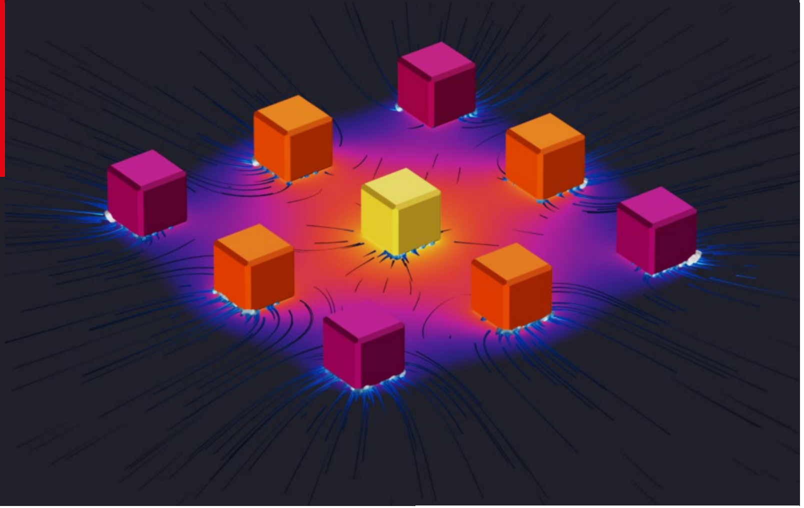
Heat and Mass Transfer
ME-341



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École
polytechnique
fédérale
de Lausanne

Spring Semester

Conductive Heat Transfer - 2

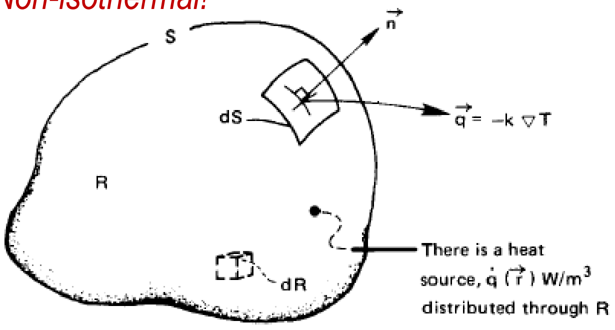
- ☑ Types of boundary conditions
- ☑ Planar and cylindrical (1D) solutions

Learning Objectives:

- ☑ Identify the possible boundary conditions
- ☑ Express mathematically the various boundary conditions
- ☑ Calculate the temperature profile in a planar or cylindrical wall

Heat Diffusion Equation – 3D

Non-isothermal!



$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

To solve the equation we need:

- Initial condition: $\mathrm{T(t=0)} = T_i(x, y, z)$
- Boundary conditions

Assumption 1: incompressible medium
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material
Assumption 4: *k* is independent of *T*

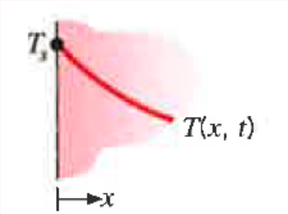
Assumption 5: steady-state ($\partial/\partial t = 0$)

Assumption 6: no heat sources ($\dot{q} = 0$)

$$\boldsymbol{\nabla^2 T = 0}$$

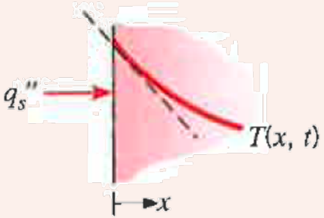
Heat Diffusion Equation – Boundary Conditions

B.C. of the 1st kind (*Dirichlet condition*):
constant surface temperature

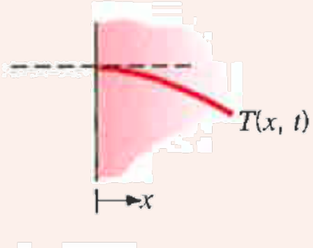


$$T(\vec{r})_{\vec{r}=\vec{r}_i} = T_w$$

B.C. of the 2nd kind (*Neumann condition*):
known heat flux



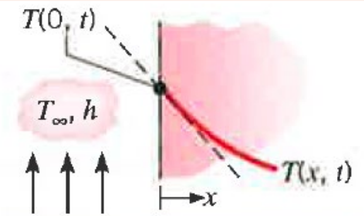
$$-k\left(\frac{\partial T}{\partial x}\right)_{x=x_i} = q_w''$$



$$-k\left(\frac{\partial T}{\partial x}\right)_{x=x_i} = 0$$

(adiabatic)

B.C. of the 3rd kind (*Robin condition*):
convection surface condition



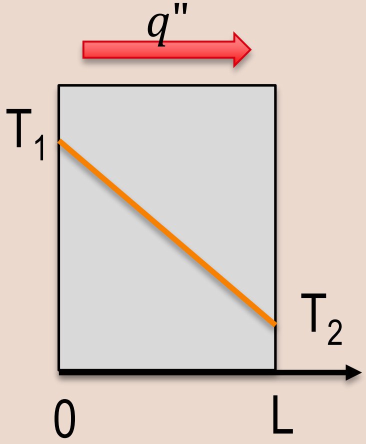
$$-k\left(\frac{\partial T}{\partial x}\right)_{x_i} = h(T(x_i, t) - T_\infty)$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources, Dirichlet's BC

Planar Wall

$$T(x) = \frac{T_2 - T_1}{L}x + T_1$$

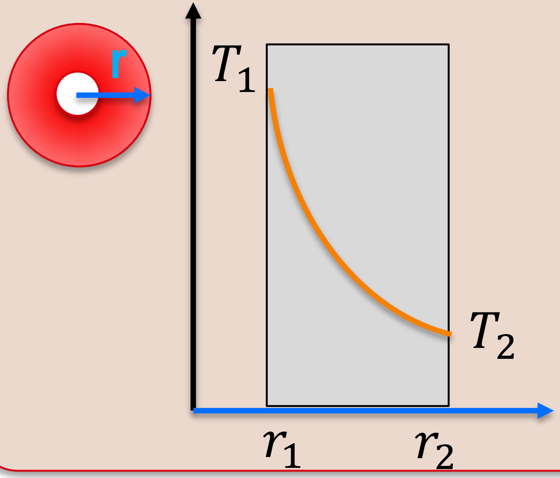
Linear !!



