Solid Mechanics: Elements of linear viscoelasticity

In linear viscoelasticity we study the constitutive response of materials when it is dominated by linear elastic and viscous characteristics.

We aim to model materials such as elastomers, gels, polymers, composites, and other materials with time dependent mechanical response.

We suppose that the strains are infinitesimal:

$$\left\| \frac{\partial u_i}{\partial x_j} \right\| = O(\varepsilon) << 1; \quad F_{ij} = \delta_{ij} + O(\varepsilon); \quad J \approx 1 + O(\varepsilon); \quad (i, j = 1, 2, 3)$$

F_{ij}: Deformation gradient tensor

J: Jacobian

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LINEAR ELASTICITY

A linear elastic response, described by Hook's law, describes a time independent linear relationship between the stresses and strains.

Furthermore, it implies that the response to a given input is instantaneously realized, or there is no phase lag between input and output.

LINEAR VISCOELASTICITY

A linear viscoelastic response also suggests a linearity between stresses and strains. This relationship, however, is a functional of the load-time history.

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The basic criteria in the linear theory of viscoelasticity are two:

1. proportionality
2. superposition

Define the input $\boldsymbol{I}$ and corresponding response $\boldsymbol{R}$

$$R = R[I]$$

For a linear viscoelastic material we have:

Proportionality: $R[cI] = cR[I]$

Superposition: $R[I_a + I_b] = R[I_a] + R[I_b]$

Based on two criteria we can define the stress-strain relationship, known as the Boltzmann superposition integral.

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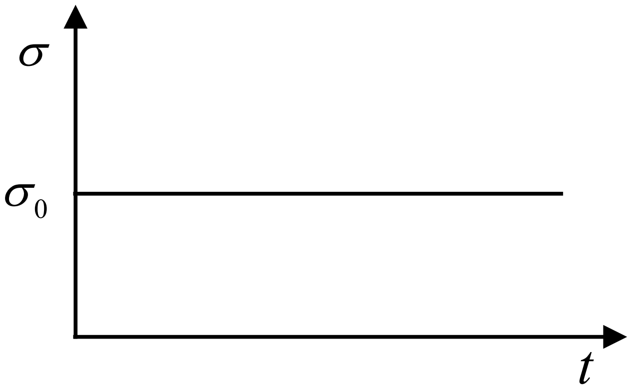
In viscoelasticity, the characteristics of time-dependent material response are conventionally identified by:

CREEP Testing



Input stress in known,
measure the resulting strain

$$J(t) = \varepsilon(t) / \sigma_0$$

RELAXATION Testing

Input strain in known,
measure the resulting stress

$$G(t) = \sigma(t) / \varepsilon_0$$

The objective of viscoelasticity is
to model the strain or stress response

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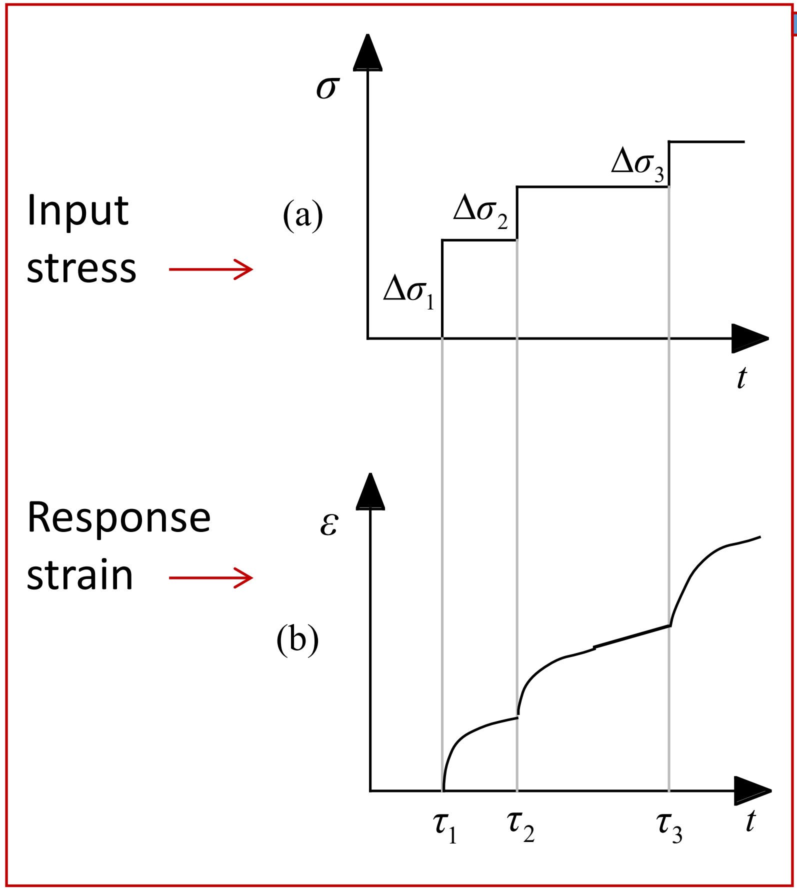
BOLTZMANN SUPERPOSITION INTEGRALS:

