Exercise 5.1

Consider steady, parallel flow of atmospheric air over a flat plate. The air has a temperature and free stream velocity of $300K$ and $25m/s$

a) Evaluate the boundary layer thickness at distances of $x = 1, 10, 100mm$ from the leading edge.

b) If a second plate were installed parallel to and at a distance of $3mm$ from the first pate, what is the distance from the leading edge at which boundary layer merging would occur?

c) Evaluate the surface shear stress and the y-velocity component at the outer edge of the boundary layer for the single plate at $x = 1, 10, 100mm$

d) Comment on the validity of the boundary layer model

TABLE 7.1 Flat plate laminar boundary layer functions [3]

$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2 f}{d\eta^2}$
0	0	0	0.332
0.4	0.027	0.133	0.331
0.8	0.106	0.265	0.327
1.2	0.238	0.394	0.317
1.6	0.420	0.517	0.297
2.0	0.650	0.630	0.267
2.4	0.922	0.729	0.228
2.8	1.231	0.812	0.184
3.2	1.569	0.876	0.139
3.6	1.930	0.923	0.098
4.0	2.306	0.956	0.064
4.4	2.692	0.976	0.039
4.8	3.085	0.988	0.022
5.2	3.482	0.994	0.011
5.6	3.880	0.997	0.005
6.0	4.280	0.999	0.002
6.4	4.679	1.000	0.001
6.8	5.079	1.000	0.000

Solution

Assumptions: 1) steady flow, 2) boundary layer approximations are valid, 3) flow is laminar.

Properties: Air(300K, 1atm): $\rho = 1.161 kg/m^3$, $\nu = 15.89 \times 10^{-6} m^2/s$.

Analysis:

a) We have seen that the boundary layer evolution can be solved for by defining two similarity variables. Eventually,for laminar flow, we have seen that we can write:(Formula from W5L1-1h Slides, page 20)

$$\delta \equiv df'(\eta)/d\eta = \frac{u(y=\eta)}{u_\infty} = 0.99$$

We know that:

$$\eta = y\sqrt{u_\infty/\nu x}$$

From the table of $[\eta, f, f']$ we found that:

$$\eta(\delta) = \eta(f'(\eta) = 0.99) = 4.92 = \delta\sqrt{u_\infty/\nu x}$$

and we have thus obtained:

$$\delta = \frac{4.92x}{\mathrm{Re}_x^{1/2}} = \frac{4.92}{(u_\infty/\nu)^{1/2}} x^{1/2} = \frac{4.92x^{1/2}}{(25m/s\,/15.89 \times 10^{-6}m^2/s)^{1/2}} = 3.99 \times 10^{-3} x^{1/2}$$

At the requested positions along the plate we thus obtain:

x [m]	0.001	0.01	0.1
δ [mm]	0.126	0.399	1.262

b) As the plate are separated by $d = 3mm$ the boundary layers developing on each plate merge at $x = x_m$ for which $\delta = 1.5$mm. Given that:

$$x^{1/2} = \frac{\delta}{3.99 \times 10^{-3} m^{1/2}}$$

we can calculate:

$$x_m^{1/2} = \frac{0.0015m}{3.99 \times 10^{-3} m^{1/2}} = 0.376 m^{1/2}, \qquad x_m = 141mm$$

c) We have also seen that for the laminar case the friction coefficient is:

$$C_f \equiv \frac{\tau_w}{\rho u_\infty^2/2} = \frac{0.664}{Re_x^{1/2}}$$

Therefore the shear stress is

$$\tau_{s,x} = 0.664\frac{\rho u_\infty^2/2}{\mathrm{Re}_x^{1/2}} = 0.664\frac{\rho u_\infty^2/2}{(u_\infty/\nu)^{1/2}x^{1/2}} = \frac{0.664\times 1.161kg/m^3(25m/s)^2/2}{(25m/s\,/15.89\times 10^{-6}m^2/s)^{1/2}x^{1/2}} = \frac{0.192}{x^{1/2}}N/m^2$$

x [m]	0.001	0.01	0.1
$\tau_{s,x}$ [N/m^2]	6.07	1.92	0.61

We have also seen that the velocity distribution in the boundary layer can be expressed as:

$$v = 0.5\left(\frac{\nu u_\infty}{x}\right)^{1/2}[\eta f'(\eta) - f]$$

Always from the table of $[\eta, f, f']$ we have that for $y=\delta$:

$$\eta = 4.92$$

$$f = 3.2$$

$$\mathrm{d}f/\mathrm{d}\eta = f' = 0.99$$

and therefore

$$v = \frac{0.5}{x^{1/2}}(15.89\times 10^{-6}m^2/s\times 25m/s)^{1/2}(4.92\times 0.99 - 3.2) = (0.0167/x^{1/2})m/s$$

x [m]	0.001	0.01	0.1
v [m/s]	0.528	0.167	0.053

Comments:

1. $v \ll u_\infty$ and $\delta \ll x$ are consistent with BL approximations. Note, $v \to \infty$ as $x \to 0$ and approximations breakdown very close to the leading edge.
2. Since $\mathrm{Re}_{x_m} = 2.22\times 10^5$, laminar BL model is valid.
3. Above expressions are approximations for flow between parallel plates, since $\mathrm{d}u_\infty/\mathrm{d}x > 0$ and $\mathrm{d}p/\mathrm{d}x < 0$.

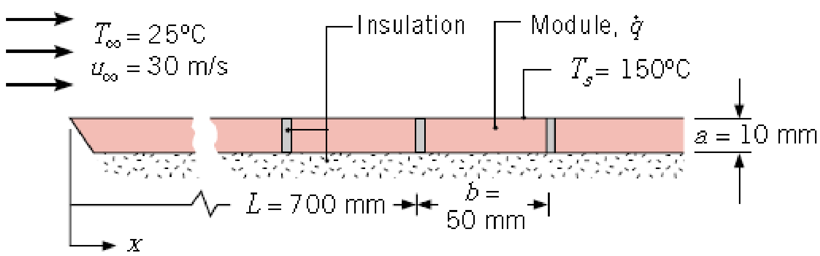
Exercise 5.2

A flat plate of width $W = 1m$ is maintained at a uniform surface temperature of $T_s = 150C$ by using independently controlled, heat-generating rectangular modules of thickness $a = 10mm$ and length $b = 50mm$. Each module is insulated from its neighbors, as well as on its back side. Atmospheric air at $25C$ flows over the plate at a velocity of $30m/s$. The thermophysical properties of the module are $k = 5.2W/mK$, $c_p = 320J/kgK$, and $\rho = 2300kg/m^3$.

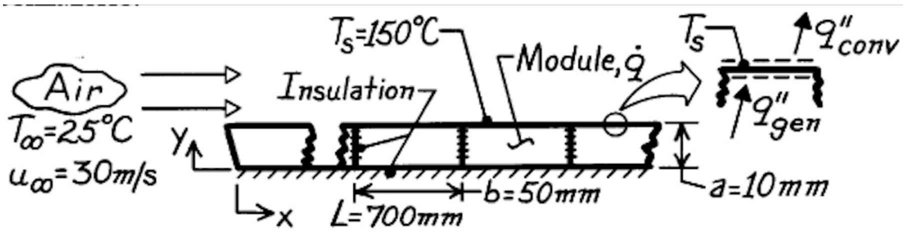
a) Find the required power generation $\dot{q}W/m^3$, in a module positioned at a distance $L = 700mm$ from the leading edge

b) Find the maximum temperature T_{max} in the heat-generating module

Note: Air properties are $k_{air} = 0.0308W/mK$, $\nu = 22.02 \cdot 10^{-6}m^2/s$, $Pr = 0.698$



Solution



Assumptions: 1) Laminar flow at leading edge of the plate, 2) transitions Reynolds number of 5×10^5, 3) heat transfer is one-dimensional in y-direction within each module, 4) $\dot{q}$ is uniform within module, 5) negligible radiation heat transfer.

Properties: Module material (given): $k = 5.2W/mK$, $c_p = 320J/kgK$, $\rho = 2300kg/m^3$; Air($\overline{T}_f = (T_s + T_\infty)/2 = 360$K, 1atm): $k = 0.0308W/mK$, $\nu = 22.02 \times 10^{-6}m^2/s$, Pr = 0.698.

