Exercise 7.1

Consider a large vertical plate with a uniform surface temperature of $130°C$ suspended in quiescent air at $25°C$ and at atmospheric pressure.

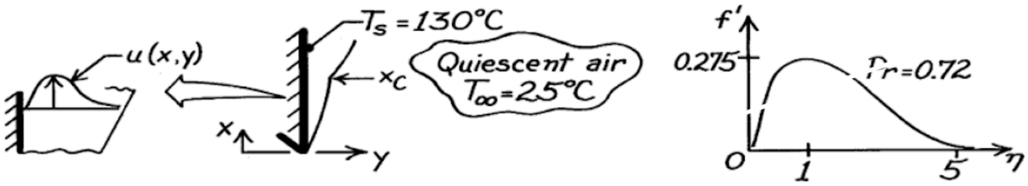
a) Estimate the boundary layer thickness at a location $0.25m$ measured from the lower edge

b) What is the maximum velocity in the boundary layer at this location and at what position in the boundary layer does the maximum occur?

c) Using the similarity solution result determine the heat transfer coefficient $0.25m$ from the lower edge

d) At what location on the plate measured from the lower edge will the boundary layer become turbulent?

Solution



Assumptions: (1) isothermal, vertical surface in an extensive, quiescent medium, (2) boundary layer assumptions valid.

Properties: Air ($T_f = (T_s + T_\infty)/2 = 350K$, $1atm$): $\nu = 20.92 \times 10^{-6} m^2/s$, $k = 0.030 W/mK$, $\text{Pr} = 0.700$.

Analysis:

a) From similarity solution results (see above right), the boundary layer thickness ($f' \approx 0$) corresponds to a value of $\eta \approx 5$.

$$y = \eta x (\text{Gr}_x/4)^{-1/4}$$

$$\text{Gr}_x = g\beta(T_s - T_\infty)\frac{x^3}{\nu^2} = 9.8m/s^2 \times \frac{1}{350K}(130-25)K\frac{x^3}{(20.92 \times 10^{-6}m^2/s)^2} = 6.718 \times 10^9 x^3$$

$$y \approx 5(0.25m)(6.718 \times 10^9 \times 0.25^3/4)^{-1/4} = 1.746 \times 10^{-2} m = 17.5mm$$

b) From the similarity solution shown above, the maximum velocity occurs at $\eta \approx 1$ with $f'(\eta) = 0.275$. We have then:

$$u = \frac{2\nu}{x}\text{Gr}_x^{1/2} f'(\eta) = \frac{2 \times 20.92 \times 10^{-6} m^2/s}{0.25m}(6.718 \times 10^9 \times 0.25^3)^{1/2} \times 0.275 = 0.47 m/s$$

The maximum velocity occurs at a value of $\eta = 1$, using the equation for y found in point (a), it follows that this corresponds to a position in the boundary layer given as:

$$y_{max} = 1/5(17.5mm) = 3.5mm$$

c) The local heat transfer coefficient at $x = 0.25m$ is:

$$\text{Nu}_x = \frac{h_x x}{k} = (\text{Gr}_x/4)^{1/4} g(\text{Pr}) = (6.718 \times 10^9 \times 0.25^3/4)^{1/4} 0.50 = 35.7$$

$$h_x = \text{Nu}_x \frac{k}{x} = 35.7 \times \frac{0.030W/mK}{0.25m} = 4.3 W/m^2K$$

The value for $g(\text{Pr})$ is determined for $\text{Pr} = 0.700$.

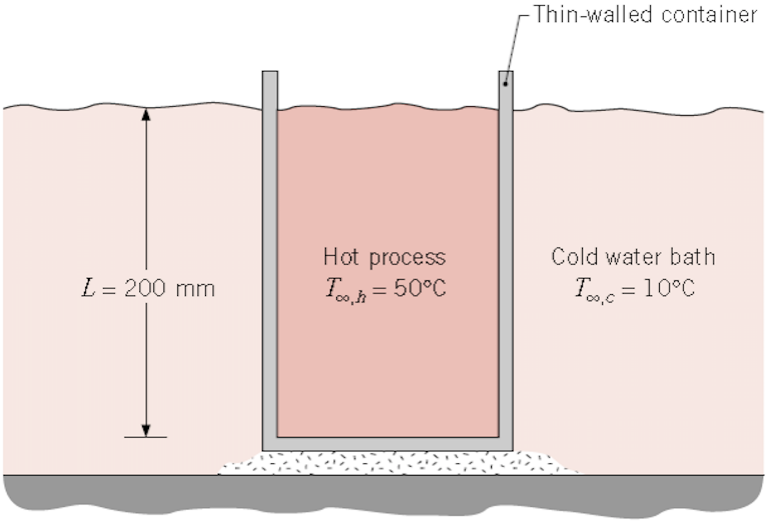
d) The boundary layer becomes turbulent at x_c given as:

