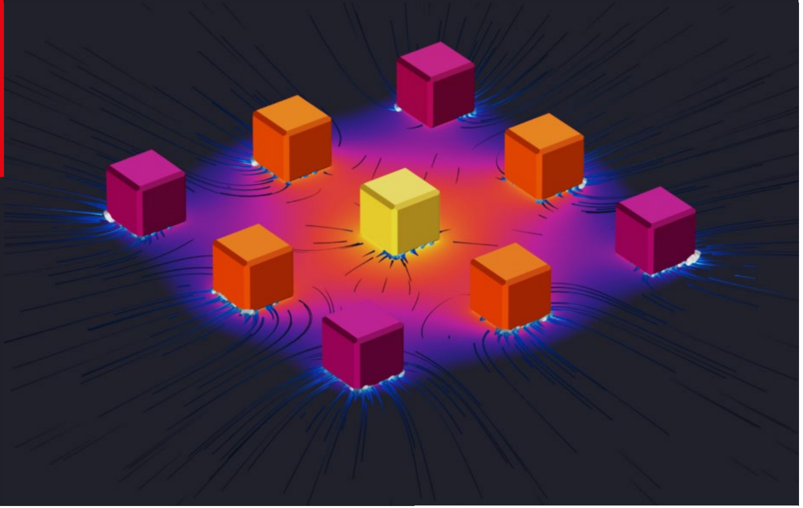
Heat and Mass Transfer
ME-341

Instructor: Giulia Tagliabue



École
polytechnique
fédérale
de Lausanne

Spring Semester

Conductive Heat Transfer - 1

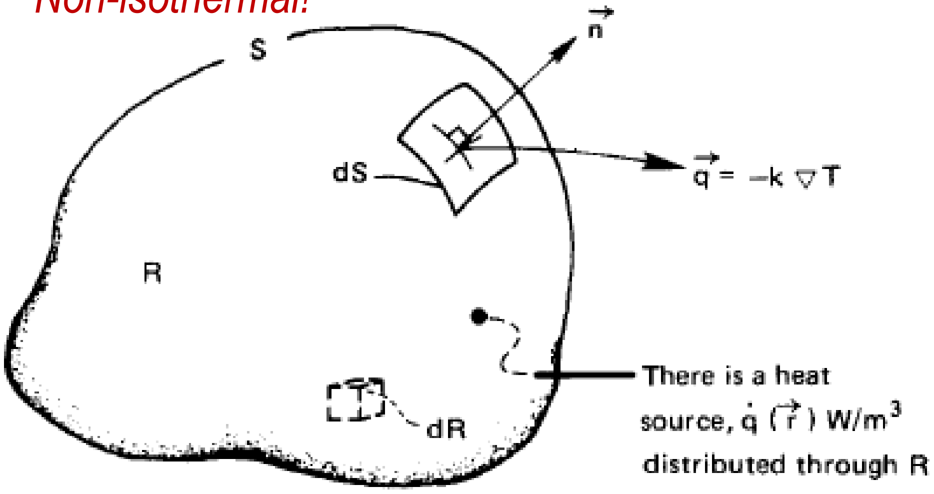
- ☑ Heat diffusion equation
- ☑ Thermal conductivity and diffusivity

Learning Objectives:

- ☑ Understand material parameters and know their magnitude
- ☑ Derive the general heat diffusion equation

Heat Diffusion Equation – 3D

Non-isothermal!



Assumption 1: incompressible medium
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material

Heat Diffusion Equation:

$$\boldsymbol{\nabla \cdot (k\, \nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t}}$$

Assumption 4: *k* is independent of *T*

$$k\nabla^2 T + \dot{q} = \rho c \frac{\partial T}{\partial t}$$

$$\boldsymbol{\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

$$\alpha = \frac{k}{\rho c} = thermal\ diffusivity\ [\frac{m^2}{s}]$$

Conductive Heat Transfer - 2

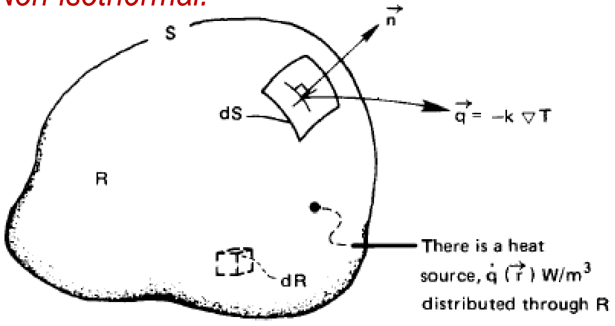
- ❑ Types of boundary conditions
- ❑ Planar and cylindrical (1D) solutions

Learning Objectives:

- ❑ Identify the possible boundary conditions
- ❑ Express mathematically the various boundary conditions
- ❑ Calculate the temperature profile in a planar or cylindrical wall

Heat Diffusion Equation – 3D

Non-isothermal!