(consecutive creep loads)



$$\varepsilon(t) = \Delta\sigma_1 J(t-\tau_1) + \Delta\sigma_2 J(t-\tau_2) + \Delta\sigma_3 J(t-\tau_3)$$

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau$$

Creep compliance

Similarly for relaxation experiment

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau$$

Relaxation function

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BOLTZMANN SUPERPOSITION INTEGRALS:

They are transformed in ordinary differential equations using Laplace Transforms:

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau$$

$$b_o\sigma(t) + b_1\frac{d\sigma(t)}{dt} + b_2\frac{d^2\sigma(t)}{dt^2} + \ldots + b_n\frac{d^n\sigma(t)}{dt^n}$$
$$= a_o\varepsilon(t) + a_1\frac{d\varepsilon(t)}{dt} + a_2\frac{d^2\varepsilon(t)}{dt^2} + \ldots + a_n\frac{d^2\varepsilon(t)}{dt^2}$$

coefficients $b_o, b_1, b_2 \ldots, a_o, a_1, a_2 \ldots$ are constant

linear viscoelastic response can be described either by the Boltzmann's superposition integrals or a differential equation.

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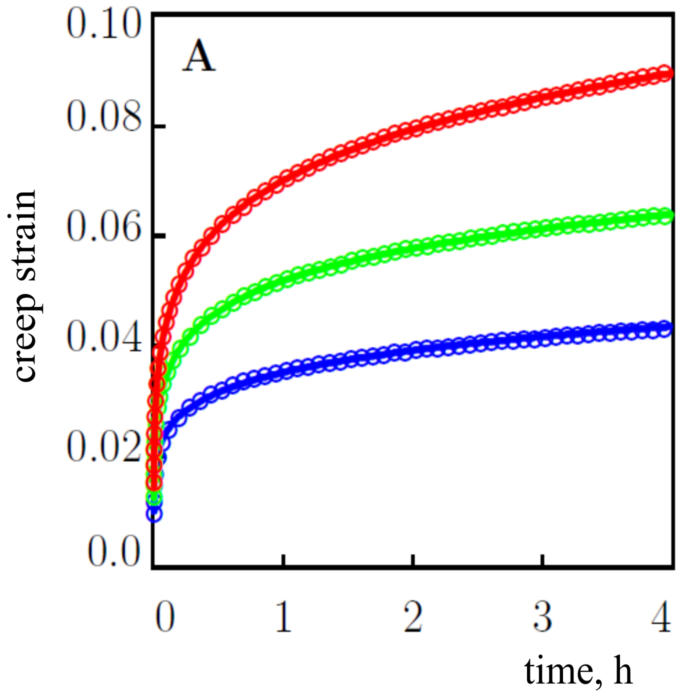
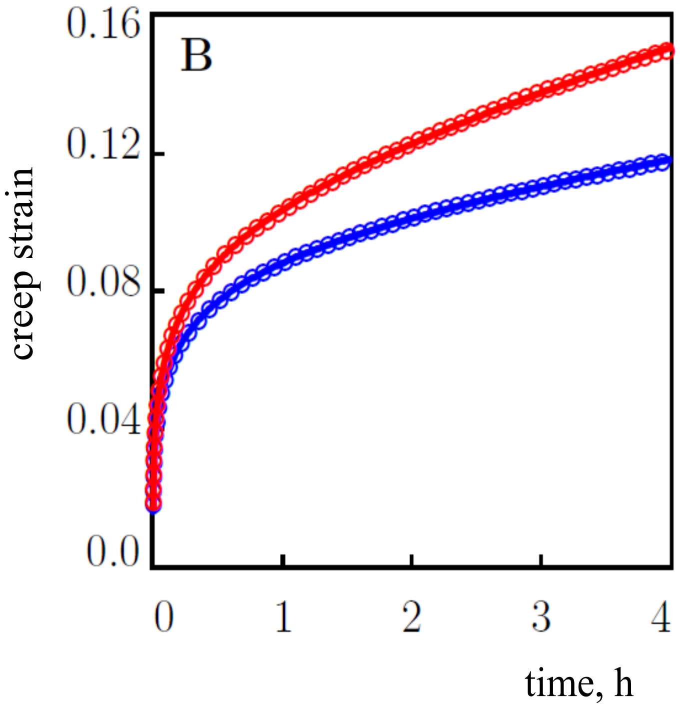
Experimental Observations