$$\text{Ra}_{x,c} = \text{Gr}_{x,c}\text{Pr} \approx 10^9, \qquad x_c \approx (10^9/6.718 \times 10^9 \times 0.700)^{1/3} = 0.60m$$

Comments: Note that $\beta = 1/T_f$ is a suitable approximation for air.

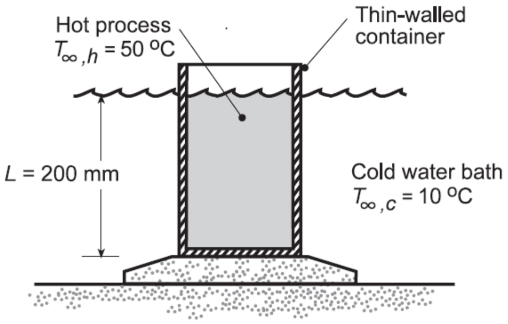
Exercise 7.2

A thin walled container with a hot process fluid at 50°C is placed in quiescent cold water bath at 10°C. Heat transfer at the inner and outer surfaces of the container may be approximated by free convection from a vertical plate. Determine the overall heat transfer coefficient between the hot process fluid and the cold water bath. Assume the properties of the hot process fluid are those of water.



Note: Estimate the temperature of the wall as the average between the hot and cold fluid temperatures. Estimate the physical properties of the hot and cold films at 310K and 295K respectively (how does this compare to the exact average temperature for each film?).

Solution



Assumptions: (1) steady-state conditions, (2) heat transfer at the surface approximated by free convection from a vertical plane, (3) fluids are extensive and quiescent, (4) hot process fluid thermophysical properties approximated as those of water, (5) negligible container wall thermal resistance.

Properties:

- Water (assume $T_{f,h} = 310K$): $\rho_h = 1/1.007 \times 10^{-3} = 993kg/m^3$, $c_{p,h} = 4178J/kgK$, $\nu_h = \mu_h/\rho_h = 695 \times 10^{-6}Ns/m^2/993kg/m^3 = 6.999 \times 10^{-7}m^2/s$, $k_h = 0.628W/mK$, $\text{Pr}_h = 4.62$, $\alpha_h = k_h/\rho_h c_{p,h} = 1.514 \times 10^{-7}m^2/s$, $\beta_h = 361.9 \times 10^{-6}K^{-1}$.
- Water (assume $T_{f,c} = 295K$): $\rho_c = 1/1.002 \times 10^{-3} = 998kg/m^3$, $c_{p,c} = 4181J/kgK$, $\nu_c = \mu_c/\rho_c = 959 \times 10^{-6}Ns/m^2/998kg/m^3 = 9.609 \times 10^{-7}m^2/s$, $k_c = 0.606W/mK$, $\text{Pr}_c = 6.62$, $\alpha_h = k_h/\rho_h c_{p,h} = 1.452 \times 10^{-7}m^2/s$, $\beta_h = 227.5 \times 10^{-6}K^{-1}$.