$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

To solve the equation we need:

- Initial condition: $\mathrm{T}(\mathrm{t}=0) = T_i(x, y, z)$
- Boundary conditions

<u>Assumption 1</u>: incompressible medium
<u>Assumption 2</u>: medium at rest (no convection)
<u>Assumption 3</u>: isotropic material
<u>Assumption 4</u>: *k* is independent of *T*

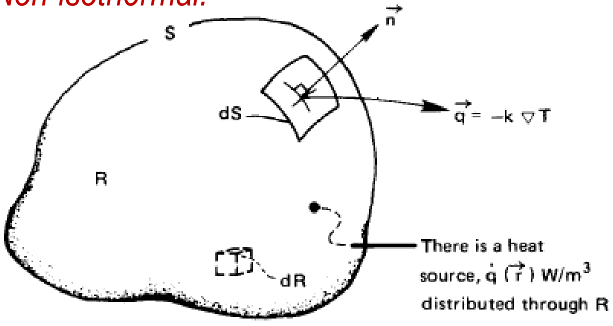
<u>Assumption 5</u>: steady-state ($\partial/\partial t = 0$)

<u>Assumption 6</u>: no heat sources ($\dot{q} = 0$)

$$\boldsymbol{\nabla^2 T = 0}$$

Heat Diffusion Equation – 3D, steady-state, no-heat sources

Non-isothermal!



Assumption 1: incompressible medium (solid)
Assumption 2: medium at rest (no convection)
Assumption 3: isotropic material
Assumption 4: *k* independent of *T*
Assumption 5: steady state
Assumption 6: no heat sources

$$\nabla^2 T = 0$$

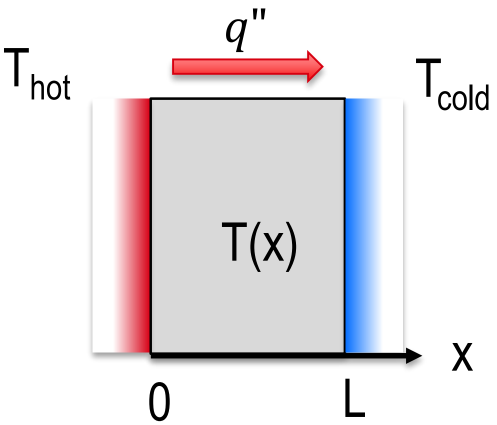
Cartesian coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(k\frac{\partial T}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = 0$$

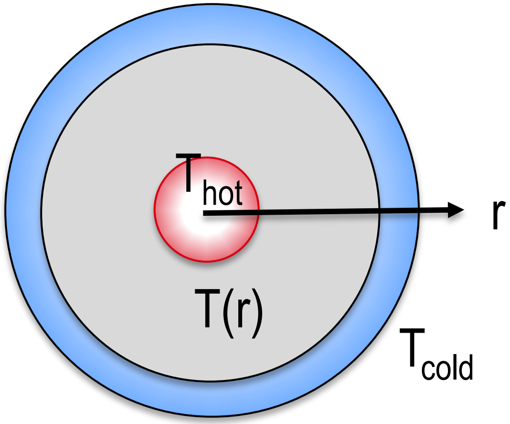
Heat Diffusion Equation – 1D, steady-state, no-heat sources



Cartesian coordinates:

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

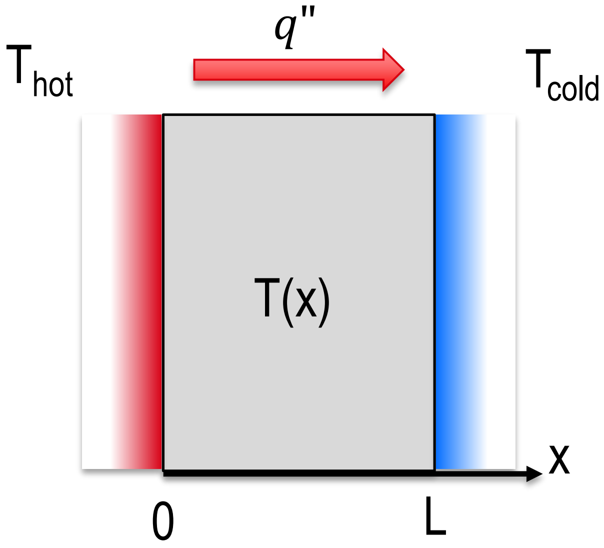
Boundary Conditions ???



Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0 \quad \Rightarrow \quad T(r) = C_1 \ln(r) + C_2$$

Heat Diffusion Equation – Boundary Conditions



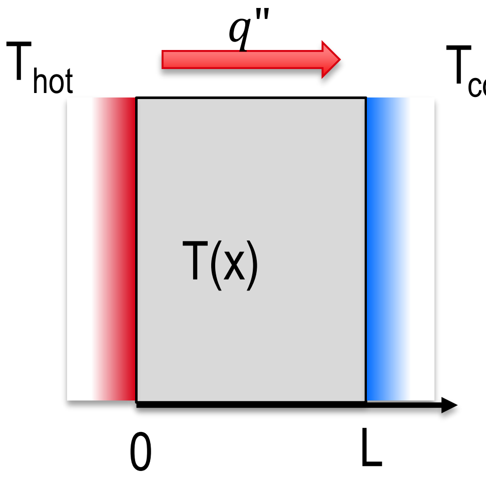
$$T(x) = C_1 x + C_2$$

What boundary conditions can we define at $x = 0$ and $x = L$?