Analysis:

a) The module power generation follows from an energy balance on the module surface,

$$q''_{conv} = q''_{gen}$$

$$\overline{h}(T_s - T_\infty) = \dot{q} \cdot a \quad \text{or} \quad \dot{q} = \frac{\overline{h}(T_s - T_\infty)}{a}$$

where we have used the AVERAGE convection coefficient over the module to write the energy balance. To select a convection correlation for estimating $\overline{h}$, first we need to find the Reynolds number at $x = L$:

$$\text{Re}_L = \frac{u_\infty L}{\nu} = \frac{30m/s \times 0.70m}{22.02 \times 10^{-6}m^2/s} = 9.877 \times 10^5 > 5 \cdot 10^5$$

therefore the flow is turbulent.

We also find that the onset of turbulent flow occurs at:

$$L_{o,turb} = Re_{turb}\nu/u_\infty = 5 \cdot 10^5 \times 22.02 \cdot 10^{-6}/30 = 0.367m$$

Therefore the flow is turbulent over six modules before the one of interest and it is thus appropriate to estimate the average convection coefficient using the local Re value at the center of the module of interest, i.e. $\overline{h} \approx h_x(L + b/2)$ is appropriate, with:

$$\text{Re}_{L+b/2} = \frac{30m/s \times (0.700 + 0.050/2)m}{22.02 \times 10^{-6}m^2/s} = 9.877 \times 10^5$$

Using the correlation for forced convection/external/turbulent flow evaluated at $x = L + b/2 = 0.725m$:

$$\text{Nu}_x = \frac{h_x x}{k} = 0.0296 \cdot \text{Re}_x^{4/5}\text{Pr}^{1/3} = 0.0296 \cdot (9.877 \times 10^5)^{4/5}(0.698)^{1/3} = 1640$$

$$\overline{h} \approx h_x = \frac{\text{Nu}_x k}{x} = \frac{1640 \times 0.0308W/mK}{0.725} = 69.7W/m^2K$$

Hence,

$$\dot{q} = \frac{69.7W/m^2K(150 - 25)K)}{0.010m} = 8.713 \times 10^5$$

b) This module is insulated on all sides and heat can flow only towards the air. Also, we have uniform heat generation in the module. Therefore, based on our analysis of conduction in a wall with uniform heat sources, we know that the maximum temperature within the module occurs at the surface next to the insulation ($y = 0$). Furthermore, we remember that for one-dimensional conduction with thermal energy generation, the temperature profile within the wall is parabolic:

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

To solve for the actual temperature profile we have to apply the proper boundary conditions. For a wall of thickness a with an adiabatic insulation at $x = 0$ and a surface temperature of T_s at the surface, the boundary conditions are:

$$\frac{\partial T}{\partial x}|_{x=0} = 0$$
$$T(a) = T_s$$

and we obtain:

$$T(x) = -\frac{\dot{q}}{2k}x^2 + \frac{\dot{q}a^2}{2k} + T_s$$

Hence we can easily get $T(0)$:

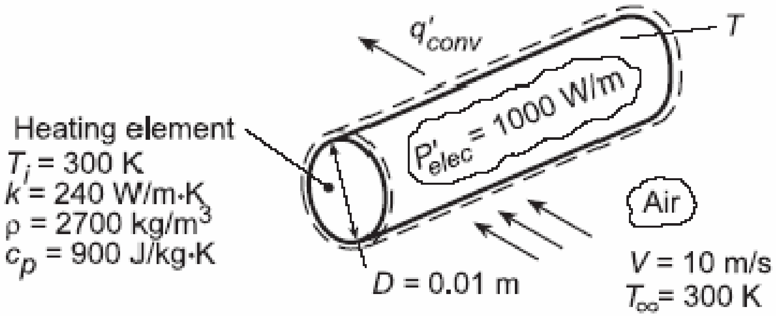
$$T(0) = \frac{\dot{q}a^2}{2k} + T_s = \frac{8.713 \times 10^5 W/m^3 \times (0.010m)^2}{2 \times 5.2W/mK} + 150°C = 158.4°C$$

Exercise 5.3

A long, cylindrical, electrical heating element of diameter $D = 10mm$, thermal conductivity $k = 240W/mK$, density $\rho = 2700\text{kg/m}^3$ and specific heat $c_p = 900J/kgK$ is installed in a duct for which air moves in cross flow over the heater at a temperature and velocity of $27°C$ and $10m/s$ respectively.