CREEP Testing





Creep of High Density Polyethylene at 21^{o}C under different stresses
(A): 9.0 (blue), 11 (green), 13 (red); (B) 14. (blue), 15 (red). (units in MPa).

FROM: A. D. Drozdov, et al, Lifetime Predictions for High-Density Polyethylene under Creep: Experiments and Modeling, *Polymers* **2023**, 15, 334.

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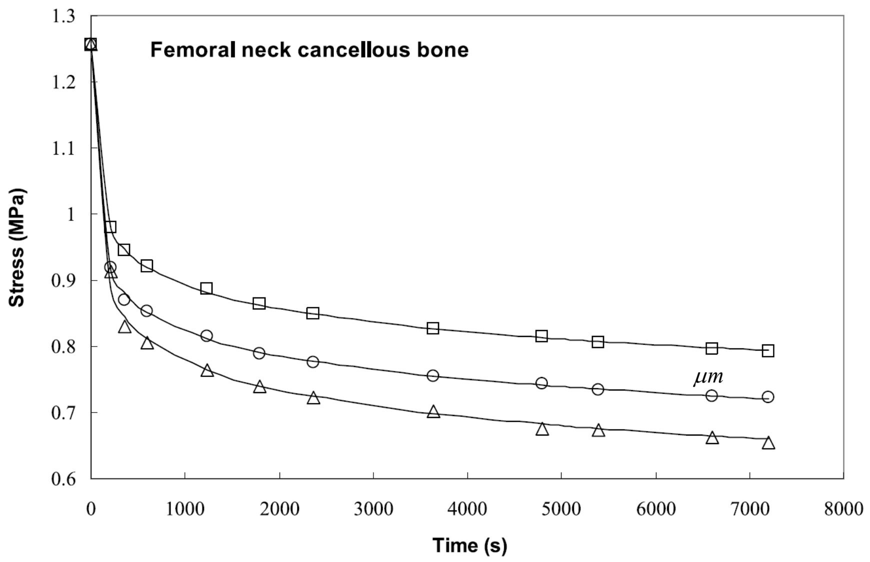
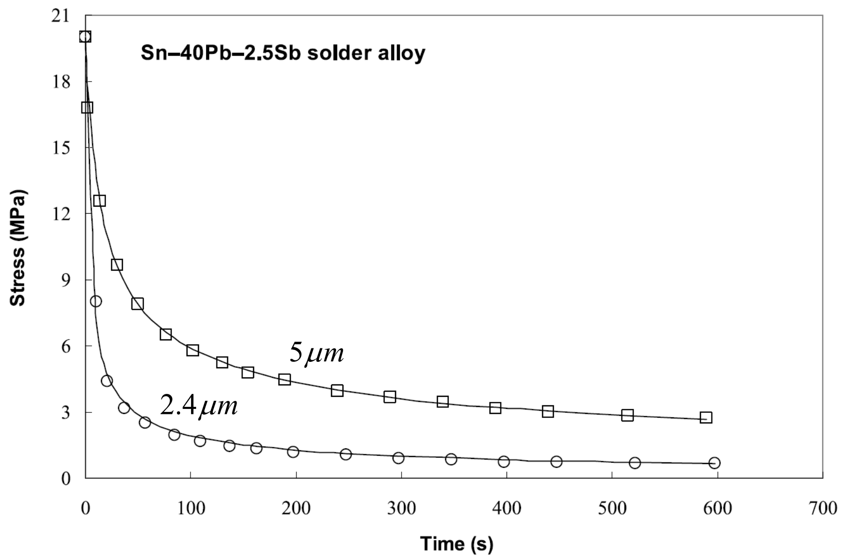
Experimental Observations

RELAXATION Testing



Stress relaxation of bone at 21^{o}C under three initial strains.



Stress relaxation solder alloy at 21^{o}C with two different grain size.