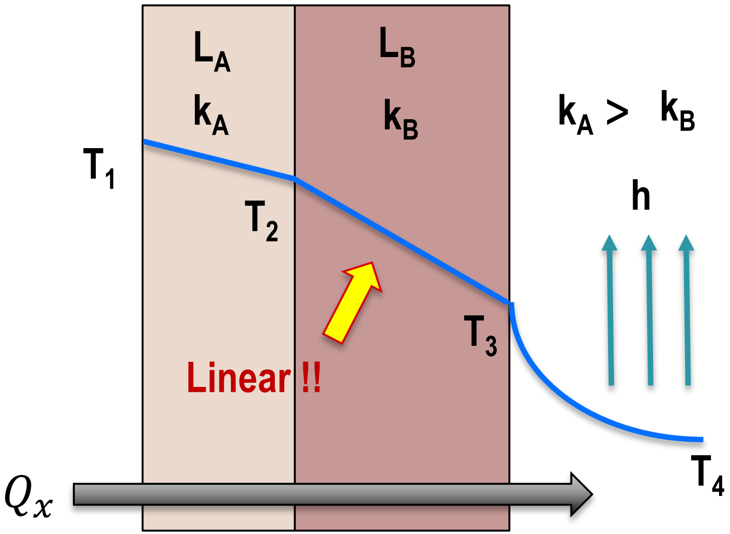
Radial System

$$T(r) = \frac{T_1 - T_2}{\ln(r_1/r_2)}\ln\left(\frac{r}{r_2}\right) + T_2$$

Logarithmic !!



Heat Diffusion - Example



$A = 4m^2$

$T_1 = 900C = 1173K$
$T_4 = 10C = 283K$

$h = 50 \, {}^{W}/_{m^2K}$

$L_A = 10 \ cm = 0.1m$
$L_B = 50 \text{ cm} = 0.5\text{m}$

$k_A = 100 \, {}^{W}/_{mK}$
$k_B = 0.1 \, {}^{W}/_{mK}$

- **Calculate Q**
- **Calculate T_3, T_2**
- **Draw the temperature profile**

$$Q_x = hA(T_3 - T_4) \qquad Q_x = {}^{k_A A}/_{L_A}(T_1 - T_2) \qquad Q_x = {}^{k_B A}/_{L_B}(T_2 - T_3)$$

Q = 708.6 W, T2 = 899.8 C, T3 = 13.54 C

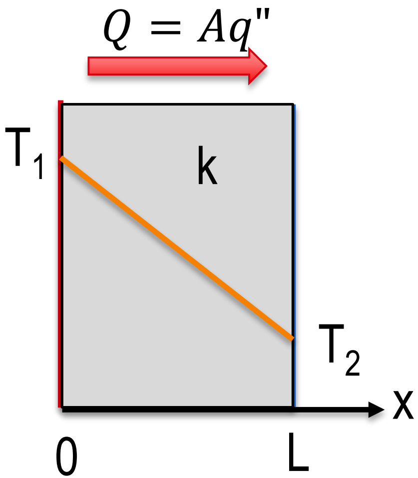
Conductive Heat Transfer - 3

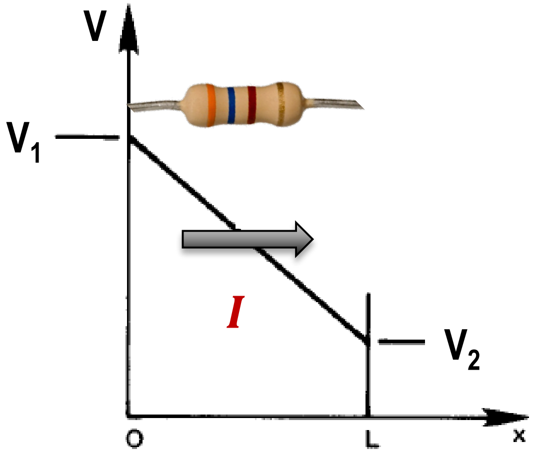
- ❑ Thermal resistance
- ❑ Bi Number
- ❑ Intro to Thermal Circuits

Learning Objectives:

- ❑ Calculate the thermal resistances
- ❑ Calculate the Bi number
- ❑ Solve 1D problems using thermal circuits