- Neglecting radiation, estimate the steady-state surface temperature when, per unit length of the heater, electrical energy is being dissipated at a rate of $1000W/m$. To estimate the thermophysical properties use $T_f = 450K$.
- If the heater is activated from an initial temperature of $27°C$, estimate the time required for the surface to come within $10°C$ of its steady state value.

Solution



Assumptions: (1) Uniform heater temperature, (2) Negligible radiation

Properties: *Table A.4*, air (assume $T_f \approx 450\,K$), $Pr = 0.686$, $\nu = 32.39{\cdot}10^{-6}\,m^2/s$, $k = 0.0373\,W/mK$.

Analysis:

a) Performing an energy balance fro steady-state conditions, we obtain:

$$\mathrm{q'_{conv}} = \overline{\mathrm{h}}(\pi \mathrm{D})\,(\mathrm{T} - \mathrm{T_\infty}) = \mathrm{P'_{elec}} = 1000\mathrm{W/m}$$

With

$$\mathrm{Re}_D = \frac{\mathrm{VD}}{\nu} = \frac{(10\mathrm{m/s})0.01\mathrm{m}}{32.39 \times 10^{-6}\mathrm{m^2/s}} = 3,087$$

For forced external convection over a cylinder we can use the Churchill and Bernstein correlation for all values of Re, therefore:

$$\overline{\mathrm{Nu}}_\mathrm{D} = 0.3 + \frac{0.62\mathrm{Re}_\mathrm{D}^{1/2}\mathrm{Pr}^{1/3}}{\left[1 + (0.4/\mathrm{Pr})^{2/3}\right]^{1/4}}\left[1 + \left(\frac{\mathrm{Re}_\mathrm{D}}{282,000}\right)^{5/8}\right]^{4/5}$$

$$\overline{\mathrm{Nu}}_\mathrm{D} = 0.3 + \frac{0.62(3087)^{1/2}(0.686)^{1/3}}{\left[1 + (0.4/0.686)^{2/3}\right]^{1/4}}\left[1 + \left(\frac{3087}{282,000}\right)^{5/8}\right]^{4/5} = 28.2$$

$$\overline{\mathrm{h}} = \frac{\mathrm{k}}{\mathrm{D}}\overline{\mathrm{Nu}}_\mathrm{D} = \frac{0.0373\mathrm{W/m \cdot K}}{0.010\mathrm{m}}28.2 = 105.2\mathrm{W/m^2 \cdot K}$$

Hence the steady-state temperature is

$$\mathrm{T} = \mathrm{T_\infty} + \frac{\mathrm{P'_{elec}}}{\pi \mathrm{Dh}} = 300\mathrm{K} + \frac{1000\mathrm{W/m}}{\pi(0.01\mathrm{m})105.2\mathrm{W/m^2 \cdot K}} = 603\mathrm{K}$$

b) With $Bi = \overline{h}r_0/2k = 105.2W/m^2 \cdot K(0.005m)/2 \cdot 240\mathrm{W/m \cdot K} = 0.0011 < 0.1$, a lumped capacitance analysis may be performed.

Yet, due to the heat sources in the body, we cannot use the simple expression for the time evolution of the body temperature we derived in class. Instead we need to go back and write the time-dependent energy balance onto the cylinder:

$$\rho c V \frac{\partial T}{\partial t} = -Q_{conv} + \dot{q}V$$

$$\rho c V \frac{\partial T}{\partial t} = -\overline{h}A_s(T - T_\infty) + \dot{q}V$$

We write:

$$\frac{\partial(T-T_\infty)}{\partial t} = -\frac{4\bar{h}}{\rho c D}(T-T_\infty) + \frac{\dot{q}}{\rho c}$$

We can write the volumetric heat generation as:

$$\dot{q} = \frac{P'_{elec}L}{V} = \frac{4P'_{elec}}{\pi D^2}$$

And so we write:

$$\frac{\partial(T-T_\infty)}{\partial t} = -\frac{4\bar{h}}{\rho c D}(T-T_\infty) + \frac{4\bar{h}}{\rho c D}\frac{P'_{elec}}{\pi D\bar{h}}$$