FROM: Q. Liu, et al, The compression stress relaxation and creep study of three heading on femoral neck cancellous bone. *J. Biomed. Eng. Res*. **27**(2), 93–96, 2008.

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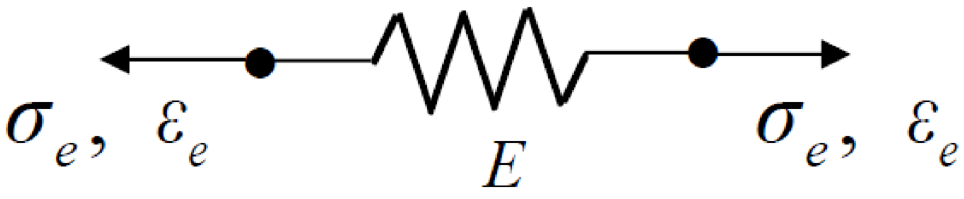
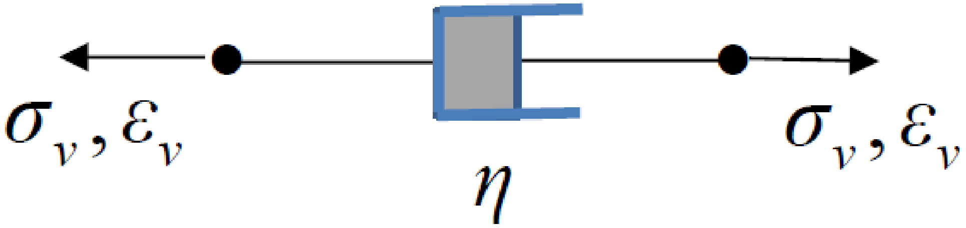
Linear theory of viscoelasticity builds models based on two mechanical analoges: Spring and dashpot elements:



Spring : Linear Elastic

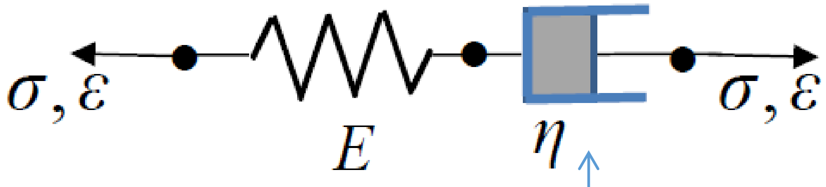
$$\sigma_e = E\varepsilon_e$$



Dashpot : Linear Viscous Respone

$$\sigma_v = \eta \frac{d\varepsilon_v}{dt}, \quad \Rightarrow \varepsilon_v = \frac{\sigma_v}{\eta} t$$

Maxwell model



$$\sigma_e = E\varepsilon_e \qquad \sigma_v = \eta \frac{d\varepsilon_v}{dt}$$

$$\varepsilon = \varepsilon_e + \varepsilon_v \qquad \sigma = \sigma_e = \sigma_v$$

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{d\varepsilon_e}{dt} + \frac{d\varepsilon_v}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}.$$

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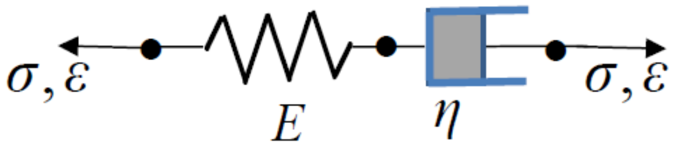
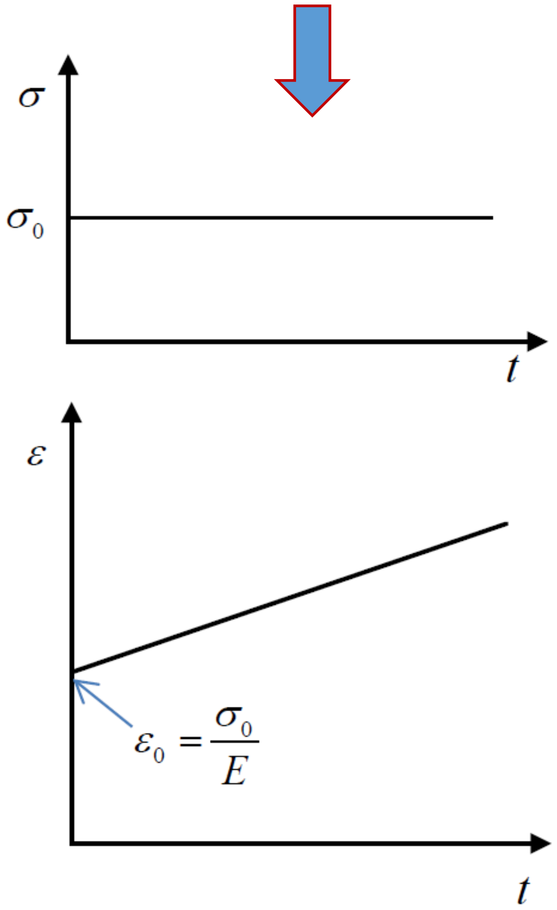
Maxwell model



