Exercise 3.1

Turbine blades mounted to a rotating disk in a gas turbine engine are exposed to a gas stream that is at $T_\infty = 1200°\text{C}$ and maintains a convection coefficient of $h = 250\ \text{W/m}^2\text{K}$ over the blade.

The blades are fabricated from Inconel with $k = 20\ \text{W/mK}$ and have a length of 50 mm. The blade profile has a constant cross-sectional area of $A_c = 0.0006\ \text{m}^2$ and a perimeter $P = 110$ mm. A proposed blade cooling scheme, which involves routing air through the supporting disc, is able to maintain the base of each blade at a temperature $T_b = 300°\text{C}$.

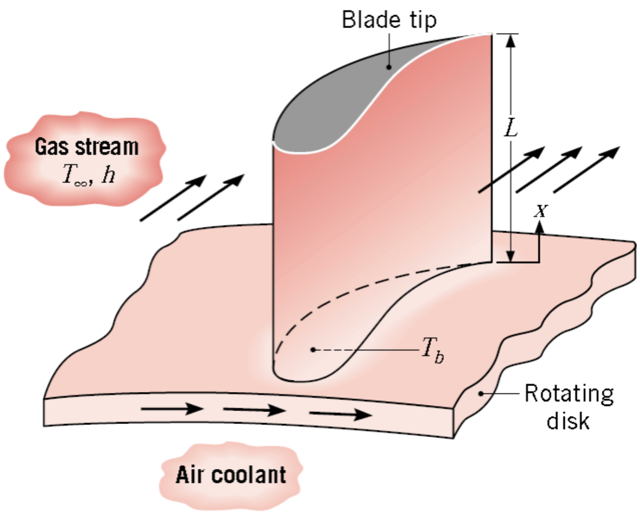
a) If the maximum allowable blade temperature is 1050°C, is the proposed cooling scheme satisfactory?

   Hint 1: treat the turbine blade as a fin of constant cross-section and assume that the tip of the blade is adiabatic.

   Hint2: carefully consider the physical problem and imagine where the temperature would be maximum in this fin!!

b) For the proposed cooling scheme, what is the rate at which heat is transferred from each blade to the coolant?

If necessary use the hyperbolic function table at the end of the document to determine the necessary values.



Solution

a) The blade can be considered a fin with adiabatic tip. Hence the equation that describes the temperature profile is case B of the table:

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b		Fin Heat Transfer Rate q_f	
A	Convection heat transfer: $h\theta(L) = -kd\theta/dx\vert_{x=L}$	$\frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	(3.70)	$M\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$	(3.72)
B	Adiabatic $d\theta/dx\vert_{x=L} = 0$	$\frac{\cosh m(L-x)}{\cosh mL}$	(3.75)	$M \tanh mL$	(3.76)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	(3.77)	$M\frac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$	(3.78)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx}	(3.79)	M	(3.80)

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$
$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\theta_b$

The peak temperature in this fin will be maximum at the tip of the blade. In fact, contrary to the case seen in class, in this case the fin removes heat from the gas stream towards the disk. So the peak temperature will be at the farthest point from the fin base. Equation do not change however, for $x = L$ we obtain:

$$\frac{T(L) - T_\infty}{T_b - T_\infty} = \frac{1}{\cosh mL}$$

$$m = \left(\frac{hP}{kA_c}\right)^{1/2} = \left(\frac{250\text{W/m}^2\text{K} \times 0.11\text{m}}{20\text{W/mK} \times 6 \times 10^{-4}\text{m}^2}\right)^{1/2}$$

$$m = 47.87\text{m}^{-1} \quad \text{and} \quad mL = 47.87\text{m}^{-1} \times 0.05\text{m} = 2.39$$

From the hyperbolic function table at the end of the document, we get: $\cosh mL = 5.51$

Hence: $T(L) = 1200°\text{C} + (300 - 1200)°\text{C}/5.51 = 1037°\text{C}$

The operating conditions are therefore acceptable.

b) Heat transfer is: $Q_f = M \tanh mL = -517\text{W} \times 0.983 = -508\text{W}$

Notice the negative sign of Q. It flows opposite to the x-axis we used to derive the equations, so from the tip towards the base of the fin.