Thermal resistance

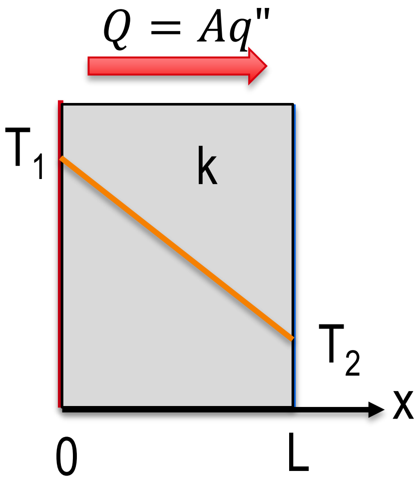


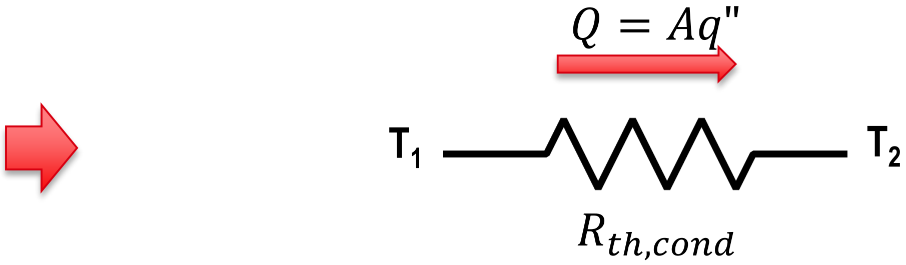


$$T(x) = \frac{T_2 - T_1}{L} x + T_1$$

$$Q = -\frac{kA}{L}(T_2 - T_1)$$

Thermal resistance – Planar Wall - Conduction



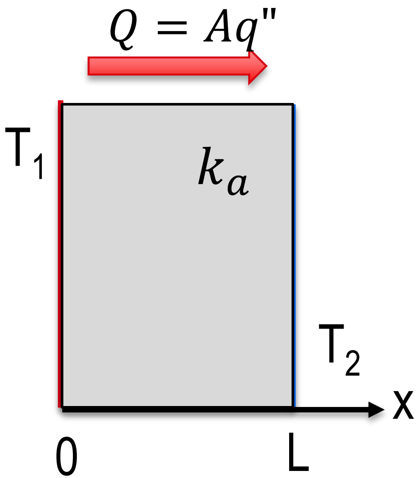


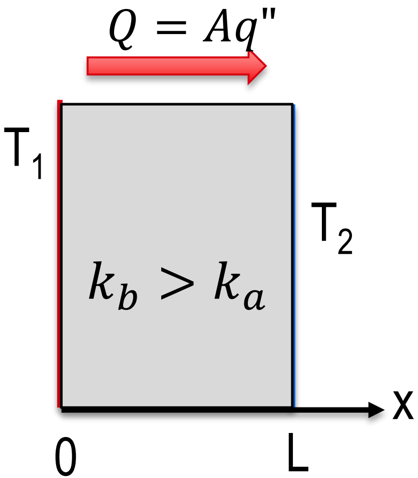
$$T(x) = \frac{T_2 - T_1}{L}x + T_1$$

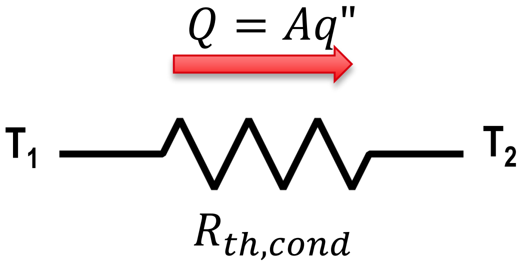
$$Q = -\frac{kA}{L}(T_2 - T_1) = \frac{(T_1 - T_2)}{R_{th,cond}}$$

$$R_{th,cond} = \frac{L}{kA} \qquad [\text{ K/W }]$$

Thermal resistance – Planar Wall - Conduction





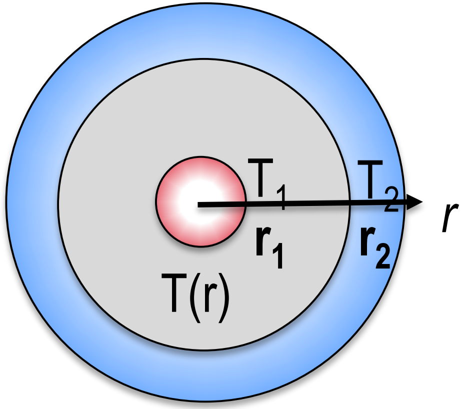


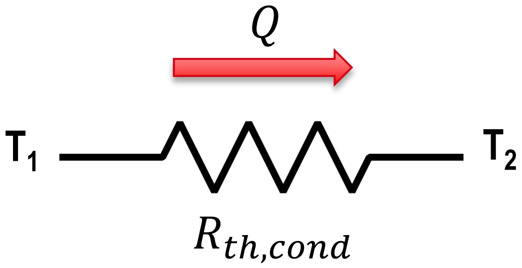
$$T(x) = \frac{T_2 - T_1}{L}x + T_1$$

$$Q = -\frac{kA}{L}(T_2 - T_1) = \frac{(T_1 - T_2)}{R_{th,cond}}$$

$$R_{th,cond} = \frac{L}{kA}$$ [K/W]

Thermal resistance – Cylindrical Pipe - Conduction

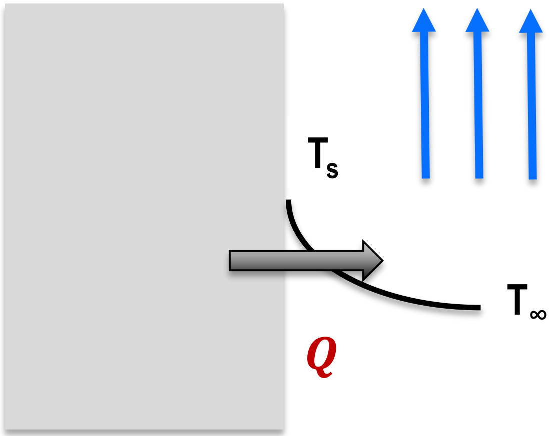


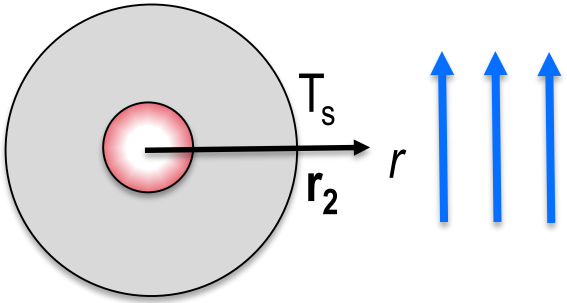


$$T(r) = \frac{T_1 - T_2}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_2$$

$$Q_r = \frac{2\pi Lk}{\ln(r_2/r_1)}(T_1 - T_2) = \frac{(T_1 - T_2)}{R_{th,cond-cyl}}$$