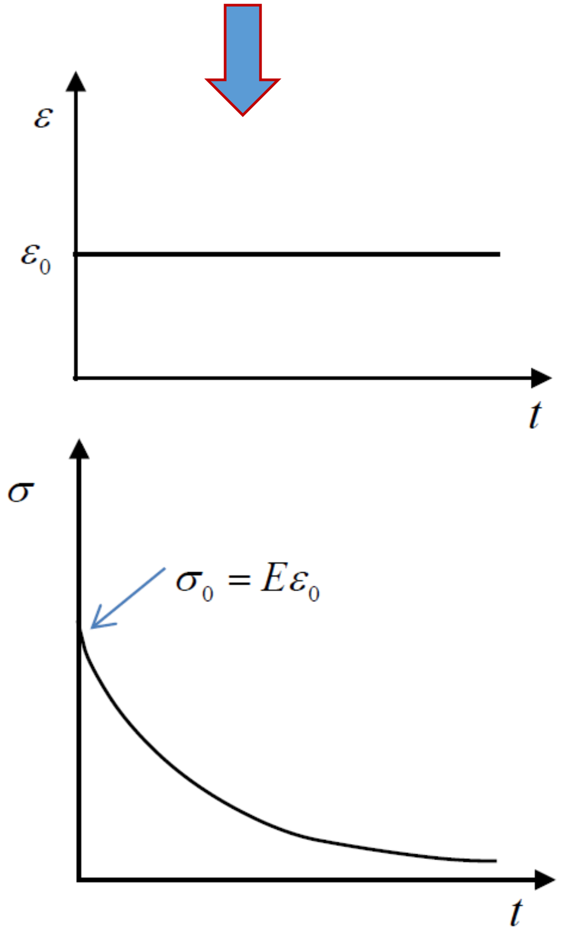
$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}.$$

Creep response



$$\varepsilon(t) = \left(\frac{t}{\eta} + \frac{1}{E}\right)\sigma_o$$

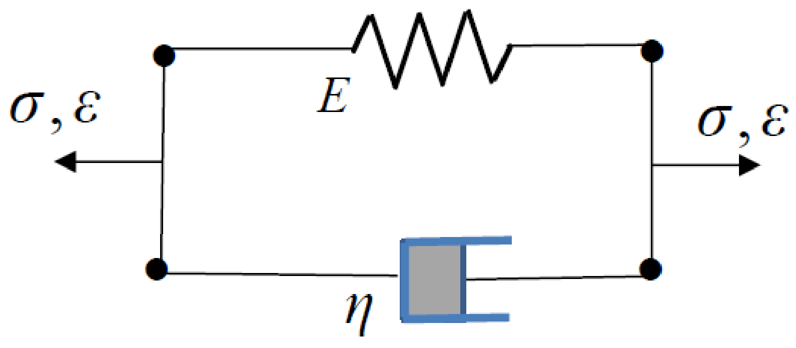
Relaxation response



$$\sigma(t) = \sigma_o \exp(-tE/\eta)$$

Kelvin-Voight model

$$\sigma_e = E\varepsilon_e$$



$$\sigma_v = \eta \frac{d\varepsilon_v}{dt}$$

$$\sigma = \sigma_e + \sigma_v \, . \quad \varepsilon = \varepsilon_e = \varepsilon_v$$