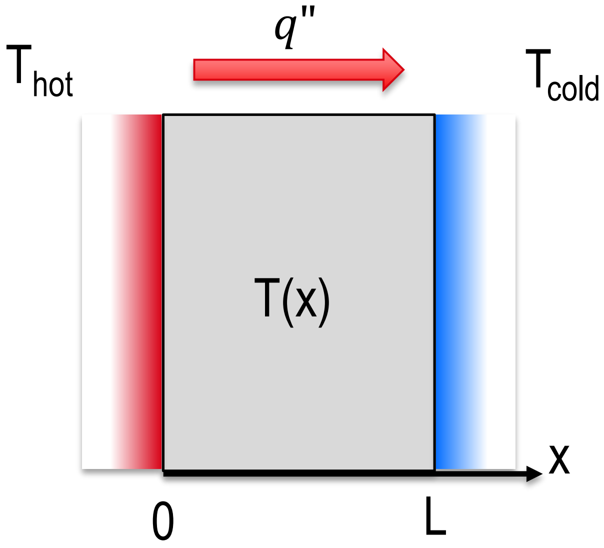
1. *Think in mathematical terms: what conditions can constrain a function at a given point?*

2. *Think in physical terms: what physical quantity is correlated with the first derivative of the temperature?*

What boundary conditions can we define at $x = 0$ and $x = L$?



Heat Diffusion Equation – Boundary Conditions



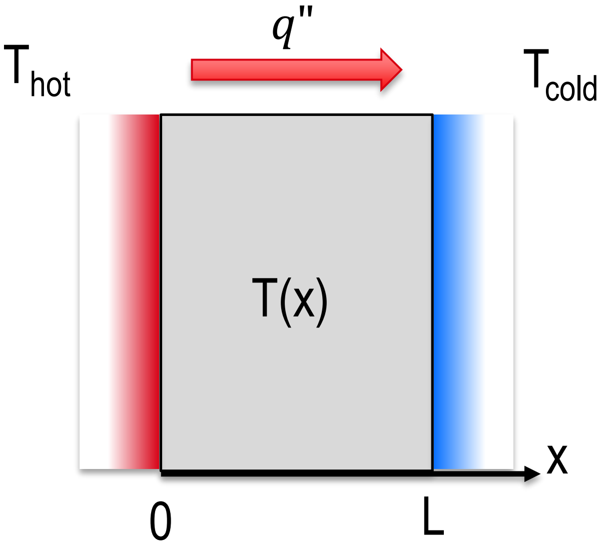
$$T(x) = C_1 x + C_2$$

What boundary conditions can we define at $x = 0$ and $x = L$?

1. *Think in mathematical terms: what conditions can constrain a function at a given point?*

- Set the value of the function (i.e. $\mathrm{T}(\mathrm{x} = 0) = T_1$)
- Set the derivatives of the function (i.e. $\frac{dT}{dx}(\mathrm{x} = 0) = C_1$)

Heat Diffusion Equation – Boundary Conditions



$$T(x) = C_1 x + C_2$$

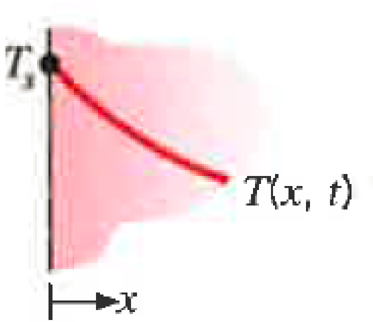
What boundary conditions can we define at $x = 0$ and $x = L$?

2. *Think in physical terms: what physical quantity is correlated with the first derivative of the temperature?*

   - From Fourier's law we know that: $q'' \propto \frac{dT}{dx}$ or Q $\propto A\frac{dT}{dx}$

Heat Diffusion Equation – Boundary Conditions (1D)

- B.C. of the 1st kind (Dirichlet condition): constant surface temperature

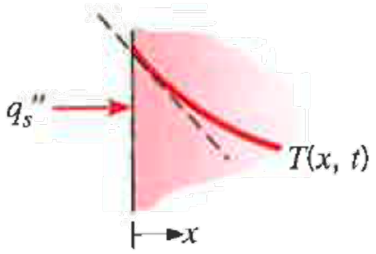


$$T(x, y, z)_{x=x_S, y=y_S, z=z_S} = T_S$$

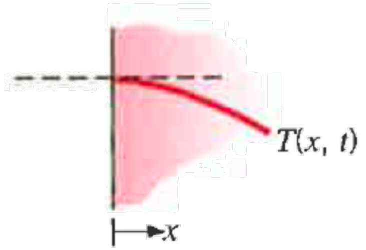
Heat Diffusion Equation – Boundary Conditions

- B.C. of the 2nd kind (Neumann condition): known ∇T

$$Q = \int q'' dA = -\int k \nabla T dA$$ ⇨ known heat flux/heat transfer rate



$$-k\left(\frac{\partial T}{\partial x}\right)_{x=x_i} = q_s''$$

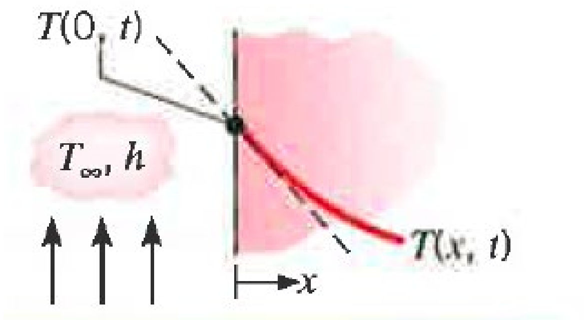


$$-k\left(\frac{\partial T}{\partial x}\right)_{x=x_i} = 0$$

Adiabatic wall or symmetry plane!

