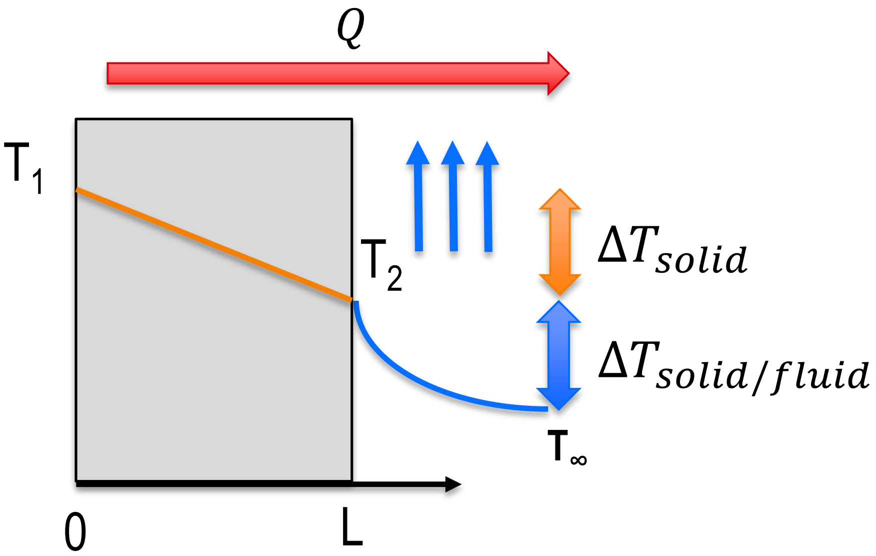
$h = 50\ {}^{W}/_{m^2K}$

$$\boldsymbol{Q_x} = \frac{T_1 - T_4}{R_{tot,series}} = \boldsymbol{UA(T_1 - T_4)}$$

***U* = Overall heat transfer coefficient**

$$U = \frac{1}{R_{tot,series}A} = \frac{1}{R''_{tot,series}}$$

Biot Number



$$Q = \frac{(T_1 - T_2)}{R_{th,cond}} = \frac{(T_2 - T_\infty)}{R_{th,conv}}$$

$$\Rightarrow \frac{(T_1 - T_2)}{(T_2 - T_\infty)} = \frac{R_{th,cond}}{R_{th,conv}} \equiv Bi$$

$$\Rightarrow \boldsymbol{Bi \equiv \frac{hL}{k}}$$

Note: L can be generalized to be a characteristic dimension of a body (e.g. diameter of a sphere)

This Lecture

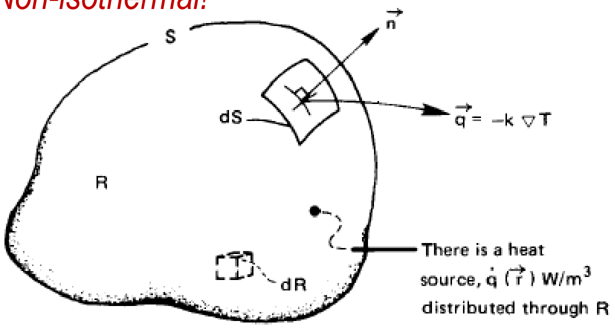
- ❑ 1D steady-state conduction with heat sources

Learning Objectives:

- ❑ Solve 1D steady state heat conduction problems in different geometries, with heat sources

Heat Diffusion Equation – From 3D to 1D, stead state, WITH heat sources

Non-isothermal!



Assumption 1: incompressible medium
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material
Assumption 4: k is independent of T

$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

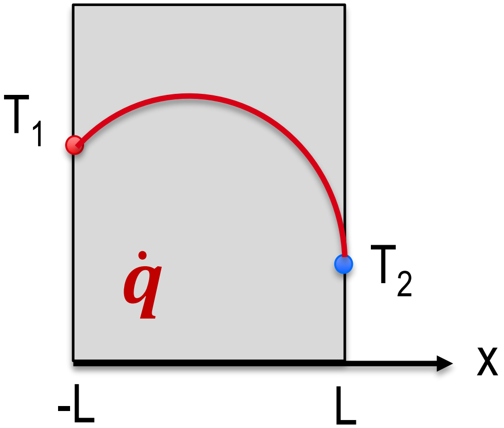
Assumption 5: steady-state ($\partial/\partial t = 0$)

$$\boldsymbol{\nabla^2 T + \frac{\dot{q}}{k} = 0}$$

1D, Cartesian coordinates: $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\dot{q}}{k} = 0$