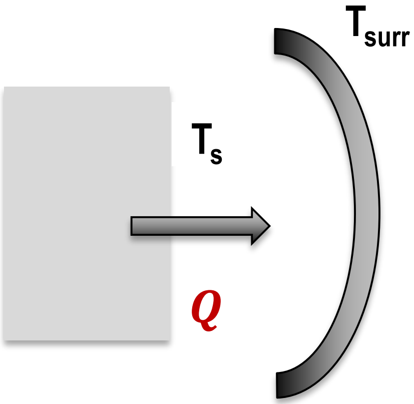
Thermal resistance – Convection

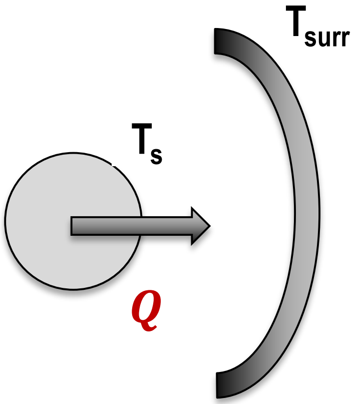




$$Q = hA(T_s - T_\infty)$$

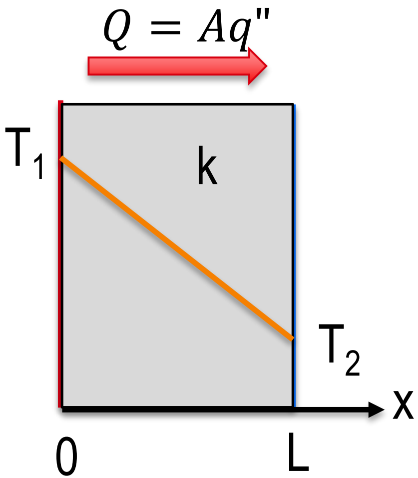
Thermal resistance –Radiation

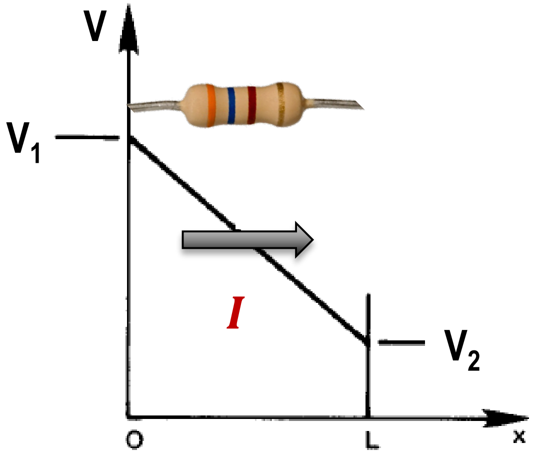




$$Q = A\varepsilon\sigma(T_s^4 - T_{surr}^4)$$

Thermal resistance





$$T(x) = \frac{T_2 - T_1}{L} x + T_1$$

$$Q = -\frac{kA}{L}(T_2 - T_1) = \frac{(T_1 - T_2)}{R_{th,cond}}$$

$$R_{th,cond} = \frac{L}{kA} \quad [\,\mathrm{K/W}\,]$$

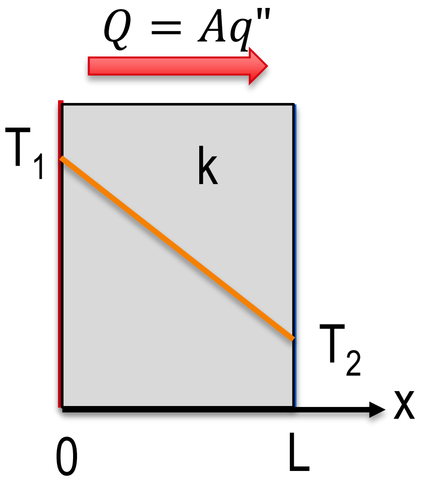
$$\mathrm{V}(x) = \frac{V_2 - V_1}{L} x + V_1$$

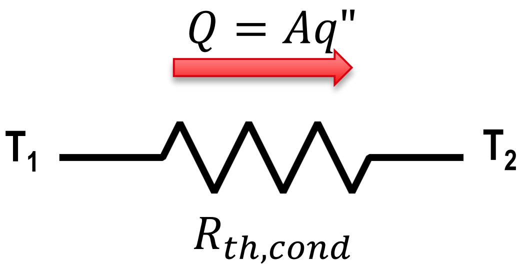
$$I = \frac{(V_1 - V_2)}{R_{el}}$$

$$R_{el} = \frac{L}{\sigma A}$$

$\sigma = electrical\ conductivity$
$A = wire\ cross\ section$

Thermal resistance – Planar Wall - Conduction





$$T(x) = \frac{T_2 - T_1}{L} x + T_1$$

$$Q = -\frac{kA}{L}(T_2 - T_1) = \frac{(T_1 - T_2)}{R_{th,cond}}$$

$$R_{th,cond} = \frac{L}{kA} \quad [\,\mathrm{K/W}\,]$$

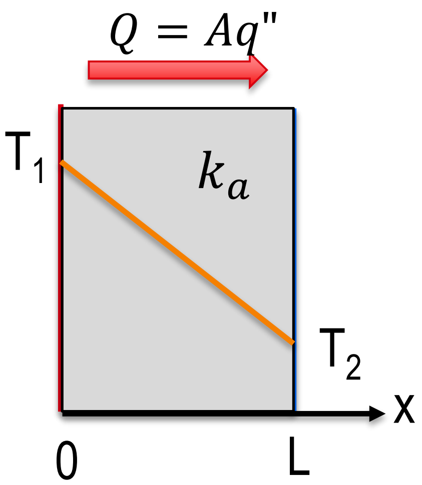
For a planar wall:

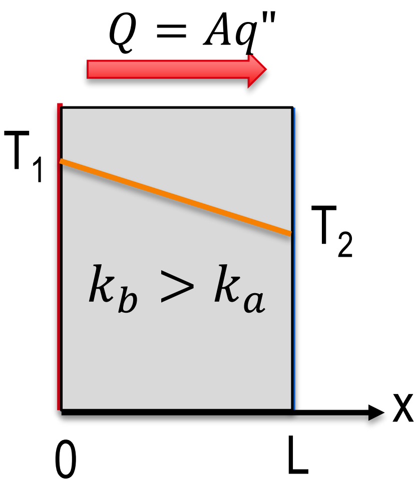
$$q'' = \frac{Q}{A} = \frac{(T_1 - T_2)}{A R_{th,cond}} = \frac{(T_1 - T_2)}{R''_{th,cond}}$$

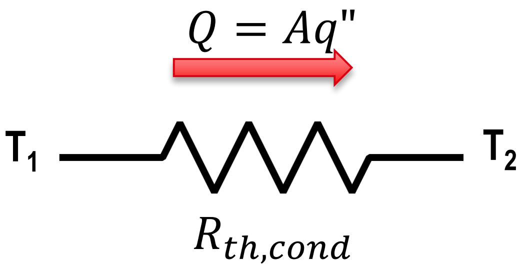
$$R''_{th,cond} = R_{th,cond} A$$

$$R''_{th,cond} = \frac{L}{k} \quad [\,\mathrm{m^2K/W}\,]$$

Thermal resistance – Planar Wall - Conduction







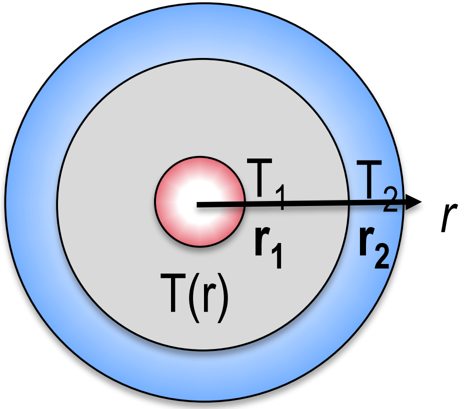
$$T(x) = \frac{T_2 - T_1}{L} x + T_1$$

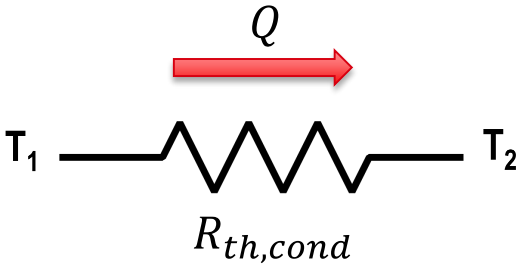
$$Q = -\frac{kA}{L}(T_2 - T_1) = \frac{(T_1 - T_2)}{R_{th,cond}}$$

$$R_{th,cond} = \frac{L}{kA} \quad [\,\mathrm{K/W}\,]$$

The higher the thermal conductivity the lower the thermal resistance, the lower the temperature drop