Heat Diffusion Equation – Boundary Conditions

- B.C. of the 3rd kind (Robin condition): convection surface condition

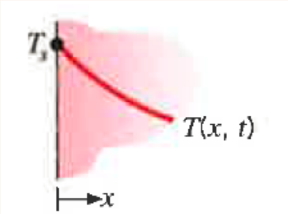


$$Q = Aq_w'' = -k\left(\frac{\partial T}{\partial x}\right)_{x=x_i} = h(T(x_i,t) - T_\infty)$$

Proportional to the temperature difference between the fluid and the wall

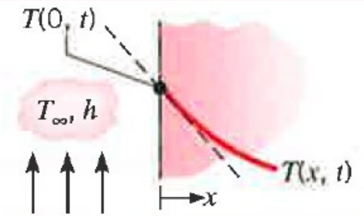
Heat Diffusion Equation – Boundary Conditions

B.C. of the 1st kind (*Dirichlet condition*):
constant surface temperature



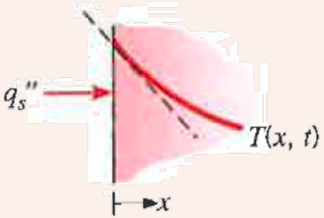
$$T(\vec{r})_{\vec{r}=\vec{r}_i} = T_w$$

B.C. of the 3rd kind (*Robin condition*):
convection surface condition

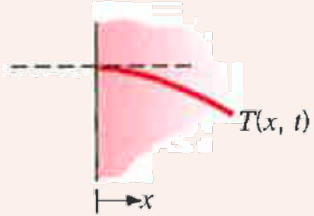


$$-k\left(\frac{\partial T}{\partial x}\right)_{x_i} = h(T(x_i, t) - T_\infty)$$

B.C. of the 2nd kind (*Neumann condition*):
known heat flux



$$-k\left(\frac{\partial T}{\partial x}\right)_{x=x_i} = q_w''$$



$$-k\left(\frac{\partial T}{\partial x}\right)_{x=x_i} = 0$$

(adiabatic)

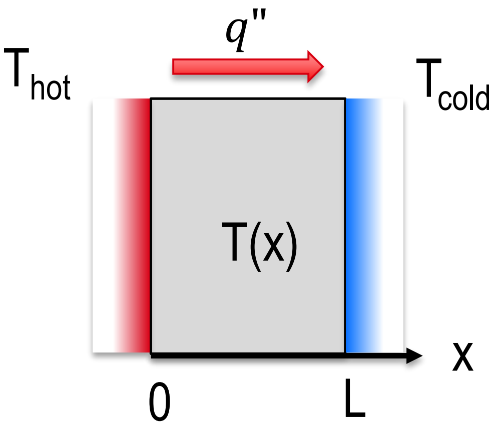
Conductive Heat Transfer - 2

- [x] Types of boundary conditions
- [ ] Planar and cylindrical (1D) solutions

Learning Objectives:

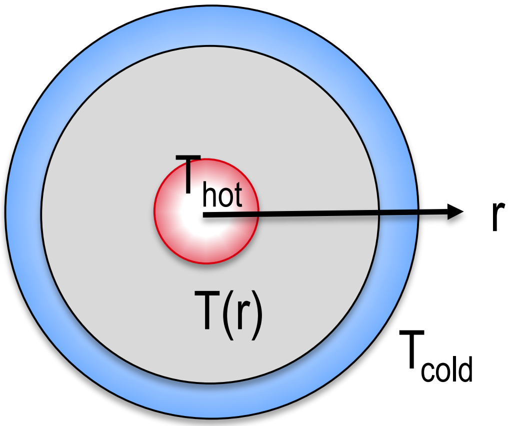
- [x] Identify the possible boundary conditions
- [x] Express mathematically the various boundary conditions
- [ ] Calculate the temperature profile in a planar or cylindrical wall

Heat Diffusion Equation – 1D, steady-state, no-heat sources



Cartesian coordinates:

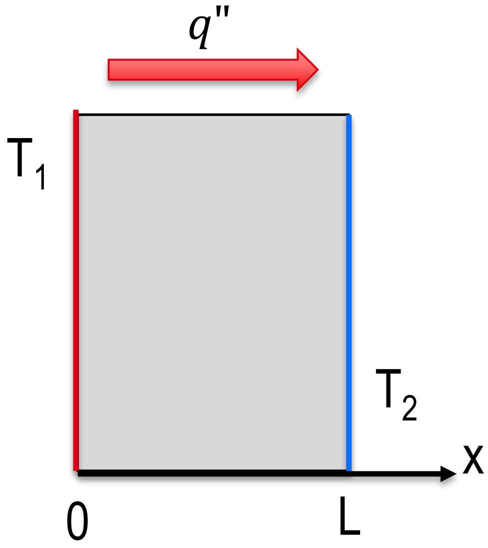
$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$



Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0 \quad \Rightarrow \quad T(r) = C_1 \ln(r) + C_2$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Planar Wall