Analysis: The overall heat transfer coefficient between the hot process fluid, $T_{\infty,h}$, and the cold water bath fluid, $T_{\infty,c}$ is:

$$U = (1/\overline{h}_h + 1/\overline{h}_c)^{-1}$$

where we have neglected the conduction through the wall due to its negligible thickness. The average free convection coefficients can be estimated from the vertical plate correlation (surface temperature boundary condition) with the Rayleigh number:

$$\overline{\text{Nu}}_L = \left[0.825 + \frac{0.387\text{Ra}_L^{1/6}}{\left(1 + (0.492/\text{Pr})^{9/16}\right)^{8/27}}\right]^2, \qquad \text{Ra}_L = \frac{g\beta\Delta T L^3}{\nu\alpha}$$

To affect a solution, assume $T_s = (T_{\infty,h} + T_{\infty,c})/2 = 30°C = 303K$, so that the hot and cold fluid film temperature are $T_{f,h} = 313K \approx 310K$ and $T_{f,c} = 293K \approx 295K$. From an energy balance across the container walls,

$$\overline{h}_h(T_{\infty,h} - T_s) = \overline{h}_c(T_s - T_{\infty,c})$$

the surface temperature T_s can be determined. Evaluating the correlation parameters, find:

Hot process fluid:

$$\text{Ra}_{L,h} = \frac{9.8m/s^2 \times 361.9 \times 10^{-6}K^{-1}(50 - 30)K(0.200m)^3}{6.999 \times 10^{-7}m^2/s \times 1.514 \times 10^{-7}m^2/s} = 5.357 \times 10^9$$

$$\overline{\text{Nu}}_{L,h} = \left[0.825 + \frac{0.387(5.357 \times 10^9)^{1/6}}{\left(1 + (0.492/4.62)^{9/16}\right)^{8/27}}\right]^2 = 251.5$$

$$\overline{h}_h = \overline{\text{Nu}}_{L,h}\frac{k_h}{L} = 251.5 \times 0.628W/mK/0.200m = 790W/m^2K$$

Cold water bath:

$$\mathrm{Ra}_{L,c} = \frac{9.8m/s^2 \times 227.5 \times 10^{-6} K^{-1}(30-10)K(0.200m)^3}{9.609 \times 10^{-7} m^2/s \times 1.452 \times 10^{-7} m^2/s} = 2.557 \times 10^9$$

$$\overline{\mathrm{Nu}}_{L,c} = \left[0.825 + \frac{0.387(2.557 \times 10^9)^{1/6}}{\left(1 + (0.492/6.62)^{9/16}\right)^{8/27}}\right]^2 = 203.9$$

$$\overline{h}_c = \overline{\mathrm{Nu}}_{L,c} \frac{k_c}{L} = 203.9 \times 0.606 W/mK / 0.200m = 618 W/m^2K$$

Finally, we get the overall heat transfer:

$$U = (1/790 + 1/618)^{-1} W/m^2K = 347 W/m^2K$$

Exercise 7.3

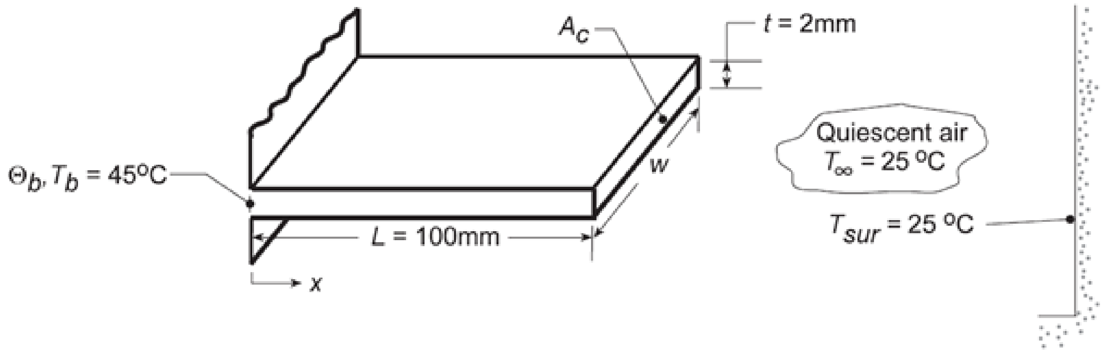
It is desired to estimate the effectiveness of a horizontal straight fin of rectangular cross section when applied to a surface operating at $45°C$ in an environment for which the surroundings and ambient air are at $25°C$. The fin is to be fabricated from aluminum alloy $(2024 - T6)$ with an anodized finish $(\epsilon = 0.82)$ and is $2mm$ thick and $100mm$ long.