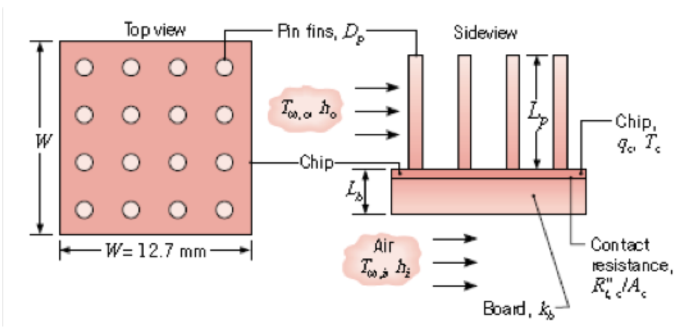
Exercise 3.2

The maximum operation temperature for an electronic chip is 75°C. To maximize the dissipation from a square chip with side length $W = 12.7mm$, it is proposed that a 4x4 array of copper fins ($k = 400$W/mK) be metallurgically joined to the outer surface of the chip.

a) Sketch the equivalent thermal circuit for the pin-chip-board assembly assuming 1D steady state conditions and negligible contact resistance between the pins and the chip. Hint: consider the heat removed from a unit cell of the periodic arrangement of fins. Consider the thermal resistance of one fin and do not forget to account for convection at the exposed surface of the chip which is not covered by the fins.

b) Write the expression for all the thermal resistances involved in the problem. Consider the convection heat transfer at the tip of each fin.

c) the pin diameter and length are D_p = 1.5mm and L_p = 15mm respectively and the contact resistance is $R_t'' = 10^{-4}$ $[\frac{m^2K}{W}]$, find what is the maximum heat dissipation rate Q_c when T_c = 75°C and h = 1000W/m2K



Solution

Assumptions:

a) Steady-state conditions

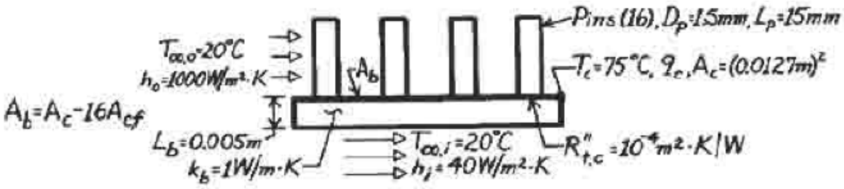
b) One-dimensional heat transfer in chipboard assembly

c) Negligible Pin chip contact resistance

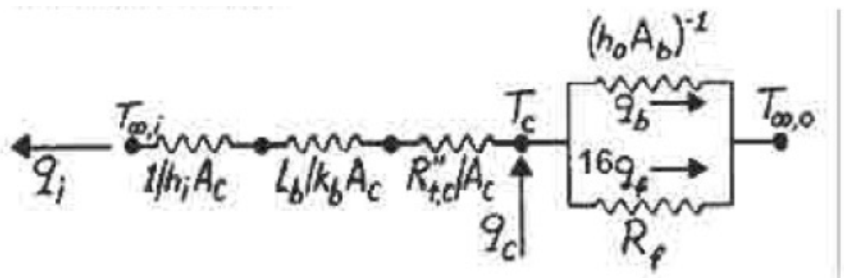
d) Constant Properties

e) Negligible chip thermal resistance

f) Uniform chip temperature



a) The system can be modelled as:



Note that we have indicated separate parallel paths for heat transfer through the fin and through the base of the fins. Also we have introduced the contact resistance between the chip and the fin.

b) The thermal resistances involved are due to convection at the bottom of the chip, conduction through the chip, contact resistance, convection at the base of the chip and conduction/convection on the fins. The first four are expressed as:

$$R_{conv,i} = 1/(h_i A_i)$$
$$R_{cond,b} = L_b/A_c k_b$$
$$R_{t,c} = R^{"}_{t,c}/A_c$$
$$R_{conv,b} = 1/(h_o A_b)$$

Using the convection boundary condition for the tip of the fin, R_f is computed as:

$$R_f = \frac{\theta_b}{16 Q_f} = \frac{\cosh(mL) + \frac{h_0}{mk} \cdot \sinh(mL)}{16 \cdot (h_0 \cdot P \cdot k \cdot A_{c,l})^{1/2} \cdot (\sinh(mL) + \frac{h_0}{mk} \cdot \cosh(mL)}$$

c) The analysis of the electrical equivalent circuit give us an expression for the heat transfer rate from the chip Q_c:

$$Q_c = 16Q_f + Q_b + Q_i$$

with

$$Q_f = M \cdot \frac{\sinh(mL) + \frac{h_0}{mk} \cdot \cosh(mL)}{\cosh(mL) + \frac{h_0}{mk} \cdot \sinh(mL)}$$