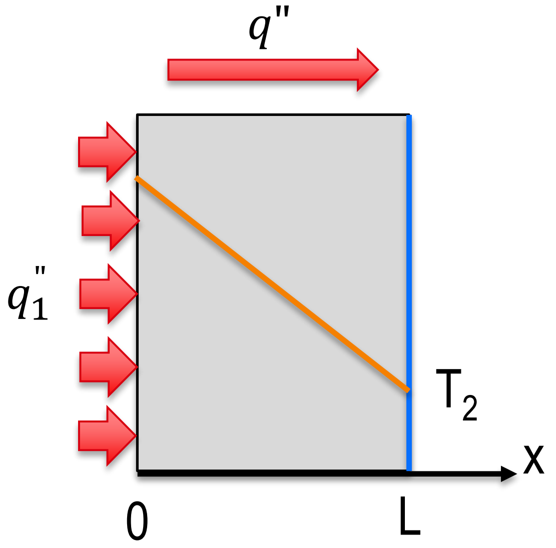


$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

CASE 1: Temperature boundary condition (BC)

$$T(x = 0) = T_1 \qquad T(x = L) = T_2$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Planar Wall



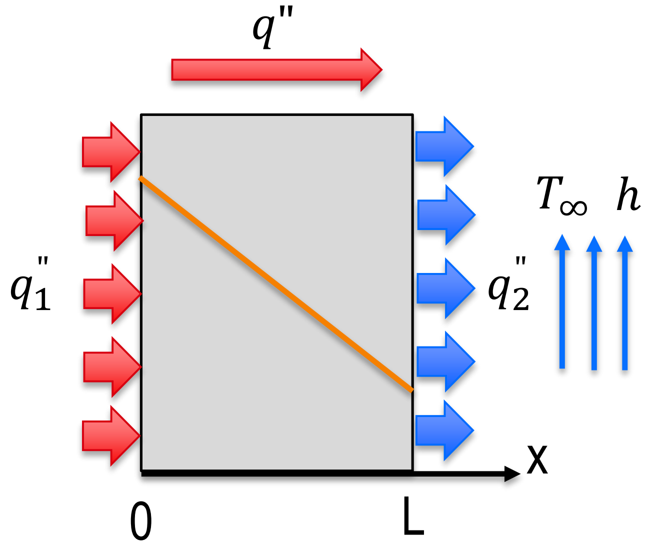
$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

CASE 2: Constant heat flux and temperature BC

$$q_1'' = -k \left.\frac{dT}{dx}\right|_{x=0} \quad \Rightarrow \quad \left.\frac{dT}{dx}\right|_{x=0} = -\frac{q_1''}{k}$$

$$T(x = L) = T_2$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Planar Wall



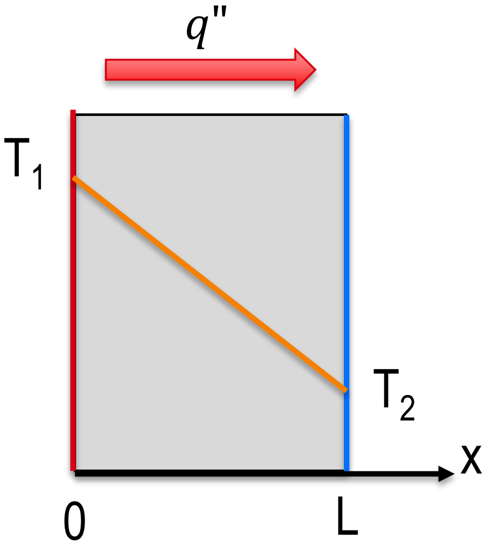
$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

CASE 3: Constant heat flux and convection BC

$$q_1'' = -k \left.\frac{dT}{dx}\right|_{x=0} \quad \Rightarrow \quad \left.\frac{dT}{dx}\right|_{x=0} = -\frac{q_1''}{k}$$

$$q_2'' = -k \left.\frac{dT}{dx}\right|_{x=L} = h(T_2 - T_\infty)$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Planar Wall



$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

CASE 1: Temperature boundary condition (BC)

$$T(x = 0) = T_1 \qquad T(x = L) = T_2$$

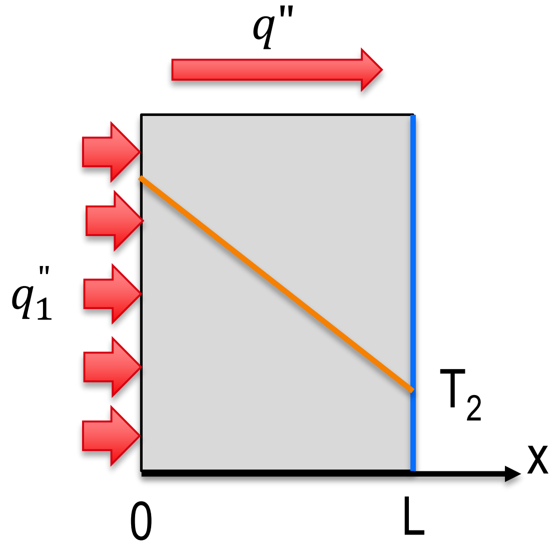
$$\Rightarrow \begin{cases} C_2 = T_1 \\ C_1 = \dfrac{T_2 - T_1}{L} \end{cases}$$

$$\Rightarrow \quad T(x) = \frac{T_2 - T_1}{L} x + T_1$$

LINEAR T profile

$$\Rightarrow \quad q_x'' = -k\frac{\partial T}{\partial x} = -k\frac{T_2 - T_1}{L} \qquad Q = -kA\frac{T_2 - T_1}{L}$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Planar Wall