a) Considering only free convection from the fin surface and estimating an average heat transfer coefficient, determine the effectiveness of the fin. *Hint 1*: be careful to pick the correct tip boundary condition for the fin; *Hint 2*: the convection coefficient for the entire fin can be obtained as the average of the natural convection coefficient on the top and bottom surfaces of the fin. *Hint 3*: Assume $w \gg L$ and that the characteristic length is $L_c \approx \frac{L}{2}$.

b) Estimate the effectiveness of the fin including the influence of radiation exchange with the surroundings.[difficult question]

Note: **Properties:** Aluminum alloy 2024-T6 $(T \approx (45 + 25)/2 = 35°\text{C} \approx 300\text{K})$, $\text{k} = 177\ \text{W/m} \cdot \text{K}$; $\epsilon = 0.82$;

Solution



Assumptions: (1) Steady-state conditions, (2) Constant properties, (3) One dimensional conduction in fin, (4) Width of fin much larger than length $w \gg L$, (5) Uniform heat transfer coefficient over length for parts (a) and (b).

Properties: *Table A-4*, Air ($\mathrm{T_f} \approx 300\mathrm{K}$), $\nu = 15.89 \cdot 10^{-6}\mathrm{m^2/s}, \mathrm{k} = 26.3 \cdot 10^{-3}\mathrm{W/m \cdot K}, \alpha = 22.5 \cdot 10^{-6}\mathrm{m/s}, \beta = 1/\mathrm{T_f} = 33.3 \cdot 10^{-3}\mathrm{K^{-1}}$

Analysis: (a) The effectiveness of a fin is determined from

$$\varepsilon = \mathrm{Q_f}/(\overline{\mathrm{h}}\mathrm{A_{c,b}}\theta_\mathrm{b})$$

where $\overline{\mathrm{h}}$ is the average heat transfer coefficient. For the case of convection at the tip, the fin heat transfer follows from

$$\mathrm{Q_f} = \mathrm{M}\frac{\sinh \mathrm{mL} + (\mathrm{h/mk}) \cosh \mathrm{mL}}{\cosh \mathrm{mL} + (\mathrm{h/mk}) \sinh \mathrm{mL}}$$

where

$$\mathrm{M} = (\mathrm{hPkA_c})^{1/2}\, \theta_\mathrm{b} \quad \text{and} \quad \mathrm{m} = (\mathrm{hP/kA_c})^{1/2}.$$

Horizontal, flat plate correlations assuming $\mathrm{T_f} = (\mathrm{T_b} + \mathrm{T_\infty})/2 \approx 300\mathrm{K}$ may be used to estimate $\overline{\mathrm{h}}$. Calculate first the Rayleigh number

$$\mathrm{Ra_{L_C}} = \frac{\mathrm{g}\beta\left(\overline{\mathrm{T}}_\mathrm{s} - \mathrm{T_\infty}\right)\mathrm{L_c^3}}{v\alpha}$$

where $\overline{\mathrm{T}}_\mathrm{s}$ is the average temperature of the fin surface and $\mathrm{L_c}$ is the characteristic length from

$$\mathrm{L_c} \equiv \frac{\mathrm{A_s}}{\mathrm{P}} = \frac{\mathrm{L} \times \mathrm{w}}{2\mathrm{L} + 2\mathrm{w}} \approx \frac{\mathrm{L}}{2}.$$

Substituting numerical values,

$$\mathrm{Ra_{L_c}} = \frac{9.8\mathrm{m/s^2} \times 1/300\mathrm{K} \times (310 - 298)\mathrm{K}\left(100 \cdot 10^{-3}/2\right)^3 \mathrm{m}^3}{22.5 \cdot 10^{-6}\mathrm{m^2/s} \times 15.89 \cdot 10^{-6}\mathrm{m^2/s}} = 1.37 \cdot 10^5$$

where $\overline{\mathrm{T}}_\mathrm{s} \approx (\mathrm{T_b} + \mathrm{T_f})/2 = 310\mathrm{K}$. Recognize the importance of this assumption which must be justified for a precise result. For the upper and lower surfaces respectively,

$$\mathrm{Nu_{L_c}} = 0.54\left(1.37 \cdot 10^5\right)^{1/4} = 10.4, \quad \overline{\mathrm{h}}_\mathrm{u} = \mathrm{Nu_{L_C}} \times \frac{\mathrm{k}}{\mathrm{L_c}} = \frac{0.0263\mathrm{W/m \cdot K}}{(100 \cdot 10^{-3}/2)\,\mathrm{m}} = 5.47\mathrm{W/m^2 \cdot K}$$

$$\mathrm{Nu_{L_C}} = 0.27\left(1.37 \cdot 10^5\right)^{1/4} = 5.20, \quad \overline{\mathrm{h}}_\ell = 2.73\mathrm{W/m^2 \cdot K}$$