Thermal resistance – Cylindrical Pipe - Conduction





$$T(r) = \frac{T_1 - T_2}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_2$$

$$Q_r = \frac{2\pi L k}{\ln(r_2/r_1)}(T_1 - T_2) = \frac{(T_1 - T_2)}{R_{th,cond-cyl}}$$

$$R_{th,cond-cyl} = \frac{ln(r_2/r_1)}{2\pi L k} \quad [\text{ K/W }]$$

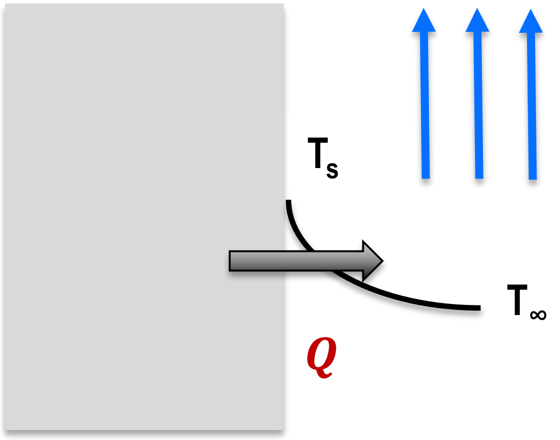
For a cylindrical pipe:

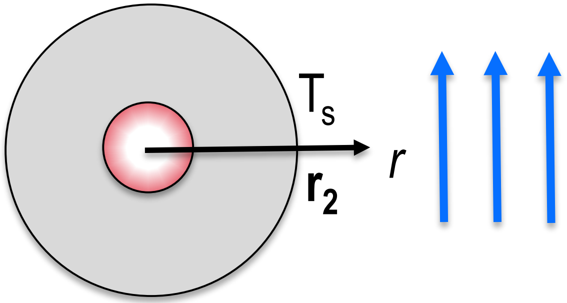
$$q' = \frac{Q}{L} = \frac{(T_1 - T_2)}{L R_{th,cond-cyl}} = \frac{(T_1 - T_2)}{R'_{th,cond-cyl}}$$

$$R'_{th,cond-cyl} = R_{th,cond-cyl} L$$

$$R'_{th,cond} = \frac{ln(r_2/r_1)}{2\pi k} \quad [\text{ mK/W }]$$

Thermal resistance – Convection





$$Q = hA(T_s - T_\infty) = \frac{(T_s - T_\infty)}{R_{th,conv}}$$

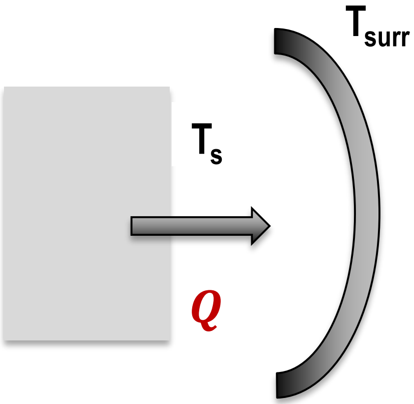
$$R_{th,conv} = \frac{1}{hA} \quad [\text{ K/W }]$$

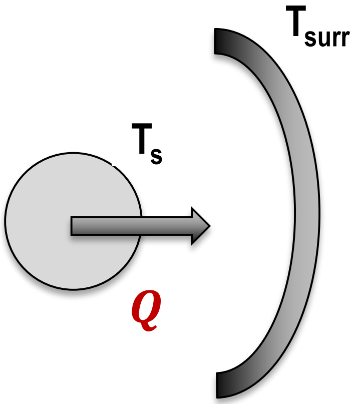
$$R''_{th,cond} = \frac{1}{h} \quad [\text{ m}^2\text{K/W }]$$

$$R_{th,conv} = \frac{1}{h2\pi r_2 L} \quad [\text{ K/W }]$$

$$R'_{th,conv-cyl} = \frac{1}{h2\pi r_2} \quad [\text{ mK/W }]$$

Thermal resistance –Radiation





$$Q = A\varepsilon\sigma(T_s^4 - T_{surr}^4) = A\varepsilon\sigma(T_s^2 + T_{surr}^2)\left(T_s + T_{surr}\right)\left(T_s - T_{surr}\right)$$

$$Q = Ah_{rad}\left(T_s - T_{sur}\right) = \frac{(T_s - T_{sur})}{R_{th,conv}}$$

$$R_{th,rad} = \frac{1}{h_{rad}A} \quad [\text{ K/W }]$$

$$R''_{th,rad} = \frac{1}{h_{rad}} \quad [\text{ m}^2\text{K/W }]$$

$$R_{th,rad-cyl} = \frac{1}{h_{rad}2\pi rL} \quad [\text{ K/W }]$$

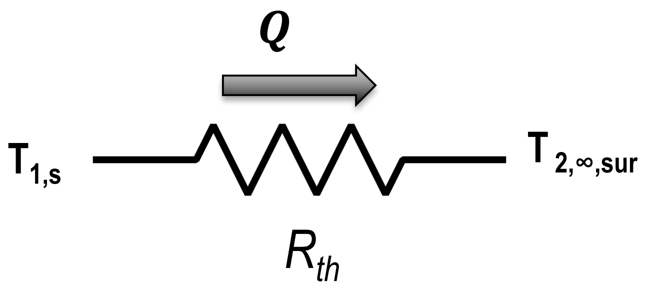
$$R'_{th,rad-cyl} = \frac{1}{h_{rad}2\pi rL} \quad [\text{ mK/W }]$$

Thermal Resistances for a Planar Wall

Conduction: $$R_{th,cond} = \frac{L}{kA}$$ [K/W]