$$m = \left(\frac{h_0 \cdot P}{kA_{c,f}}\right)^{1/2} = \left(\frac{4 \cdot h_0}{k \cdot D_p}\right)^{1/2} = \left(\frac{4 \cdot 1000\text{W/m}^2\text{K}}{400\text{W/mK} \cdot 0.0015\text{m}}\right)^{1/2} = 81.7\text{m}^{-1}$$

$$mL = 1.23 \quad \sinh(mL) = 1.57 \quad \cosh(mL) = 1.86$$

$$\frac{h_o}{mk} = \frac{1000}{81.7 \cdot 400} = 0.0306$$

$$M = (h_0 \cdot \pi \cdot D_p \cdot k \cdot \pi \cdot \frac{D_p^2}{4})^{1/2} \cdot \theta_b$$

$$M = (1000 \cdot K(\pi^2/4) \cdot 0.0015^3 \cdot 400)^{1/2} \cdot 55 = 3.17\text{W}$$

So we get $Q_f = 2.703\text{W}$

The heat transfer rate from the base of the fin is equal to:

$$Q_b = h_o A_b (T_c - T_\infty) = 1000W/mK[0.0127m^2 - 16\pi D^2/4]55 = 7.32W$$

The heat transfer rate from the board is:

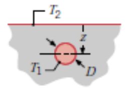
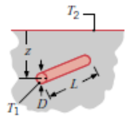
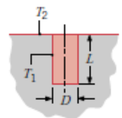
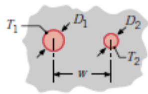
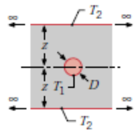
$$Q_i = \frac{T_c - T_{\infty,i}}{\left(\frac{1}{h_i} + R''_{t,c} + \frac{L_b}{k_b}\right) \cdot \frac{1}{A_c}} = \frac{0.0127 \cdot 55}{(\frac{1}{40} + 10^{-4} + 0.005)} = 0.29\text{W}$$

Hence the maximum chip heat rate is:

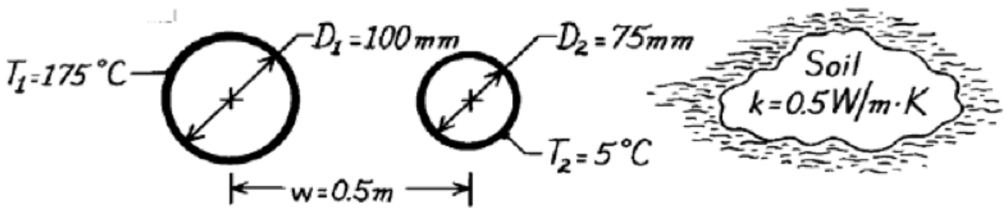
$$Q_c = (16 \cdot 2.703 + 7.32 + 0.29) = 50.9\text{W}$$

Exercise 3.3

Two parallel pipelines spaced 0.5m apart are buried in soil having a thermal conductivity of 0.5W/mK. The pipes have outer diameter of 100mm and 75mm with surface temperatures of 175°C and 5°C respectively. Estimate the heat transfer rate per unit length between the two pipelines.

System	Schematic	Restrictions	Shape Factor
Case 1 Isothermal sphere buried in a semi-infinite medium		$z > D/2$	$\dfrac{2\pi D}{1 - D/4z}$
Case 2 Horizontal isothermal cylinder of length L buried in a semi-infinite medium		$L \gg D$	$\dfrac{2\pi L}{\cosh^{-1}(2z/D)}$
		$L \gg D$ $z > 3D/2$	$\dfrac{2\pi L}{\ln(4z/D)}$
Case 3 Vertical cylinder in a semi-infinite medium		$L \gg D$	$\dfrac{2\pi L}{\ln(4L/D)}$
Case 4 Conduction between two cylinders of length L in infinite medium		$L \gg D_1, D_2$ $L \gg w$	$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$
Case 5 Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		$z \gg D/2$ $L \gg z$	$\dfrac{2\pi L}{\ln(8z/\pi D)}$

Solution



Assumptions:

a) Steady-State

b) Two dimensional conduction

c) Constant properties

d) Pipes are buried deeply (So the medium around it can be considered infinite)

e) The diameters are negligible in comparison with the length of the tubes.

f) $w > D_1 \quad w > D_2$

The heat transfer rate per unit of length from the hot pipe is : $q' = \frac{q}{L} = \frac{S \cdot k \cdot (T_1 - T_2)}{L}$