1D, Cylindrical coordinates: $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) + \dfrac{\dot{q}}{k} = 0$

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = 0 \quad \Rightarrow \quad T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

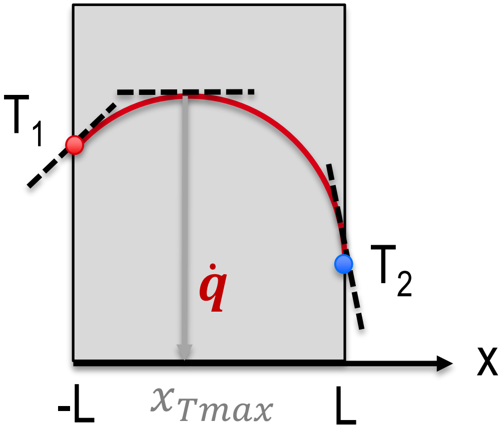
Temperature boundary condition (BC)

$$T(x=-L) = T_1 \qquad T(x=L) = T_2$$

PARABOLIC T(x) Profile

$$\Rightarrow \begin{cases} C_1 = \dfrac{T_2 - T_1}{2L} \\ C_2 = \dfrac{\dot{q}}{2k}L^2 + \dfrac{T_1 + T_2}{2} \end{cases} \quad \Rightarrow \quad T(x) = \frac{\dot{q}}{2k}L^2\left(1 - \frac{\boldsymbol{x^2}}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k}L^2\left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

$$T_{max} \rightarrow \frac{dT}{dx} = 0 \qquad \frac{dT}{dx} = -\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L}$$

$$x_{Tmax} = \frac{k}{\dot{q}}\frac{T_2 - T_1}{2L}$$

$$x_{Tmax} < 0 \; if \; (T_2 - T_1) < 0$$

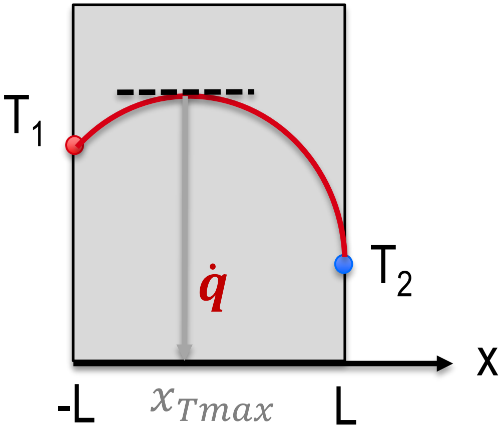
$$x_{Tmax} > 0 \; if \; (T_2 - T_1) > 0$$

$$x_{Tmax} = 0 \; if \; (T_2 - T_1) = 0$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L} > 0 \qquad x < x_{Tmax}$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L} < 0 \qquad x > x_{Tmax}$$

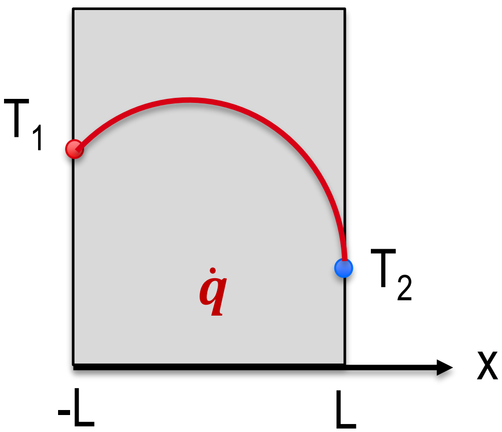
Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{\boldsymbol{x^2}}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

$$Q = -kA\frac{\partial T}{\partial x} = -kA\left(-\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L}\right) = Q(x)$$

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{\boldsymbol{x^2}}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

$$Q = -kA\frac{\partial T}{\partial x} = -kA\left(-\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L}\right) = Q(x)$$

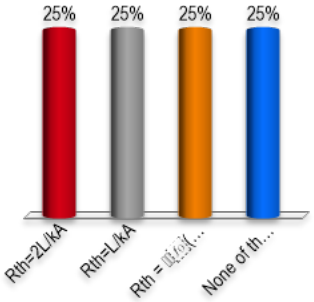
What is the expression for the thermal resistance in this layer?