The average values is estimated as $\overline{\mathrm{h}}_\mathrm{c} = \left(\overline{\mathrm{h}}_\mathrm{u} + \overline{\mathrm{h}}_\ell\right)/2 = 4.10\mathrm{W/m^2 \cdot K}$. Using this value in Eqs. $\mathrm{M} = (\mathrm{hKA_c})^{1/2}\,\theta_\mathrm{b} \quad \text{and} \quad \mathrm{m} = (\mathrm{hP/kA_c})^{1/2}$, find

$$\mathrm{M} = \left[4.10\mathrm{W/m^2 \cdot K}(2\mathrm{w})\mathrm{m} \times 177\mathrm{W/m \cdot K}\left(\mathrm{w} \times 2 \cdot 10^{-3}\right)\mathrm{m}^2\right]^{1/2}(45 - 25)^\circ\mathrm{C} = 34.1\mathrm{wW}$$

$$\mathrm{m}=\left(\overline{\mathrm{h}}_{\mathrm{c}} \mathrm{P} / \mathrm{kA}_{\mathrm{c}}\right)^{1 / 2}=\left[4.1 \mathrm{W} / \mathrm{m}^2 \cdot \mathrm{K}(2 \mathrm{w}) \mathrm{m} / 177 \mathrm{W} / \mathrm{m} \cdot \mathrm{K}\left(\mathrm{w} \times 2 \cdot 10^{-3}\right) \mathrm{m}^2\right]^{1 / 2}=4.81 \mathrm{m}^{-1}.$$

Substituting these values in the Eq. for q_f, with $mL = 0.481$ and $q_f/\mathrm{w} = q'_f$

$$\mathrm{Q}'\mathrm{f}=34.1 \mathrm{W} / \mathrm{m} \times \frac{\sinh 0.481+\left(4.1 \mathrm{W} / \mathrm{m}^2 \cdot \mathrm{K} / 4.81 \mathrm{m}^{-1} \times 177 \mathrm{W} / \mathrm{m} \cdot \mathrm{K}\right) \cosh 0.481}{\cosh 0.481+\left(4.86 \cdot 10^{-3}\right) \sinh 0.481}=15.2 \mathrm{W} / \mathrm{m}$$

and then from the first Eq. the effectiveness is

$$\varepsilon=15.2 \mathrm{W} / \mathrm{m} \times \mathrm{w} / 4.1 \mathrm{W} / \mathrm{m}^2 \cdot \mathrm{K}\left(\mathrm{w} \times 2 \cdot 10^{-3} \mathrm{m}\right)(45-25)^{\circ} \mathrm{C}=92.7$$

(b) If radiation exchange with the surroundings is considered,

$$\overline{\mathrm{h}}_{\mathrm{r}}=\epsilon \sigma\left(\overline{\mathrm{T}}_{\mathrm{s}}+\mathrm{T}_{\mathrm{sur}}\right)\left(\overline{\mathrm{T}}_{\mathrm{s}}^2+\mathrm{T}_{\mathrm{sur}}^2\right)=0.82 \times 5.67 \cdot 10^{-8} \mathrm{W} / \mathrm{m}^2 \cdot \mathrm{K}^4(310+298)\left(310^2+298^2\right) \mathrm{K}^3=5.23 \mathrm{W} / \mathrm{m}^2 \cdot \mathrm{K}.$$