The shape factor S is computed using (From the slides):

$$S = \frac{2\pi \cdot L}{\cosh^{-1}\left(\frac{4w^2 - D_1^2 - D_2^2}{2D_1 \cdot D_2}\right)}$$

Using the numerical values:

$$\frac{S}{L} = \frac{2\pi}{\cosh^{-1}\left(\frac{4 \cdot 0.5^2 - 0.1^2 - 0.075^2}{2 \cdot 0.1 \cdot 0.075}\right)} = \frac{2\pi}{\cosh(65.63)} = 1.29$$

So $q' = 1.29 \cdot 0.5 \cdot (175 - 5) = 110 \text{W/m}$

Exercise 3.4

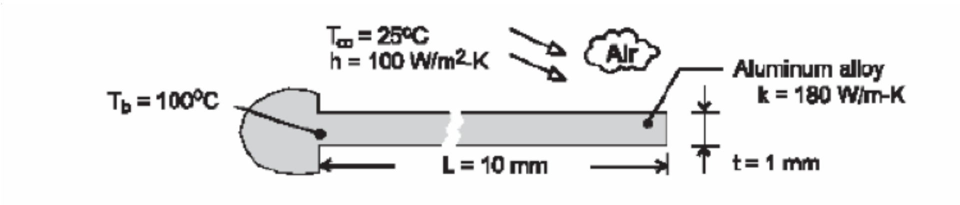
Consider an alloyed aluminum ($k = 180\text{W/mK}$) rectangular fin of length $L = 10(mm)$, thickness $t = 1\text{mm}$ and width $w >> t$. The base temperature of the fin is $T_b = 100°\text{C}$ and the fin is exposed to a fluid of temperature $T_\infty = 25°\text{C}$. Assuming a uniform convection coefficient $h = 100\text{W/m}^2\text{K}$ over the entire fin surface, determine the fin heat transfer rate per unit width q'_f, efficiency η_f, effectiveness ϵ_f and thermal resistance per unit length R'_f and the tip temperature $T(L)$ for the case of:

a) Convective heat transfer at the tip

b) Adiabatic tip

c) How do these numbers compare to the values in the infinite fin approximation?

Solution



From the knowledge of the heat transfer rate and the adimensional fin-base temperature we can compute all the other performance metrics as:

$$\eta_f = \frac{q'_f}{h\cdot(2L+t)\cdot\theta_b} \quad \epsilon_f = \frac{q'_f}{h\cdot t\cdot\theta_b} \quad R'_{t,f} = \frac{\theta_b}{q'_f}$$

With the following definitions:

$$\theta \equiv T - T_\infty, \quad \theta_b = \theta_0 = T_b - T_\infty$$

$$m \equiv \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h(2w+2t)}{kwt}} \simeq \sqrt{\frac{2h}{kt}} = 33.3\ m^{-1}$$

$$M \equiv \sqrt{hPkA_c}\theta_b = \sqrt{h(2w+2t)kwt}\theta_b \simeq \sqrt{2hw^2kt}\theta_b = 450w\ W$$

a)

$$q'_f = \frac{M}{w}\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} = 450\text{W/m}\frac{0.340 + 0.0167\times 1.057}{1.057 + 0.0167\times 0.340} = 151\text{W/m}$$

$$\eta_f = \frac{151\text{W/m}}{100\text{W/m}^2\text{K}(0.021\text{m}75^\circ\text{C}} = 0.96$$

$$\epsilon_f = \frac{151\text{W/m}}{(0.001\text{m}75^\circ\text{C}} = 20.2, \quad R'_{t,f} = \frac{75^\circ\text{C}}{151\text{W/m}} = 0.50\text{mK/W}$$

$$T(L) = T_\infty + \frac{\theta_b}{\cosh mL + (h/mk)\sinh mL} = 25^\circ\text{C} + \frac{75^\circ\text{C}}{1.057 + (0.0167)0.340} = 95.6^\circ\text{C}$$

b)

$$q'_f = \frac{M}{w}\cdot\tanh(mL) = 450\cdot 0.321 = 144\text{W/m}$$

$$\eta_f = 0.92$$

$$\epsilon_f = 19.3$$

$$R'_f = 0.52\text{mK/W}$$

$$T(L) = T_\infty + \frac{\theta_b}{\cosh(mL)} = 25 + \frac{75}{1.057} = 96^\circ\text{C}$$

c) The infinite fin approximation is equivalent to having $L \to \infty$.