A. R_{th}=2L/kA

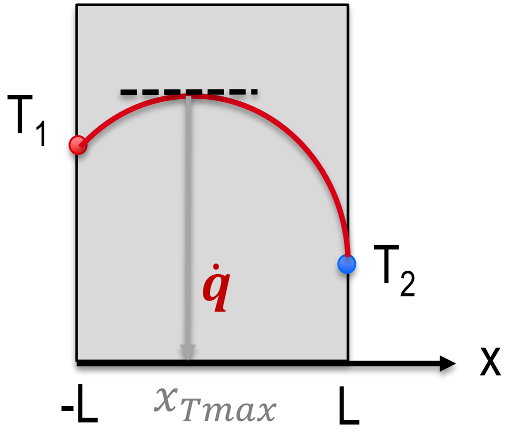
B. R_{th}=L/kA

C. R_{th} = ln(2L)/2πk

D. None of the above



Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{\boldsymbol{x^2}}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

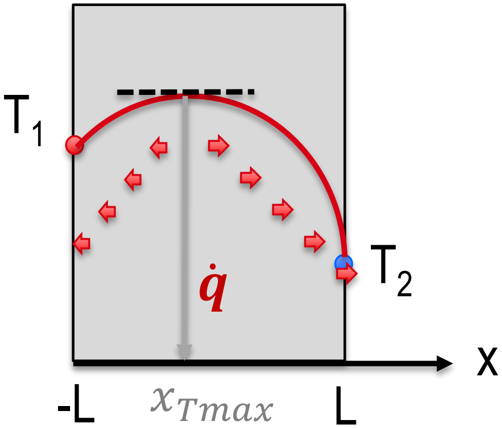
$$Q = -kA\frac{\partial T}{\partial x} = -kA\left(-\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L}\right) = Q(x)$$

The heat transfer rate is not constant!
It depends on the position along x.

$$Q(x) \neq \frac{\Delta T}{R_{th}}$$

The electrical analogy fails!
We cannot use the thermal resistance concept in layers with heat sources.

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

$$Q(x) = -kA\frac{\partial T}{\partial x} = -kA\left(-\frac{\dot{q}}{k}x + \frac{T_2 - T_1}{2L}\right)$$

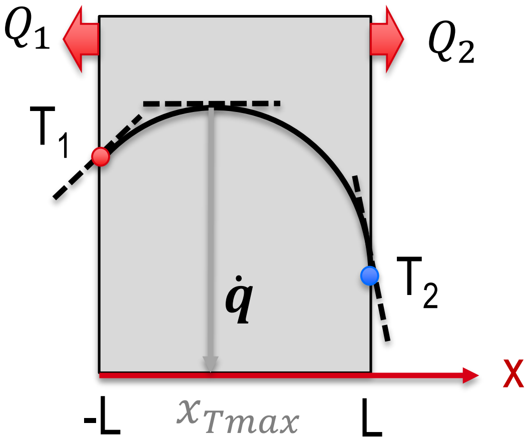
$$Q(x_{Tmax}) = 0$$

The heat transfer rate follows the temperature gradient, which has opposite signs on the right and left of x_{Tmax}. The heat generated on the right of x_{Tmax} will flow towards the right surface. The heat generated on the left side will flow towards the left side.