$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

CASE 2: Constant heat flux and temperature BC

$$q_1'' = -k \left.\frac{dT}{dx}\right|_{x=0} \quad \Rightarrow \quad \left.\frac{dT}{dx}\right|_{x=0} = -\frac{q_1''}{k}$$

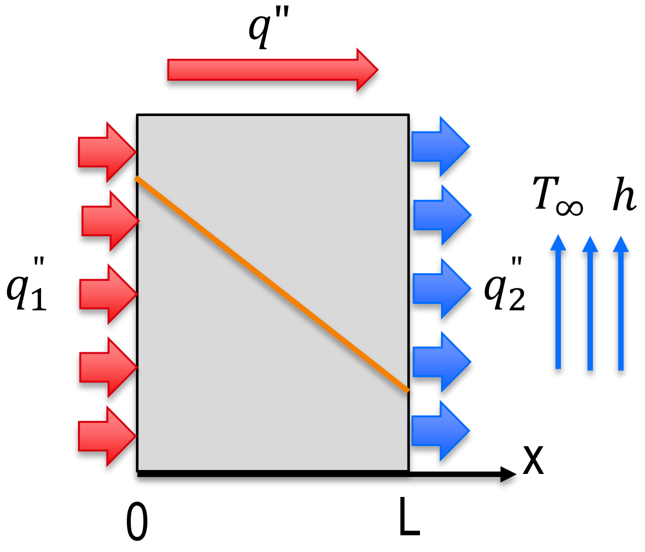
$$T(x = L) = T_2$$

$$\Rightarrow \begin{cases} C_1 = -\dfrac{q_1''}{k} \\ C_2 = T_2 + \dfrac{q_1''}{k} \mathrm{L} \end{cases}$$

$$\Rightarrow \quad T(x) = -\frac{q_1''}{k}(x - L) + T_2 \qquad T(0) = T_1 = \frac{q_1''}{k} L + T_2$$

$$\Rightarrow \quad q'' = q_1'' \qquad Q = A q_1''$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Planar Wall



$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$

CASE 3: Constant heat flux and convection BC

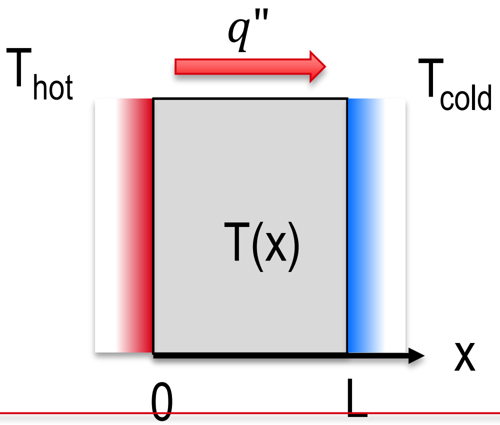
$$q_1'' = -k \left.\frac{dT}{dx}\right|_{x=0} \quad \Rightarrow \quad \left.\frac{dT}{dx}\right|_{x=0} = -\frac{q_1''}{k}$$

$$q_2'' = -k \left.\frac{dT}{dx}\right|_{x=L} = h(T_2 - T_\infty)$$

$$\Rightarrow \begin{cases} C_1 = -\dfrac{q_1''}{k} \\ -kC_1 = h(T_2 - T_\infty) \end{cases} \Rightarrow \quad T_2 = \frac{q_1''}{h} + T_\infty \quad \Rightarrow \quad T_2 = T(L) = C_1 L + C_2$$

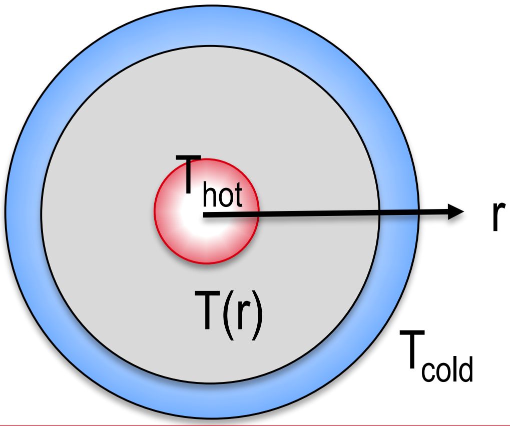
$$\Rightarrow \quad T(x) = -\frac{q_1''}{k}(x - L) + \frac{q_1''}{h} + T_\infty \qquad Q = Aq_1'' = Aq_2''$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources



Cartesian coordinates:

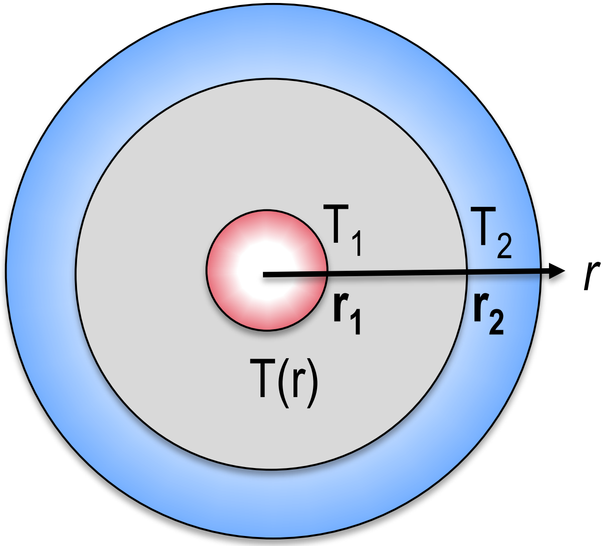
$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \Rightarrow \quad T(x) = C_1 x + C_2$$



Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0 \quad \Rightarrow \quad T(r) = C_1 \ln(r) + C_2$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Pipe

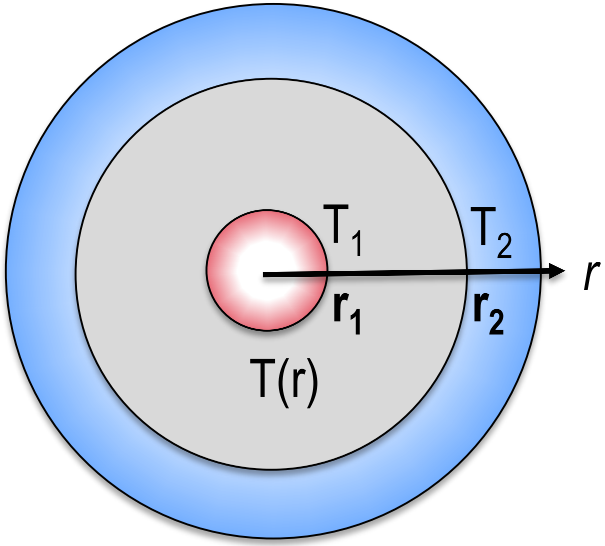


$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0 \quad \Rightarrow \quad T(r) = C_1 \ln(r) + C_2$$

CASE 1: Temperature boundary condition (BC)

$$T(r_1) = T_1 \qquad T(r_2) = T_2$$

Heat Diffusion Equation – 1D, steady-state, no-heat sources – Pipe



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0 \quad \Rightarrow \quad T(r) = C_1 \ln(r) + C_2$$

CASE 1: Temperature boundary condition (BC)

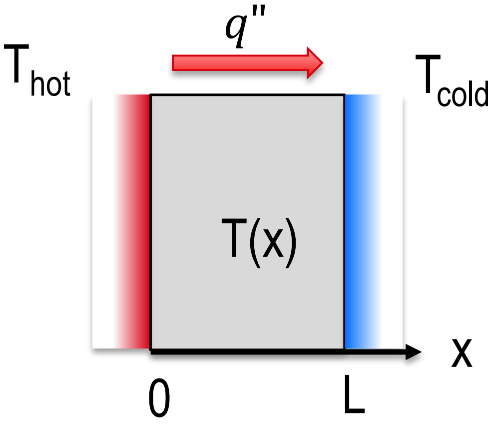
$$T(r_1) = T_1 \qquad T(r_2) = T_2$$

$$\Rightarrow \quad \boldsymbol{T(r) = \frac{T_1 - T_2}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_2}$$

$$\Rightarrow \quad Q_r = -k(2\pi r L)\frac{dT}{dr} = \frac{2\pi L k}{\ln(r_2/r_1)}(T_1 - T_2)$$

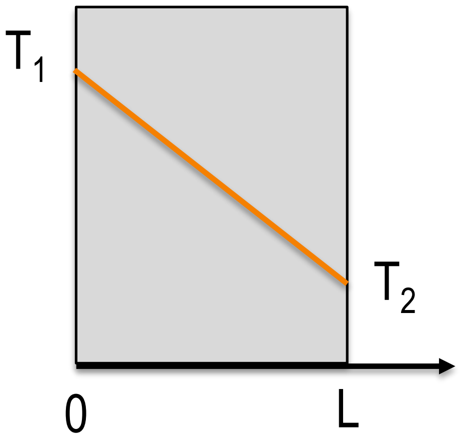
In radial system Q_r is constant but q_r'' is NOT constant because $A = A(r)$!!

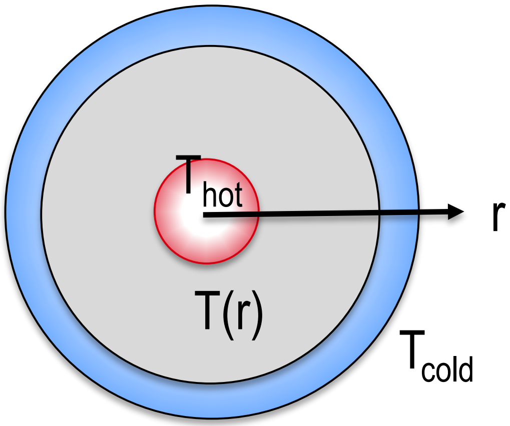
Heat Diffusion Equation – 1D, steady-state, no-heat sources



Planar Wall = Linear T(x)