$$q'_f = \frac{M}{w} = 450\text{W/m}$$

$$\eta_f = 0$$

$$\epsilon_f = 60.0$$

$$R'_f = 0.167\text{mK/W}$$

$$T(L) = T_\infty = 25^\circ\text{C}$$

Hyperbolic Functions[1]

x	$\sinh x$	$\cosh x$	$\tanh x$	x	$\sinh x$	$\cosh x$	$\tanh x$
0.00	0.0000	1.0000	0.00000	2.00	3.6269	3.7622	0.96403
0.10	0.1002	1.0050	0.09967	2.10	4.0219	4.1443	0.97045
0.20	0.2013	1.0201	0.19738	2.20	4.4571	4.5679	0.97574
0.30	0.3045	1.0453	0.29131	2.30	4.9370	5.0372	0.98010
0.40	0.4108	1.0811	0.37995	2.40	5.4662	5.5569	0.98367
0.50	0.5211	1.1276	0.46212	2.50	6.0502	6.1323	0.98661
0.60	0.6367	1.1855	0.53705	2.60	6.6947	6.7690	0.98903
0.70	0.7586	1.2552	0.60437	2.70	7.4063	7.4735	0.99101
0.80	0.8881	1.3374	0.66404	2.80	8.1919	8.2527	0.99263
0.90	1.0265	1.4331	0.71630	2.90	9.0596	9.1146	0.99396
1.00	1.1752	1.5431	0.76159	3.00	10.018	10.068	0.99505
1.10	1.3356	1.6685	0.80050	3.50	16.543	16.573	0.99818
1.20	1.5095	1.8107	0.83365	4.00	27.290	27.308	0.99933
1.30	1.6984	1.9709	0.86172	4.50	45.003	45.014	0.99975
1.40	1.9043	2.1509	0.88535	5.00	74.203	74.210	0.99991
1.50	2.1293	2.3524	0.90515	6.00	201.71	201.72	0.99999
1.60	2.3756	2.5775	0.92167	7.00	548.32	548.32	1.0000
1.70	2.6456	2.8283	0.93541	8.00	1490.5	1490.5	1.0000
1.80	2.9422	3.1075	0.94681	9.00	4051.5	4051.5	1.0000
1.90	3.2682	3.4177	0.95624	10.000	11013	11013	1.0000

[1]The hyperbolic functions are defined as

$$\sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right) \qquad \cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right) \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

The derivatives of the hyperbolic functions of the variable u are given as

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{du}{dx} \qquad \frac{d}{dx}(\cosh u) = (\sinh u)\frac{du}{dx} \qquad \frac{d}{dx}(\tanh u) = \left(\frac{1}{\cosh^2 u}\right)\frac{du}{dx}$$

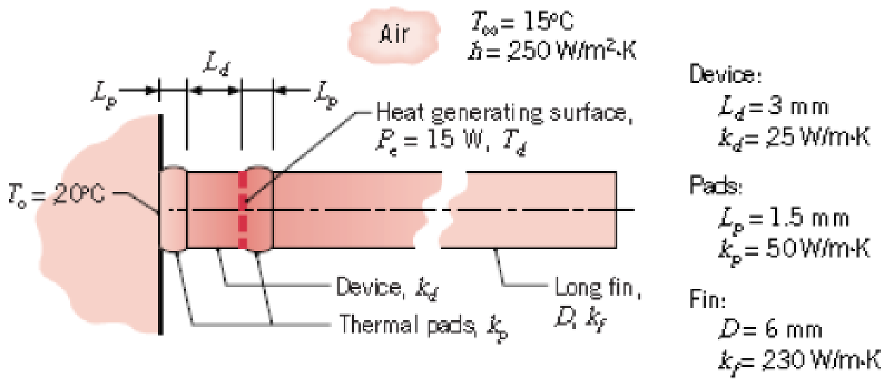
Exercise 3.5 FOR REVISION

A disk shaped electronic device of thickness L_d, diameter L and thermal conductivity k_d, dissipates electrical power at a steady rate of P_e along one of its surfaces. The device is bonded to a cooled base at T_0 using a thermal pad of thickness L_p and a thermal conductivity k_p. A long fin of diameter D and thermal conductivity k_f is bonded to the heat generating surface of the device using an identical thermal pad. The fin is cooled by an air stream which is at temperature T_∞ and provides a convection coefficient h.

a) Construct the thermal circuit of the system and write the expression of the thermal resistances involved.

b) Derive an expression for the temperature T_d of the heat-generating surface of the device in terms of the circuit thermal resistances , T_0 and T_∞

c) Calculate T_d for the prescribed conditions.