$$Q_{right} = \int_{X_{Tmax}}^{L} \dot{q}A dx = \dot{q}A[L - x_{Tmax}]$$

$$Q_{left} = \int_{-L}^{X_{Tmax}} \dot{q}A dx = \dot{q}A[x_{Tmax} + L]$$

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

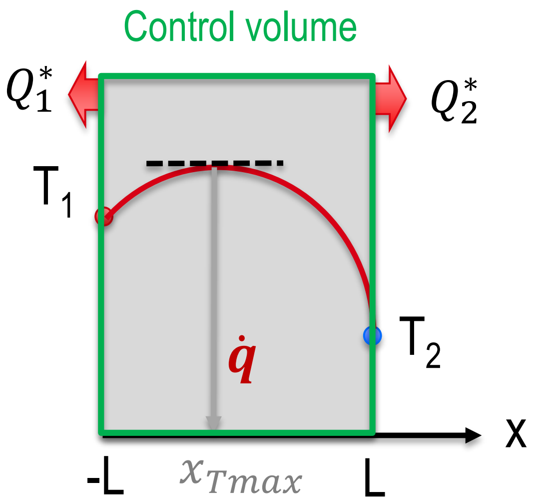
$$Q(x < x_{Tmax}) < 0 \qquad Q(-L) = -kA\left(\frac{\dot{q}}{k}L + \frac{T_2 - T_1}{2L}\right) < 0$$

$$Q(x > x_{Tmax}) > 0 \qquad Q(L) = kA\left(\frac{\dot{q}}{k}L + \frac{T_1 - T_2}{2L}\right) > 0$$

At $x = L$ the heat transfer rate Q_2 is positive and hence in the same direction as our x-axis.

At $x = -L$ the heat transfer rate Q_1 is negative. Therefore it flow in the opposite direction compared to our x-axis.

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



1st Law of Thermodynamics: $0 = Q - W + \dot{E}_{gen} - \cancel{mc\frac{dT}{dt}}$

$$0 = Q_1^* + Q_2^* + \dot{q}2LA$$

$Q_1^* < 0$

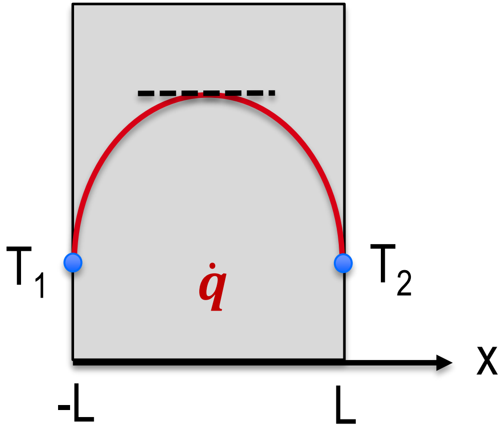
$\boldsymbol{Q_2 < 0}$

When writing the 1st law of thermodynamics the signs refer to the direction of the flow with respect to our control volume. Hence both Q_1^* and Q_2^* must be negative values because the heat is leaving the wall.

$$|Q_1^*| = |Q_1| \qquad |Q_2^*| = |Q_2|$$

Be careful to the signs, use physical intuition to check!

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k}L^2\left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

If $T_1 = T_2$ then the T profile must be symmetric and $x_{Tmax} = 0$

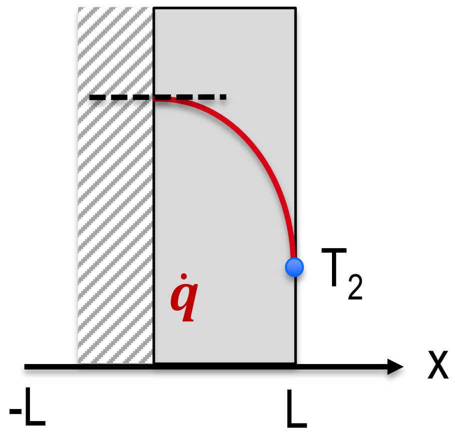
$$T(x) = \frac{\dot{q}}{2k}L^2\left(1 - \frac{x^2}{L^2}\right) + T_2 \qquad T_{max} = \frac{\dot{q}}{2k}L^2 + T_2$$

$$q''_{x=-L} = -k\left(\frac{\partial T}{\partial x}\right)_{x=-L} = -k\left(-\frac{\dot{q}}{k}x\right)_{x=-L} = -\dot{q}L$$

$$q''_{x=L} = -k\left(\frac{\partial T}{\partial x}\right)_{x=L} = -k\left(-\frac{\dot{q}}{k}x\right)_{x=L} = +\dot{q}L$$