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}}$$

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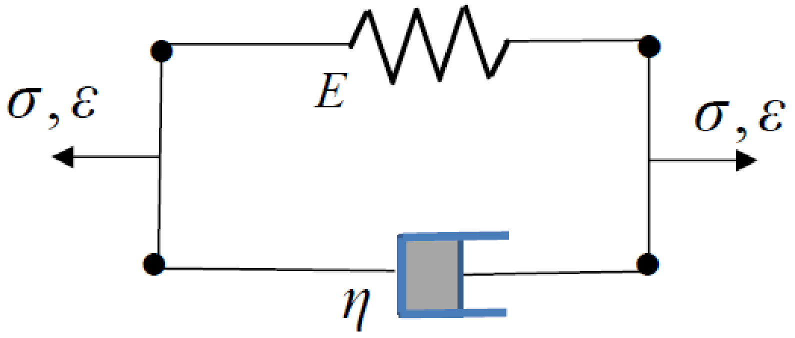
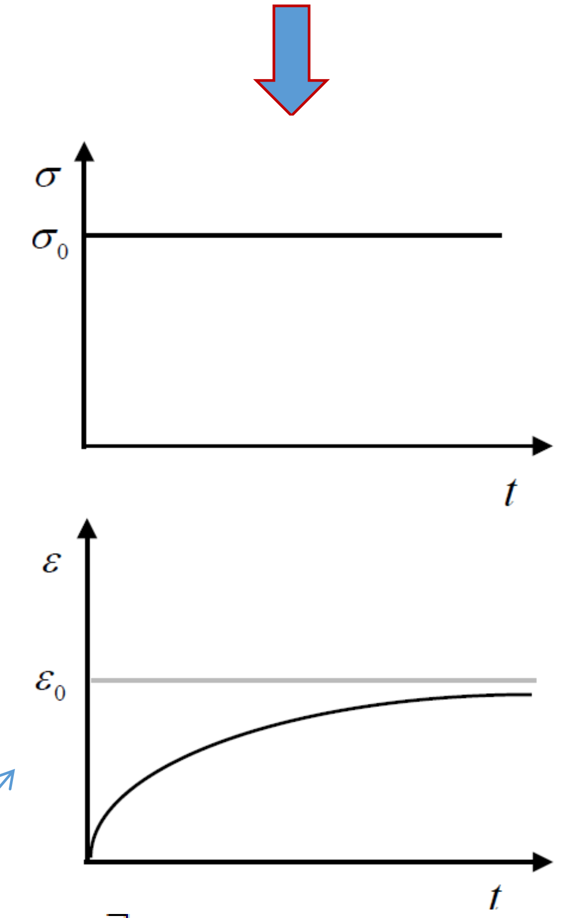
Kelvin-Voight model



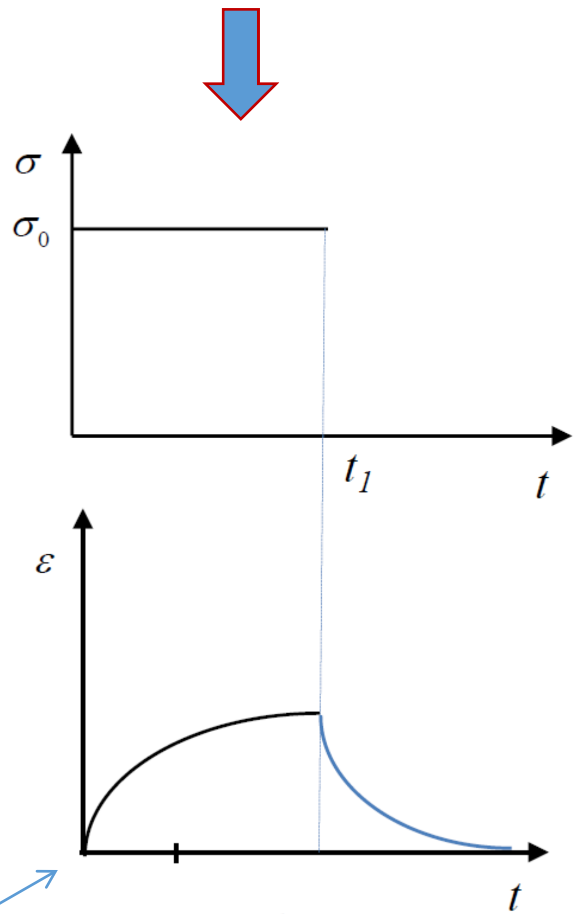
$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Creep response



$$\varepsilon(t) = \frac{\sigma_o}{E}\left[1 - e^{-t/\tau}\right]$$

Recovery response



$$\varepsilon_1(t) + \varepsilon_2(t) = \frac{\sigma_o}{E} e^{-tE/\eta}\left[e^{+t_1 E/\eta} - 1\right], \qquad t > t_1$$