Note: considered the much shorter length of the thermal pads and electronic device compared to the fin, neglect the effect of convection along these parts.

Solution

Assumptions:

a) Steady-state conditions

b) One-dimensional conduction through thermal pads and device.

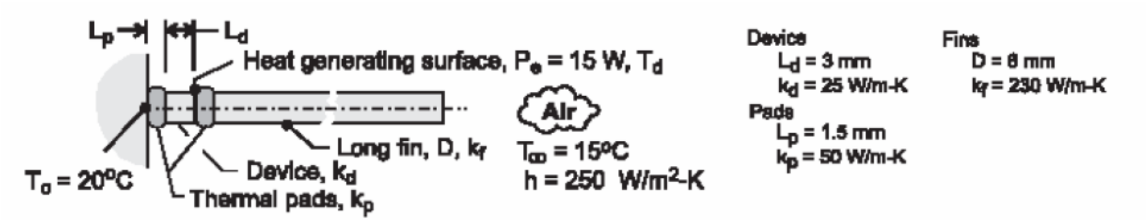
c) No lateral losses

d) Infinitely long fin

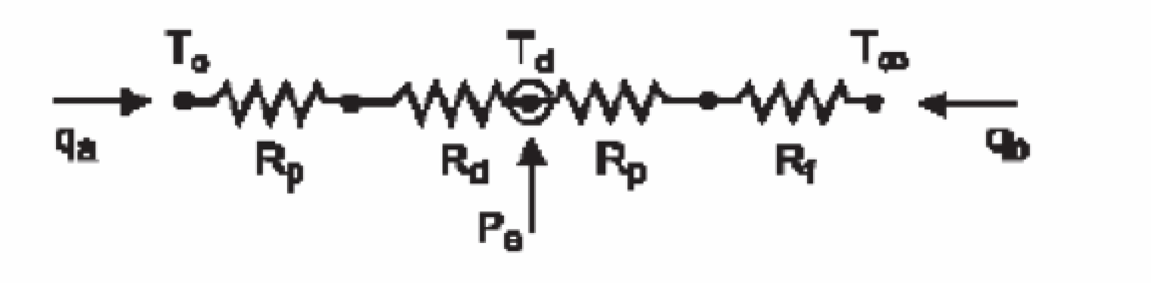
e) Negligible contact resistance between components of the system

f) Constant properties

g) Negligible radiation transfer



a) The system can be modeled with the following equivalent thermal circuit (note that the heat dissipation occurs just on a surface of the chip and hence appears as a power input in the corresponding node of the thermal circuit):



We want to first determine the thermal resistances. We know that we can neglect convection in the first part (pad + device + pad) and therefore conduction ALONG them is the only heat transfer mechanism. The thermal resistances are:

$$R_p = \frac{L_p}{k_p \cdot A_c} \quad R_d = \frac{L_d}{k_d \cdot A_d}$$

NOTE : be careful and do NOT get confused. Although these parts have a circular cross-section we use the planar resistance expressions and NOT the radial resistance expressions. In fact what matters is in which direction the heat flows. In this case the heat flows by conduction ALONG THE AXIS of the cylindrical objects (pads and devices). Therefore, it is a planar wall of circular cross-section. Instead, when the heat flows ALONG THE RADIUS of a cylindrical object, as the cross-section grows with r we need to use the radial system expressions for the thermal resistance.

For the fin we need first to identify the tip-BCs and in this case we use the infinite fin condition. We know that the fin resistance is defined as:

$$R_f = \frac{(T_p - T_\infty)}{Q_f}$$

.

where T_p is the temperature of the interface between fin and thermal pad.

For an infinite fin: $Q_f = M = \sqrt{hPk_fA_c}\theta_b = \sqrt{hPk_fA_c}(T_p - T_\infty)$

Therefore we have:

$$R_f = \frac{1}{(hPk_fA_c)^{1/2}}$$

Note: depending on the fin boundary condition it can be convenient to determine the fin resistance through the definition or through the equivalent expression:

$$R_f = 1/(hA_f\eta_f)$$

where $\eta_f = Q_f/Q_{f,max} = Q_f/(hA_f\theta_b)$.