$$q''_{out} = |q''_1| + |q''_2| = 2\dot{q}L \qquad e_{gen} = \dot{q}2L$$

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

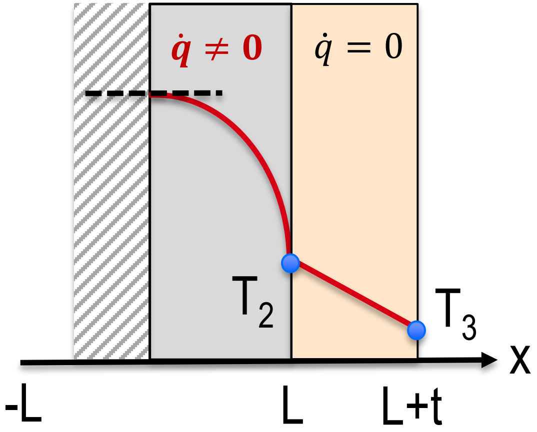
If $T_1 = T_2$ then the T profile must be symmetric and $x_{Tmax} = 0$

$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{x^2}{L^2}\right) + T_2 \qquad T_{max} = \frac{\dot{q}}{2k} L^2 + T_2$$

$$q''_{x=0} = -k\left(\frac{\partial T}{\partial x}\right)_{x=0} = -k\left(-\frac{\dot{q}}{k}x\right)_{x=0} = 0$$

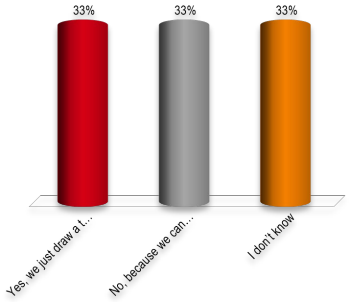
At the centerline of the wall effectively the heat flux is zero, satisfying the symmetry of the problem.
This is equivalent to having a perfectly insulated boundary at x=0.

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{\boldsymbol{x^2}}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$

Can we draw an equivalent electrical circuit for this problem?

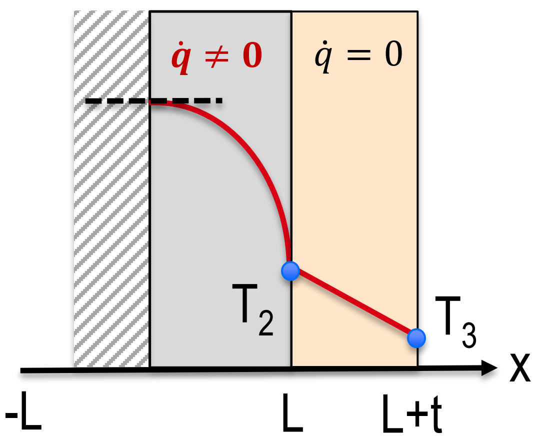


A. Yes, we just draw a thermal resistance for each layer

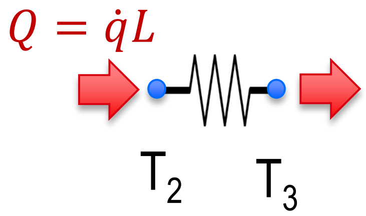
B. No, because we cannot define the thermal resistances everywhere

C. I don’t know

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



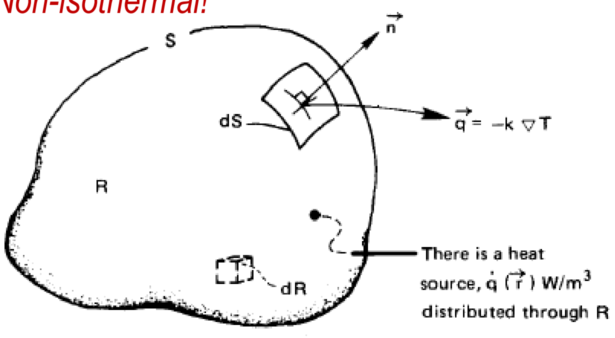
$$T(x) = \frac{\dot{q}}{2k} L^2 \left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2}$$



We can draw an equivalent electrical circuit only for the layers WITHOUT heat sources. Layers WITH heat sources inject into the equivalent circuit a certain Q

Heat Diffusion Equation – From 3D to 1D, stead state, WITH heat sources

Non-isothermal!