By writing;

$$\theta = T - T_\infty - \frac{P'_{elec}}{\pi D\bar{h}}$$

$$a = \frac{4\bar{h}}{\rho c D}$$

we can rearrange as:

$$\frac{\partial\theta}{\partial t} = -a\theta$$

Integrating from $t=0$ to t we get the solution:

$$\frac{\theta}{\theta_i} = exp(-at)$$

$$\frac{T - T_\infty - \frac{P'_{elec}}{\pi D\bar{h}}}{T_i - T_\infty - \frac{P'_{elec}}{\pi D\bar{h}}} = exp(-at)$$

That we can re-write as:

$$T - T_\infty = \frac{P'_{elec}}{\pi D\bar{h}}[1-exp(-at)] + (T_i - T_\infty)exp(-at)$$

In this case $T_i = T_\infty$ and therefore the equation reduces to:

$$\mathrm{T} = \mathrm{T}_\infty + \frac{P'_{elec}}{\pi D\bar{h}}[1-\exp(-\mathrm{at})]$$

where $a = 4\bar{\mathrm{h}}/\mathrm{D}\rho\mathrm{c_p} = \left(4\times105.2\mathrm{W/m^2\cdot K}\right)/\left(0.01\mathrm{m}\times2700\mathrm{kg/m^3}\times900\mathrm{J/kg\cdot K}\right) = 0.0173\mathrm{s^{-1}}$ and $\mathrm{P'_{elec}}/\pi\mathrm{Dh} = 1000\mathrm{W/m}/\pi\left(0.01\mathrm{m}\times105.2\mathrm{W/m^2\cdot K}\right) = 302.6\mathrm{K}$.

We want to know the time needed to reach $T = T_{steadystate} - 10 = 593K$ and thus we write:

$$[1-\exp(-0.0173\mathrm{t})] = \frac{(593-300)\mathrm{K}}{302.6\mathrm{K}} = 0.968$$