This assumes the fin is gray-diffuse and small compared to the surroundings. Using $\overline{\mathrm{h}}=\overline{\mathrm{h}}_{\mathrm{c}}+\overline{\mathrm{h}}_{\mathrm{r}}$ where $\overline{\mathrm{h}}_{\mathrm{c}}$ is the convection parameter from part (a), we find $\overline{\mathrm{h}}=(4.10+5.23) \mathrm{W} / \mathrm{m}^2 \cdot \mathrm{K}=9.33 \mathrm{W} / \mathrm{m}^2 \cdot \mathrm{K}$, $\mathrm{M}=51.4 \mathrm{w}\ \mathrm{W},\ \mathrm{m}=7.26 \mathrm{m}^{-1},\ \mathrm{Q}_{\mathrm{f}}'=31.8 \mathrm{W} / \mathrm{m}$ giving

$$\varepsilon=85.2$$

Exercise 7.4 FOR REVISION

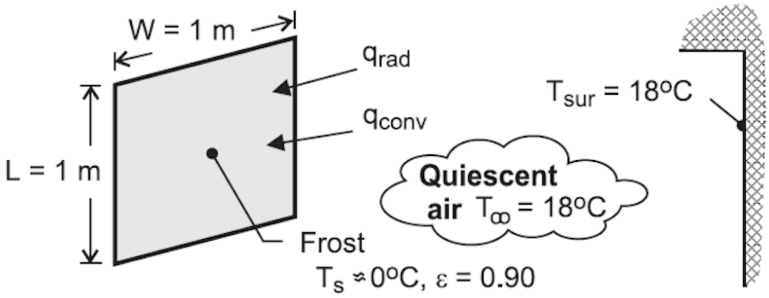
Consider a vertical single-pane window of equal width and height ($W = L = 1m$). The interior surface is exposed to the air and walls of a room, which are each at $18°C$. Under cold ambient conditions for which a thin layer of frost has formed on the inner surface, what is the heat loss though the window? Do NOT neglect radiative heat transfer. The frost might be assumed to have an emissivity $\epsilon = 0.9$.

In addition, comment on the following aspects:

a) During incipience of frost formation, where would you expect the frost to begin developing on the window? In other words, which part of the window is coldest?

b) How would your analysis be affected by a frost layer whose thickness is not negligible?

Solution



Assumptions: (1) steady-state, (2) surface of frost isothermal with $T_s \approx 0°C$, (3) radiation exchange is between a small surface (window) and a large enclosure (walls of room), (4) room air is quiescent.

Properties: Air ($T_f = 9°C = 282K$): $k = 0.0249W/mK$, $\nu = 14.3 \times 10^{-6}m^2/s$, $\alpha = 20.1 \times 10^{-6}m^2/s$, $\Pr = 0.712$, $\beta = 3.55 \times 10^{-3}K^{-1}$.

Analysis: Under steady-state conditions, the heat loss through the window corresponds to the rate of heat transfer to the frost by convection and radiation:

$$q = q_{conv} + q_{rad} = W \times L\left[\overline{h}(T_\infty - T_s) + \varepsilon\sigma(T_{sur}^4 - T_s^4)\right]$$

with $\mathrm{Ra}_L = g\beta(T_\infty - T_s)L^3/\alpha\nu = 9.8m/s^2 \times 0.00355K^{-1} \times 18K(1m)^3/(14.3 \times 20.1 \times 10^{-12}m^4/s^2) = 2.18 \times 10^9$, we can determine the average $\overline{Nu_L}$ (we are interested in the overall heat transfer from the plate) for the surface temperature boundary condition:

$$\overline{\mathrm{Nu}}_L = \left[0.825 + \frac{0.387\mathrm{Ra}_L^{1/6}}{\left(1 + (0.492/\mathrm{Pr})^{9/16}\right)^{8/27}}\right]^2 = 156.5$$

$$\overline{h} = \mathrm{Nu}_L\frac{k}{L} = 156.5\frac{0.0249W/mK}{1m} = 3.9W/m^2K$$

$$q = (1m^2)\left[3.9W/m^2K \times 18K + 0.90 \times 5.67 \times 10^{-8}W/m^2K^4(291^4 - 273^4)\right] = 70.2W + 82.5W = 152.7W$$

Comments:

a) Since the thermal boundary layer thickness is zero at the top of the window and has its maximum value at the bottom, the temperature of the glass will actually be largest and smallest at the top and bottom respectively. Hence, frost will first begin to form at the bottom.

b) The thickness of the frost layer does not affect the heat loss, since the inner surface of the layer remains at $T_s \approx 0°C$. However, the temperature of the glass/frost interface decreases with increasing thickness, from a value of $0°C$ for negligible thickness.