$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Assumption 5: steady-state ($\boldsymbol{\partial/\partial t = 0}$)

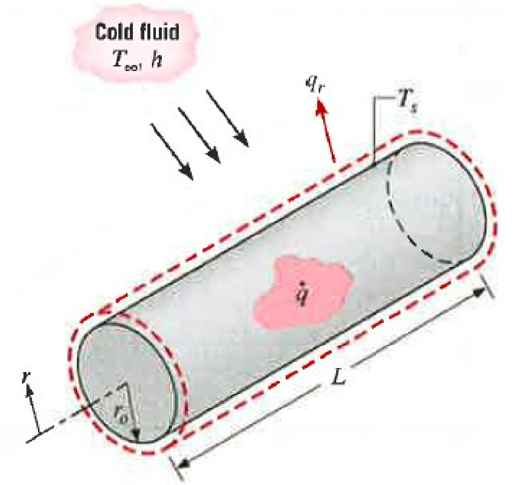
$$\boldsymbol{\nabla^2 T + \frac{\dot{q}}{k} = 0}$$

Assumption 1: incompressible medium
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material
Assumption 4: *k* is independent of *T*

1D, Cartesian coordinates: $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\dot{q}}{k} = 0$

1D, Cylindrical coordinates: $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) + \dfrac{\dot{q}}{k} = 0$

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



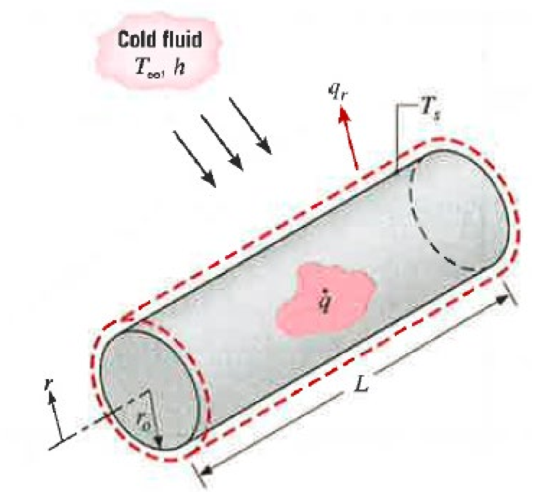
$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = -\frac{\dot{q}}{k} \Rightarrow T(r) = -\frac{\dot{q}}{4k}r^2 + C_1\ln(r) + C_2$$

Boundary Conditions:

At the surface we impose a temperature BC $T(r_0) = T_s$

For a solid cylinder, i.e. a wire, at the center we have to satisfy a symmetry condition. $\left.\frac{dT}{dr}\right|_{r=0} = 0$

Heat Diffusion Equation – 1D, steady-state, WITH heat sources



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = -\frac{\dot{q}}{k} \quad ➡ \quad T(r) = -\frac{\dot{q}}{4k}r^2 + C_1\ln(r) + C_2$$

Boundary Conditions: $T(r_0) = T_s$ $\quad \left.\frac{dT}{dr}\right|_{r=0} = 0$

$$\begin{cases} C_1 = 0 \\ C_2 = \frac{\dot{q}r_0^2}{4k} + T_s \end{cases}$$

➡ $$T(r) = \frac{\dot{q}r_0^2}{4k}\left(1 - \frac{r^2}{r_0^2}\right) + T_s$$

➡ $$Q(r) = -kA\frac{\partial T}{\partial r}$$

➡ $$Q(r_0) = -k(2\pi r_0 L)\left(-\frac{\dot{q}r_0}{2k}\right) = \dot{q}\pi r_0^2 L = \dot{q}V$$

This Lecture

☑ 1D steady-state conduction with heat sources

Learning Objectives:

☑ Solve 1D steady state heat conduction problems in different geometries, with heat sources

Next Lecture

- ❑ 2D Conduction & Shape Factor

Learning Objectives:

- ❑ Approach simple 2D conduction problems