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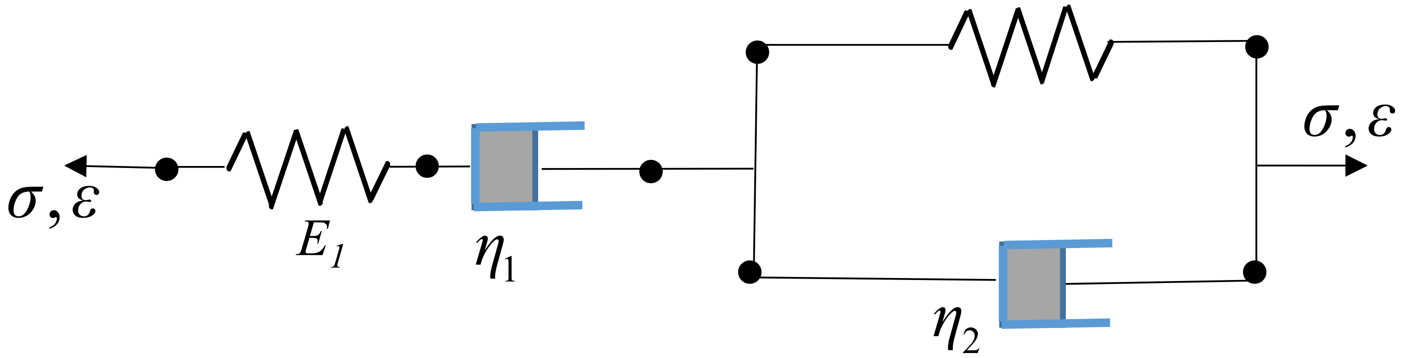
Creep of the four-parameter model
(Maxwell and Kelvin-Voigt in series)



Rheological Equation

$$\ddot{\sigma}\frac{\eta_1\eta_2}{E_1E_2}+\dot{\sigma}\left(\frac{\eta_1}{E_2}+\frac{\eta_1}{E_1}+\frac{\eta_2}{E_2}\right)+\sigma=\frac{\eta_1\eta_2}{E_2}\ddot{\varepsilon}+\eta_1\dot{\varepsilon}$$

For creep $\ddot{\sigma}=\dot{\sigma}=0$ and $\sigma=\sigma_0$

$$\ddot{\varepsilon}+\frac{E_2}{\eta_2}\dot{\varepsilon}=\frac{E_2}{\eta_1\eta_2}\sigma_0$$

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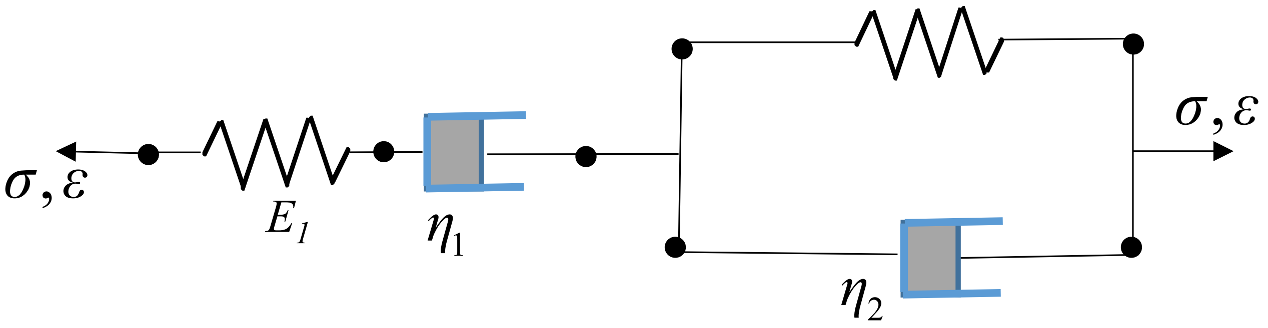

$$\ddot{\varepsilon} + \frac{E_2}{\eta_2}\dot{\varepsilon} = \frac{E_2}{\eta_1 \eta_2}\sigma_0 \quad \Rightarrow \quad \varepsilon(t) = \varepsilon_h(t) + \varepsilon_p(t) = C_1 e^{-\frac{E_2}{\eta_2}t} + C_2 + \frac{\sigma_0}{\eta_1}t$$