Convection: $$R_{th,cond} = \frac{1}{hA}$$ [K/W]

Radiation: $$R_{th,rad} = \frac{1}{h_{rad}A}$$ [K/W]



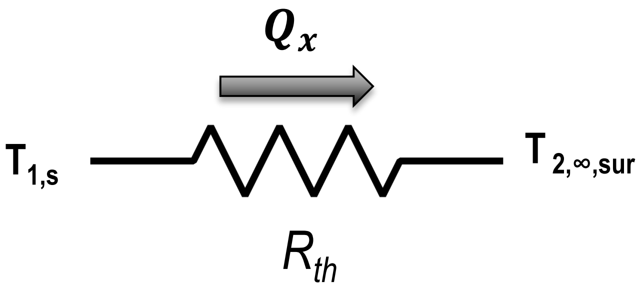
$$Q = \frac{\left(T_{1,s} - T_{2,\infty sur}\right)}{R_{th}}$$

Thermal Resistance Cylindrical (Radial) System

Conduction: $$R_{th,cond-cyl} = \frac{ln(r_2/r_1)}{2\pi Lk}$$ [K/W]

Convection: $$R_{th,conv} = \frac{1}{h2\pi rL}$$ [K/W]

Radiation: $$R_{th,rad} = \frac{1}{h_{rad}2\pi rL}$$ [K/W]



$$Q_x = \frac{(T_{1,s} - T_{2,\infty sur})}{R_{th}}$$

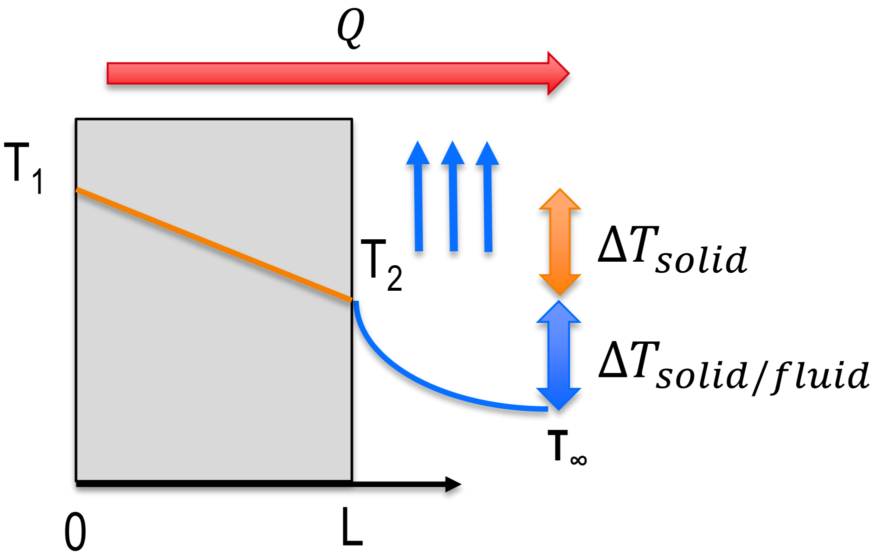
Conductive Heat Transfer - 3

- [x] Thermal resistance
- [ ] Bi Number
- [ ] Intro to Thermal Circuits

Learning Objectives:

- [x] Calculate the thermal resistances
- [ ] Calculate the Bi number
- [ ] Solve 1D problems using thermal circuits

Biot Number



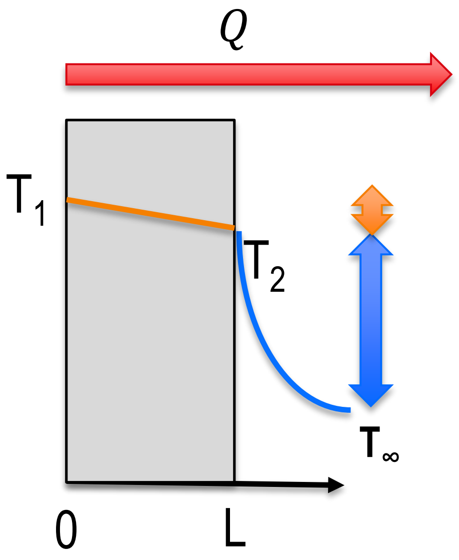
$$Q = \frac{(T_1 - T_2)}{R_{th,cond}} = \frac{(T_2 - T_\infty)}{R_{th,conv}}$$

$$\Rightarrow \frac{(T_1 - T_2)}{(T_2 - T_\infty)} = \frac{R_{th,cond}}{R_{th,conv}} \equiv Bi$$

$$\Rightarrow \boldsymbol{Bi \equiv \frac{hL}{k}}$$

Note: L can be generalized to be a characteristic dimension of a body (e.g. diameter of a sphere)

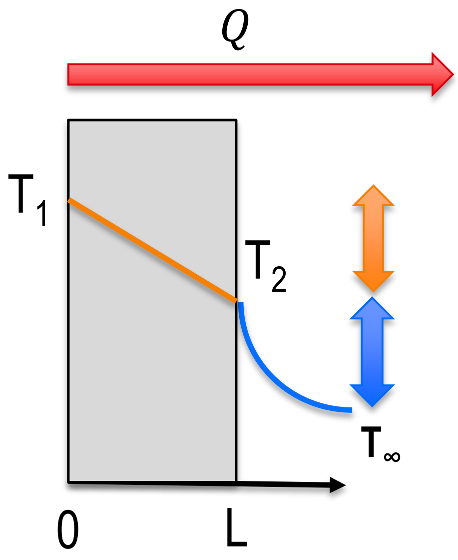
Biot Number



$$\Delta T_{solid} \ll \Delta T_{solid/fluid}$$

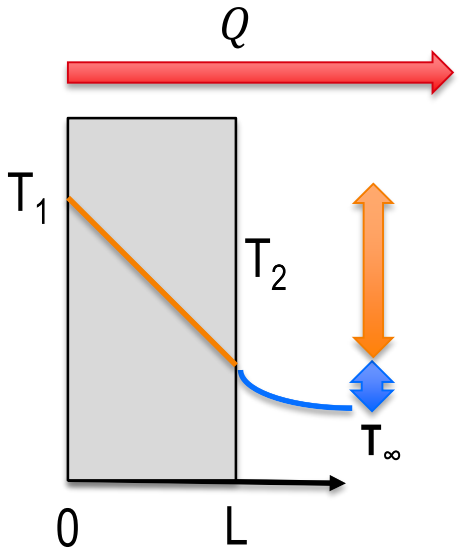
$$Bi \equiv \frac{hL}{k} \ll 1$$

Solid has ~constant temperature



$$\Delta T_{solid} \sim \Delta T_{solid/fluid}$$

$$Bi \equiv \frac{hL}{k} \sim 1$$



$$\Delta T_{solid} \gg \Delta T_{solid/fluid}$$

$$Bi \equiv \frac{hL}{k} \gg 1$$

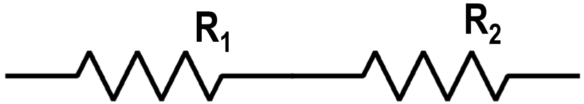
Conductive Heat Transfer - 3

- [x] Thermal resistance
- [x] Bi Number
- [ ] Intro to Thermal Circuits

Learning Objectives:

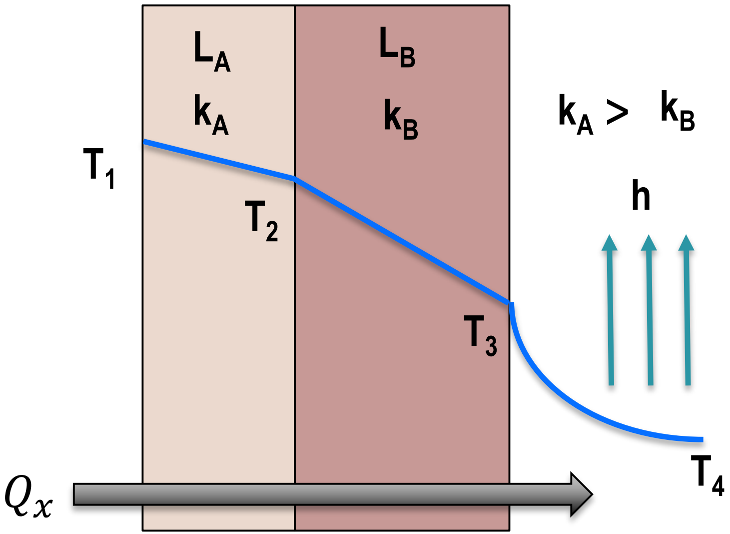
- [x] Calculate the thermal resistances
- [x] Calculate the Bi number
- [ ] Solve 1D problems using thermal circuits

Thermal Circuits



$$R_{series} = R_1 + R_2 + .. = \sum R_i$$

Thermal Circuits



$A = 4m^2$

$T_1 = 900C = 1173K$
$T_4 = 10C = 283K$

$h = 50\, {}^{W}/_{m^2K}$

$L_A = 10\ cm = 0.1m$
$L_B = 50\ \mathrm{cm} = 0.5\mathrm{m}$

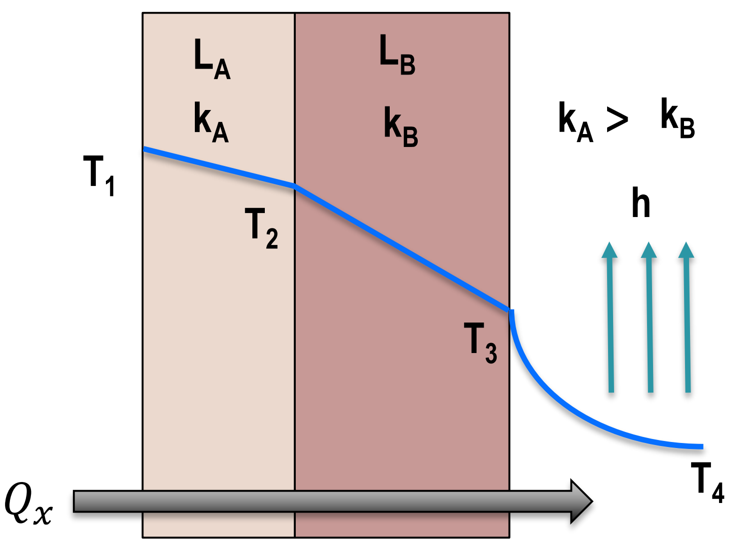
$k_A = 100\, {}^{W}/_{mK}$
$k_B = 0.1\, {}^{W}/_{mK}$

- **Calculate Q**
- **Calculate T_3, T_2**
- **Draw the temperature profile**

Draw the electrical circuit equivalent to this problem then reduce it to a single resistor.

Compare your solution to the procedure you used in Slide 21.

Thermal Circuits



$R_{th,cond,A}$ $\quad$ $R_{th,cond,B}$ $\quad$ $R_{th,conv}$

$L_A/k_A A$ $\quad$ $L_B/k_B A$ $\quad$ $1/hA$

$\mathrm{A} = 4m^2$

$T_1 = 900C = 1173K$

$T_4 = 10C = 283K$

$h = 50\, {}^{W}/_{m^2K}$

$L_A = 10\ cm = 0.1m$

$L_B = 50\ \mathrm{cm} = 0.5\mathrm{m}$

$k_A = 100\, {}^{W}/_{mK}$

$k_B = 0.1\, {}^{W}/_{mK}$

$$\boldsymbol{Q_x = \frac{T_1 - T_4}{R_{tot,series}} = \frac{T_1 - T_4}{R_{th,cond,A} + R_{th,cond,B} + R_{th,conv}}}$$

$$Q_x = \frac{T_3 - T_4}{R_{th,conv}}$$

$$Q_x = \frac{T_1 - T_2}{R_{th,cond,A}}$$

Q = 708.6 W,
T2 = 899.8 C,
T3 = 13.54 C

Conductive Heat Transfer - 3

- ☑ Thermal resistance
- ☑ Bi Number
- ☑ Intro to Thermal Circuits

Learning Objectives:

- ☑ Calculate the thermal resistances
- ☑ Calculate the Bi number
- ☑ Solve 1D problems using thermal circuits

Next Week

- ❑ Thermal Circuits & Overall Heat Transfer Coefficient
- ❑ 1D steady-state conduction with heat sources
- ❑ Introduction to Fins

Learning Objectives:

- ❑ Solve 1D steady state heat conduction problems in different geometries, with & without heat sources