b) To obtain an expression for T_d, we need to perform an energy balance about the d-node:

$$\dot{E}_{in} - \dot{E}_{out} = q_a + q_b + P_e = 0$$

Using the thermal circuit it is clear that:

$$q_a = \frac{T_0 - T_d}{R_p + R_d} \quad q_b = \frac{T_\infty - T_d}{R_p + R_f}$$

Combining the above equations, it is possible to express T_d:

$$T_d = \frac{P_e + \frac{T_0}{R_p+R_d} + \frac{T_\infty}{R_p+R_f}}{\frac{1}{R_p+R_d} + \frac{1}{R_p+R_f}}$$

c) Substituting numerical values with the foregoing relations, find:

$$R_p = 1.061\text{K/W} \quad R_d = 4.244\text{K/W} \quad R_f = 5.712\text{K/W} \quad T_d = 62.4°\text{C}$$

Exercise 3.6 FOR REVISION

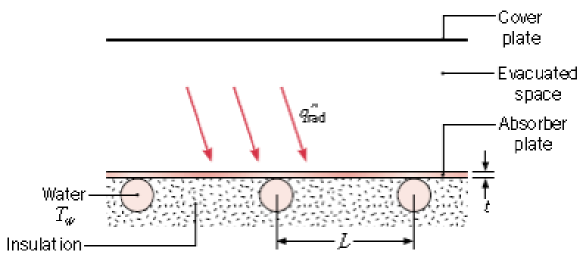
Copper tubing is joined to the absorber of a flat-plate solar collector as shown in this figure. The aluminum alloy absorber plate is 6mm thick with $k = 180$ $[\frac{W}{mK}]$ and well insulated on its bottom. The top surface of the plate is separated from a transparent cover plate by an evacuated space. The tubes are spaced at a distance $L = 0.2$m from each other. Water is circulated through the tubes to remove the collected energy and it may be assumed to have a constant temperature of $T_w = 60$°C. Under steady-state operating conditions the net radiation heat flux to the surface is $q''_{rad} = 800\text{W/m}^2$. (Note: this value accounts for both the radiation absorption by the collector plate and radiative heat exchange between the collector plate and the cover-plate).

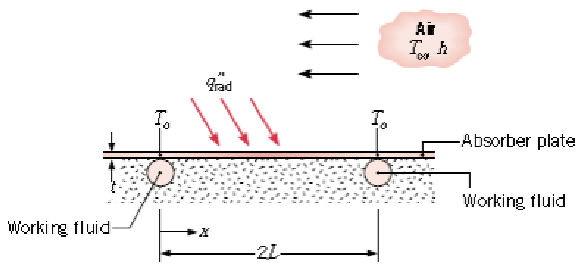
a) Draw a schematic of the flat-plate solar collector and the water tubing with all the relevant physical parameters and known temperature/heat flux values. Treating the flat-plate collector as a fin, identify an infinitesimal section of it with length dx and write the energy balance for it. Then express it as a function of the temperature. Integrate the obtained differential equation to find the function that describes the temperature profile in the collector plate.

b) Assume that the temperature of the absorber plate directly above a tube is equal to that of the water. What is the maximum temperature on the collector plate?

c) What is the heat transfer rate per unit length of tube?

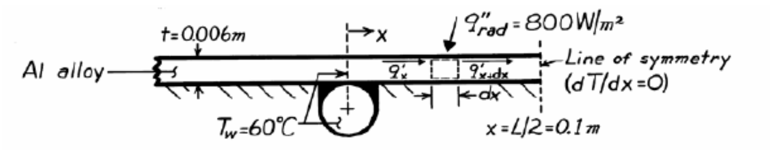
d) Let's now imagine that the cover plate is removed. In this case the surface of the collector plate is directly cooled by air flowing over it with temperature T_∞ and convection coefficient h. Derive the differential equation that governs the temperature distribution $T(x)$ in the plate and define the appropriate boundary conditions. What is the solution for this differential equation? (DIFFICULT)





Solution

a)



$$q'_x + q''_{rad}(dx) - q'_{x+dx} = 0, \quad \text{with } q'_{x+dx} = q'_x + \frac{\mathrm{d}q'_x}{\mathrm{d}x}dx \text{ and } q'_x = -kt\frac{\mathrm{d}T}{\mathrm{d}x}$$

It follows that,

$$q''_{rad} - \frac{\mathrm{d}}{\mathrm{d}x}\left[-kt\frac{\mathrm{d}T}{\mathrm{d}x}\right] = 0$$

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \frac{q''_{rad}}{kt} = 0$$

$$\Rightarrow T(x) = -\frac{q''_{rad}}{2kt}x^2 + C_1x + C_2$$

b) The boundary conditions are at $x = 0$ and $x = L/2$ are:

$$T(0) = T_w \Rightarrow C_2 = T_w$$

$$\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=L/2} = 0 \Rightarrow C_1 = \frac{q''_{rad}L}{2kt}$$

The second boundary condition accounts for the symmetry of the problem. We have indeed seen in class that a symmetry plane is equivalent to an adiabatic boundary and therefore the net heat transfer rate is zero as well as the first derivative of the temperature profile.