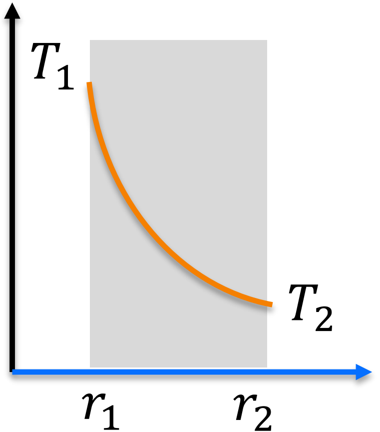
$$T(x) = C_1 x + C_2$$



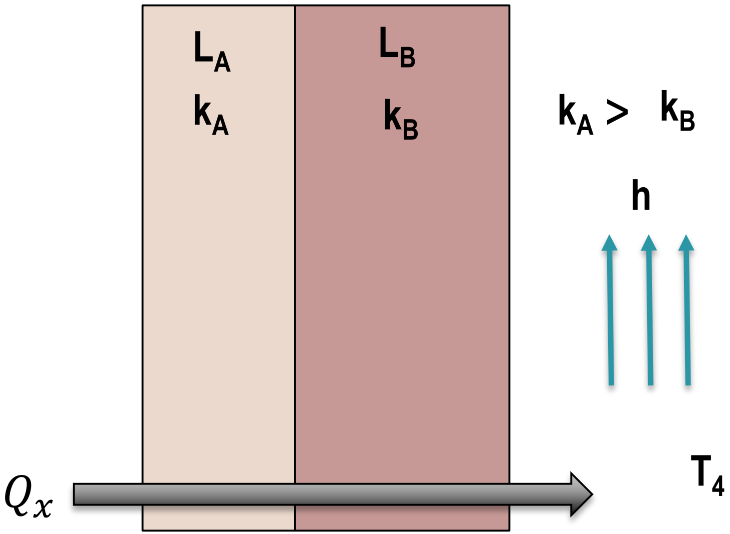


Pipe = Logarithmic T(r)

$$T(r) = C_1 \ln(r) + C_2$$



Heat Diffusion - Example



$\text{A} = 4m^2$

$T_1 = 900C = 1173K$
$T_4 = 10C = 283K$

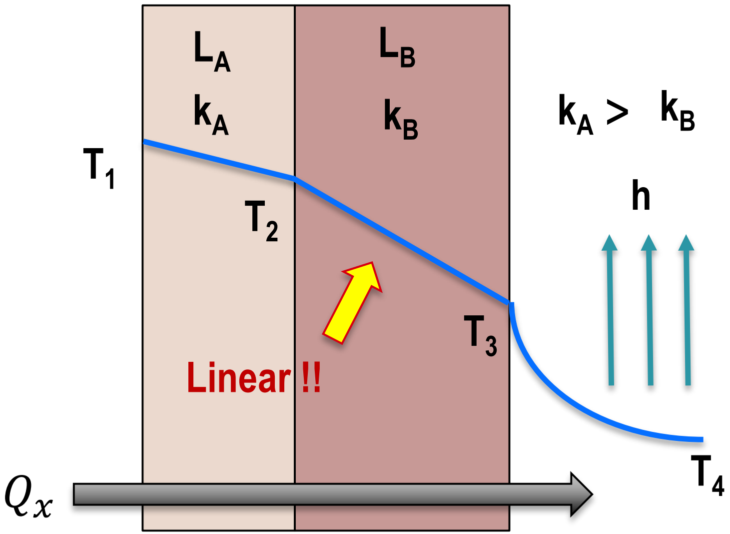
$h = 50\,{}^{W}/_{m^2K}$

$L_A = 10\ cm = 0.1m$
$L_B = 50\ \text{cm} = 0.5\text{m}$

$k_A = 100\,{}^{W}/_{mK}$
$k_B = 0.1\,{}^{W}/_{mK}$

- **Calculate Q**
- **Calculate T_3, T_2**
- **Draw the temperature profile**

Heat Diffusion - Example



$A = 4m^2$

$T_1 = 900C = 1173K$
$T_4 = 10C = 283K$

$h = 50\,{}^{W}/_{m^2K}$

$L_A = 10\ cm = 0.1m$
$L_B = 50\ \text{cm} = 0.5\text{m}$

$k_A = 100\,{}^{W}/_{mK}$
$k_B = 0.1\,{}^{W}/_{mK}$

- **Calculate Q**
- **Calculate T_3, T_2**
- **Draw the temperature profile**

$$Q_x = hA(T_3 - T_4) \qquad Q_x = {}^{k_A A}/_{L_A}(T_1 - T_2) \qquad Q_x = {}^{k_B A}/_{L_B}(T_2 - T_3)$$

Q = 708.6 W, T2 = 899.8 C, T3 = 13.54 C

Conductive Heat Transfer - 2

- ✓ Types of boundary conditions
- ✓ Planar and cylindrical (1D) solutions

Learning Objectives:

- ✓ Identify the possible boundary conditions
- ✓ Express mathematically the various boundary conditions
- ✓ Calculate the temperature profile in a planar or cylindrical wall

Conductive Heat Transfer - 3

- ❑ Thermal resistance
- ❑ Bi Number
- ❑ Overall heat Transfer Coefficient

Learning Objectives:

- ❑ Calculate the thermal resistances
- ❑ Calculate the Bi number
- ❑ Calculate the overall Heat transfer coefficient
- ❑ Solve 1D problems using thermal resistances

Supplementary slides