$$t \approx 200s$$

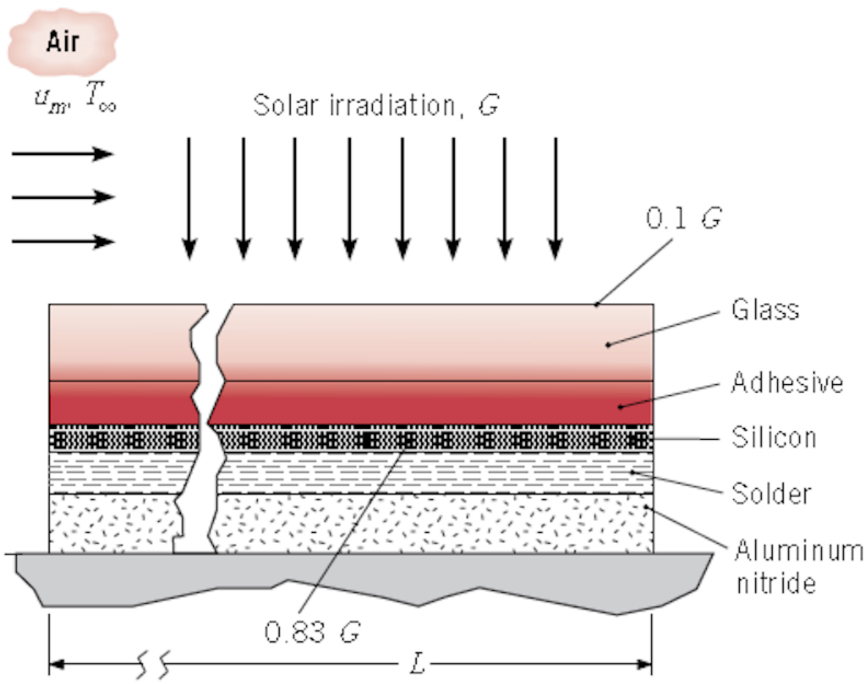
Exercise 5.4 [Difficult] FOR REVISION

A photovoltaic solar panel consists of a sandwich of (top to bottom) a $3mm$ thick ceria-doped glass($k_g = 1.4W/mK$), a $0.1mm$ thick optical grade adhesive ($k_a = 145W/mK$), a very thin silicon semiconducting material, a $0.1mm$ thick solder layer ($k_s = 50W/mK$) and a $2mm$ thick aluminum nitride substrate ($k_{an} = 120W/mK$). The solar-to-electrical conversion efficiency within the semiconductor depends on the silicon temperature, T_{si} and is described by the expression $\eta = 0.28 - 0.001T_{si}$, where T_{si} is in C for $25C < T_{si} < 250C$. Ten percent of the solar irradiation is absorbed at the top surface of the glass, while 83% of the solar irradiation is transmitted to and absorbed by the silicon (the remaining 7% is reflected away from the cell). The glass has an emissivity of 0.9. Consider an $L = 1m$ long, $w = 0.1m$ wide solar cell that is placed on an insulated surface. Air at $25C$ flows over the solar cell, parallel to the long direction, with a velocity of $4m/s$. The temperature of the surroundings is also $25C$. The solar irradiation is $G = 700W/m^2$. The boundary layer is tripped to turbulent condition at the leading edge of the panel at therefore the correlation for the turbulent case must be used to estimate the average value of Nu regardless of the Re_L value:

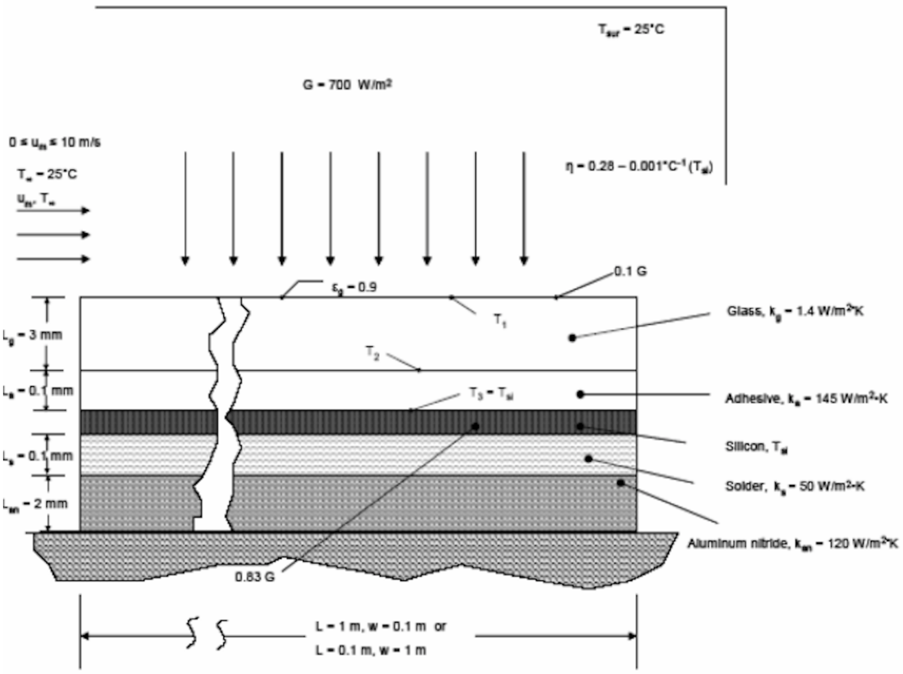
$$\overline{\mathrm{Nu}}_L = 0.037\,\mathrm{Re}_L^{4/5}\mathrm{Pr}^{1/3}$$

a) Draw the thermal circuit that represents the entire problem. Do not forget radiation heat transfer to the environment. Note that no radiation penetrates deeper than the silicon layer. Also, assume that the stack is sitting on a perfect insulating layer.

b) Determine the temperature of the silicon layer and the electric power produced by the solar cell.



Solution

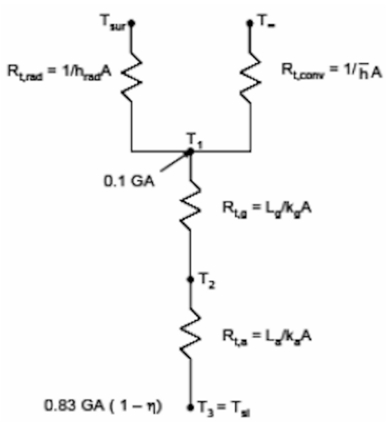


Assumptions: 1) Steady-state conditions, 2) constant properties, 3) one-dimensional heat transfer, 4) tripped and turbulent boundary layer, 5) large surroundings, 6) negligible contact resistances.