Hence,

$$T(x) = \frac{q''_{rad}}{2kt}x(L-x) + T_w$$

The location of the maximum temperature can be found by $\mathrm{d}T/\mathrm{d}x = 0 \Rightarrow x = L/2$

$$\begin{aligned} T_{max} = T(L/2) &= \frac{q''_{rad}L^2}{8kt} + T_w \\ &= \frac{800\mathrm{W/m^2}(0.2\mathrm{m})^2}{8(180\mathrm{W/mK})(0.006\mathrm{m})} + 60^\circ\mathrm{C} \\ &= 63.7^\circ\mathrm{C} \end{aligned}$$

c) Each tube collects heat from both right and left sections of the collector plate. The heat flux can be obtained by the gradient of temperature at the tube position, i.e. $x = 0$. So we write:

$$Q = 2\left[-k(tL)\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0}\right]$$

where tL is the cross section for heat transfer rate by conduction, and therefore, dividing by L:

$$q' = 2\left[-kt\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0}\right]$$

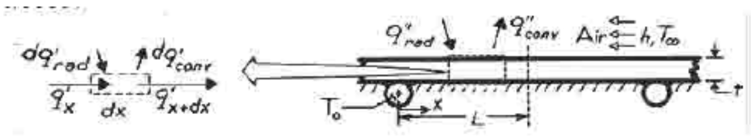
where the factor 2 arises due to heat transfer from both sides of the tube. Hence,

$$q' = -Lq''_{rad}$$

And so:

$$q' = -0.2\text{m} \times 800\text{W/m}^2 = -160\text{W/m}$$

d)



$$q'_x + dq'_{rad} = q'_{x+dx} + dq'_{conv}$$

$$\begin{aligned} \text{where } q'_{x+dx} &= q'_x + (dq'_x/dx)dx \\ dq'_{rad} &= q''_{rad}dx \\ dq'_{conv} &= h(T - T_\infty)dx \end{aligned}$$

Hence,

$$q''_{rad}dx = (dq'_x/dx)dx + h(T - T_\infty)dx$$

From Fourier's law, the conduction heat rate per unit width is:

$$q'_x = -kt\frac{\mathrm{d}T}{\mathrm{d}x} \qquad \frac{\mathrm{d}^2T}{\mathrm{d}x^2} - \frac{h}{kt}(T - T_\infty) + \frac{q''_{rad}}{kt} = 0$$

Defining $\theta = T - T_\infty$, $\mathrm{d}^2T/\mathrm{d}x^2 = \mathrm{d}^2\theta/\mathrm{d}x^2$, the differential equation becomes:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \frac{h}{kt}\theta + \frac{q''_{rad}}{kt} = 0$$

It is a second order differential equation with coefficients and a source term, its general solution is of the form:

$$\theta = C_1e^{+\lambda x} + C_2e^{-\lambda x} + \frac{S}{\lambda^2}$$

where $\lambda = \sqrt{\dfrac{h}{kt}}$ and $S = \dfrac{q''_{rad}}{kt}$

Appropriate boundary conditions are:

$$\theta(0) = T_0 - T_\infty = \theta_0 \qquad \left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=L} = 0$$

Hence,

$$\theta_0 = C_1 + C_2 + \frac{S}{\lambda^2}$$

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=L} = C_1\lambda e^{+\lambda L} - C_2\lambda e^{-\lambda L} = 0 \Rightarrow C_2 = C_1e^{2\lambda L}$$

Hence,

$$C_1 = \frac{\theta_0 - S/\lambda^2}{1 + e^{2\lambda L}} \qquad C_2 = \frac{\theta_0 - S/\lambda^2}{1 + e^{-2\lambda L}}$$

$$\theta = \left(\theta_0 - \frac{S}{\lambda^2}\right)\left[\frac{e^{\lambda x}}{1 + e^{2\lambda L}} + \frac{e^{-\lambda x}}{1 + e^{-2\lambda L}}\right] + \frac{S}{\lambda^2}$$