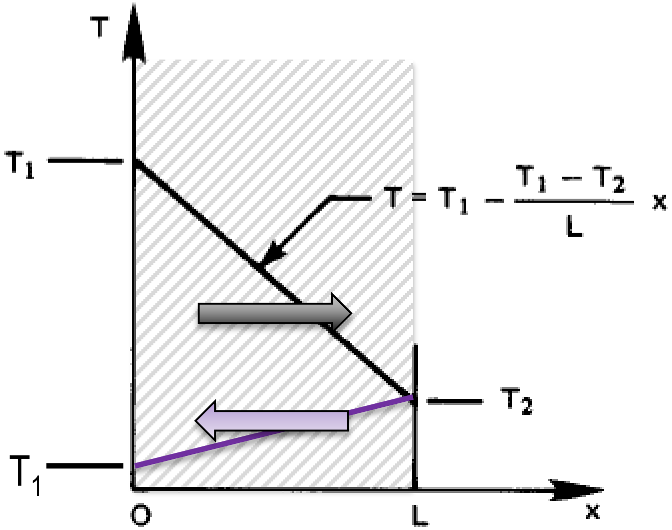
Supplementary Slides

1D Steady State Heat Diffusion & Thermal Resistance - Recap

<u>Assumption 1</u>: incompressible medium
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: k independent of T
<u>Assumption 5</u>: steady state
<u>Assumption 6</u>: **no heat sources**

Single layer	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate Q	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\mathrm{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

Be careful to the signs!



$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

Boundary Conditions:
$$T(x = 0) = T_1$$
$$T(x = L) = T_2$$

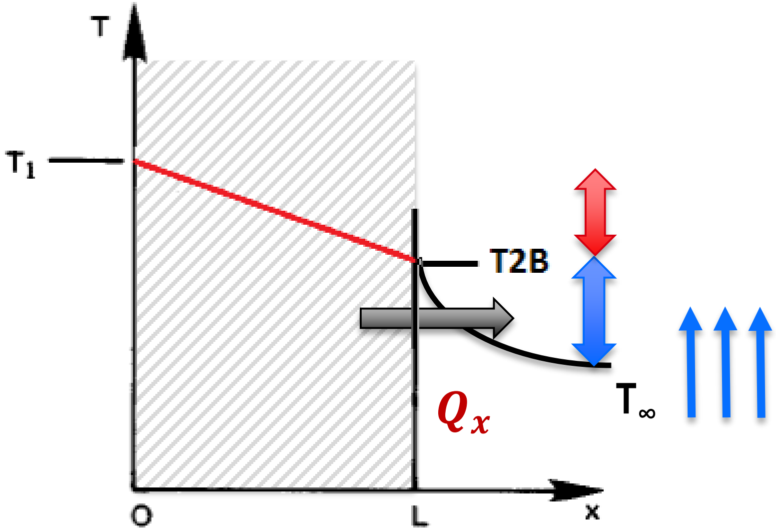
$$\Rightarrow \quad T(x) = \frac{T_2 - T_1}{L} x + T_1$$

$$\Rightarrow \quad Q_x = -kA \frac{\partial T}{\partial x} = -kA \frac{T_2 - T_1}{L}$$

If $T_1 > T_2$, $Q_x > 0$

If $T_1 < T_2$, $Q_x < 0$

Biot Number



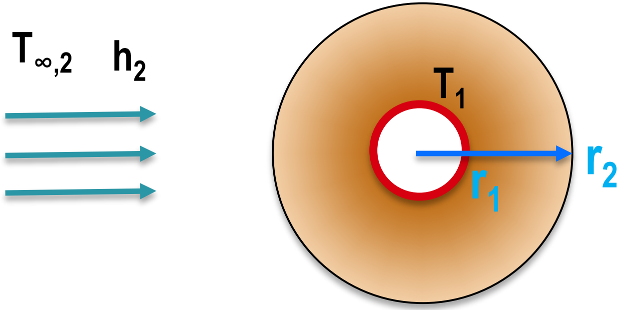
$$Q_x = \frac{(T_1 - T_{2B})}{R_{th,cond}} = \frac{(T_{2B} - T_\infty)}{R_{th,conv}}$$

$$\Rightarrow \quad \frac{(T_1 - T_{2B})}{(T_{2B} - T_\infty)} = \frac{R_{th,cond}}{R_{th,conv}} \equiv Bi$$

$$\Rightarrow \quad \boldsymbol{Bi \equiv \frac{hL}{k}}$$

Note: L can be generalized to be a characteristic dimension of a body (e.g. diameter of a sphere)

Thermal Resistance Cylindrical (Radial) System – Critical Value

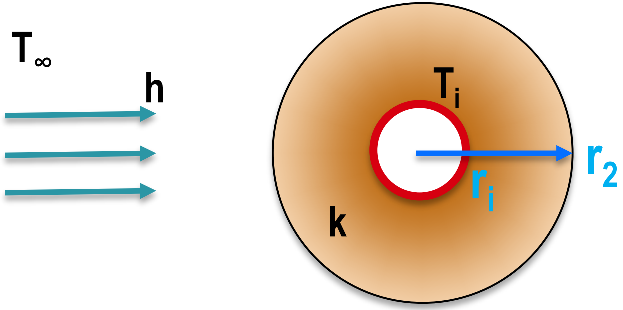


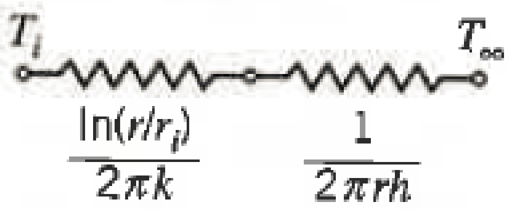
$$R'_{th,cond-cyl} = \frac{ln(r_2/r_1)}{2\pi k}$$ [mK/W] Increases as r_2 increases

$$R'_{th,conv} = \frac{1}{h2\pi r_2}$$ [mK/W] Decreases as r_2 increases

The sum of conduction and convection resistance presents an extremum for a critical value of r_2

Thermal Resistance Cylindrical (Radial) System – Critical Radius





$$R'_{\text{tot}} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h}$$

$$\frac{dR'_{\text{tot}}}{dr} = 0 \qquad \frac{1}{2\pi k r} - \frac{1}{2\pi r^2 h} = 0 \qquad r = \frac{k}{h}$$

at $r = k/h$,

$$\frac{d^2R'_{\text{tot}}}{dr^2} = \frac{1}{\pi(k/h)^2}\left(\frac{1}{k} - \frac{1}{2k}\right) = \frac{1}{2\pi k^3/h^2} > 0$$