Properties: Air(assume $T_f = 298$K, $p = 1$atm): $k = 0.0269W/mK$, $\nu = 1.58 \times 10^{-5}m^2/s$, Pr $=$ 0.706.

Analysis:

a) We begin by drawing the thermal circuit for the problem, recognizing that there is no heat transfer downward from the thin silicon layer. The thermal resistances are:



$$R_{t,g} = L_g/k_gA = 3 \times 10^{-3}m/(1.4W/mK \times 1m \times 0.1m) = 21.43 \times 10^{-3}K/W$$
$$R_{t,a} = L_a/k_aA = 0.1 \times 10^{-3}m/(145W/mK \times 1m \times 0.1m) = 6.897 \times 10^{-6}K/W$$
$$h_{rad} = \varepsilon_g\sigma(T_1 + T_{sur})(T_1^2 + T_{sur}^2)$$
$$R_{t,rad} = \frac{1}{0.9 \times 5.67 \times 10^{-8}W/m^2K^4 \times (T_1 + 298K) \times (T_1^2 + (298K)^2) \times 1m \times 0.1m}$$

For the tripped boundary layer,

$$\mathrm{Re}_L = \frac{u_m L}{\nu} = \frac{4m/s \times 1m}{1.669 \times 10^{-5} m^2/s} = 239.7 \times 10^3$$

Although $Re_L < 5 \cdot 10^5$ we know from the text that the flow is turbulent over the plate. For estimating the average convection coefficient we thus use:

$$\overline{\mathrm{Nu}}_L = 0.037\,\mathrm{Re}_L^{4/5}\mathrm{Pr}^{1/3} = 0.037 \times [239.7 \times 10^3]^{0.8} \times 0.706^{1/3} = 662.8$$

$$\bar{h} = \frac{\overline{\mathrm{Nu}}_L k}{L} = \frac{662.8 \times 0.0269 W/mK}{1m} = 17.82 W/m^2K$$

$$R_{t,conv} = \frac{1}{\bar{h}A} = \frac{1}{17.82 W/m^2K \times 1m \times 0.1m} = 561.2 \times 10^{-3} K/W$$

From the thermal circuit,

$$0.83\,GA(1-\eta) = (T_3 - T_1)/(R_{t,g} + R_{t,a}) \quad \text{or} \quad T_3 - T_1 = (R_{t,g} + R_{t,a})\,0.83\,GA(1-\eta)$$

$$T_3 - T_1 = (21.43 \times 10^{-3} K/W + 6.897 \times 10^{-6} K/W) \times 0.83 \times 700 W/m^2 \times 1m \times 0.1m \times (1-\eta) = 1.245(1-\eta)$$

We also note from the thermal circuit,

$$0.83\,GA(1-\eta) + 0.1\,GA = (T_1 - T_{sur})/R_{t,rad} + (T_1 - T_\infty)/R_{t,conv}$$

Since $T_\infty = T_{sur}$,

$$0.83\,GA(1-\eta) + 0.1\,GA = (T_1 - T_{sur})\left[\frac{1}{R_{t,rad}} + \frac{1}{R_{t,conv}}\right]$$

$$\begin{aligned} T_1 - T_{sur} &= \frac{0.83\,GA(1-\eta) + 0.1\,GA}{\left[\frac{1}{R_{t,rad}} + \frac{1}{R_{t,conv}}\right]} \\ &= \frac{0.83 \times 700 W/m^2 \times 1m \times 0.1m \times (1-\eta) + 0.1 \times 700 W/m^2 \times 1m \times 0.1m}{\frac{1}{R_{t,rad}} + 1.7819 W/K} \\ &= \frac{58.1 W(1-\eta) + 7W}{\frac{1}{R_{t,rad}} + 1.7819 W/K} \end{aligned}$$

where $\eta = 0.28 - 0.001^\circ C^{-1} \times (T_3 - 273)^\circ C$

Using the equations for $R_{t,rad}$, $T_3 - T_1$, $T_1 - T_{sur}$ and η, we get:

$$\eta = 0.2324, \quad T_3 = T_{si} = 47.6^\circ C, \quad T_1 = 46.6^\circ C, \quad R_{t,rad} = 1.661 K/W$$

The electric power is $P = 0.83\,GA\eta = 0.83 \times 700 W/m^2 \times 1m \times 0.1m \times 0.2324 = 13.50 W$.