We need two conditions to determine the two unknown coefficients

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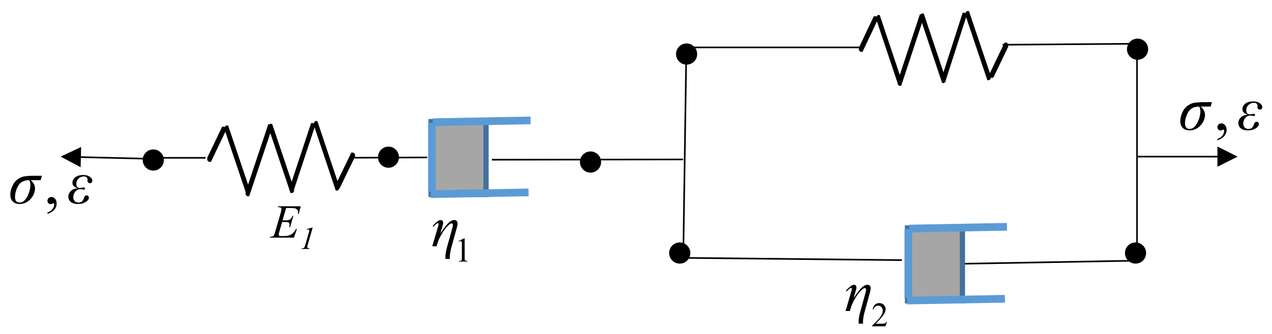

Alternatively, to apply this model in creep we consider the creep of each model.
Stress is the same in each model and strains are added to get the total strain

$$\varepsilon_1(t) = \frac{\sigma_0}{\eta_1} t + \frac{\sigma_0}{E_1}$$

$$\varepsilon_2(t) = \frac{\sigma_0}{E_2}\left(1 - e^{-\frac{E_2}{\eta_2}t}\right)$$

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) = \frac{\sigma_0}{E_2}\left(1 - e^{-\frac{E_2}{\eta_2}t}\right) + \frac{\sigma_0}{\eta_1} t + \frac{\sigma_0}{E_1} = \frac{\sigma_0}{E_2} - \frac{\sigma_0}{E_2} e^{-\frac{E_2}{\eta_2}t} + \frac{\sigma_0}{\eta_1} t + \frac{\sigma_0}{E_1}$$

$$\varepsilon(t) = -\frac{\sigma_0}{E_2} e^{-\frac{E_2}{\eta_2}t} + \frac{\sigma_0}{E_2} + \frac{\sigma_0}{E_1} + \frac{\sigma_0}{\eta_1} t$$

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Creep of the four-parameter model
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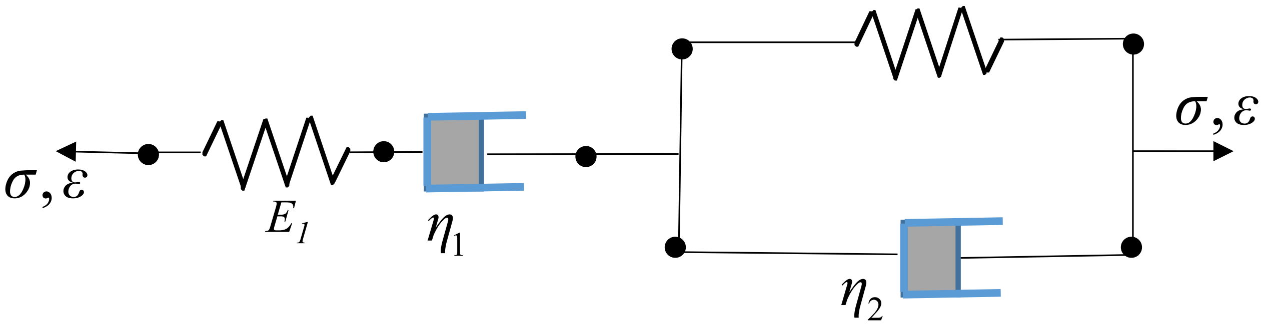

$$\varepsilon(t) = -\frac{\sigma_0}{E_2} e^{-\frac{E_2}{\eta_2}t} + \frac{\sigma_0}{E_2} + \frac{\sigma_0}{E_1} + \frac{\sigma_0}{\eta_1} t$$

There is only one oblique asymptote to this function → $\hat{\varepsilon}(t) = kt + b$

$$\begin{cases} k = \lim_{t\to\infty} \frac{\varepsilon(t)}{t} = \frac{\sigma_0}{\eta_1} = \tan\alpha \\ b = \lim_{t\to\infty} \left[\varepsilon(t) - kt\right] = \frac{\sigma_0}{E_1} + \frac{\sigma_0}{E_2} \end{cases}$$

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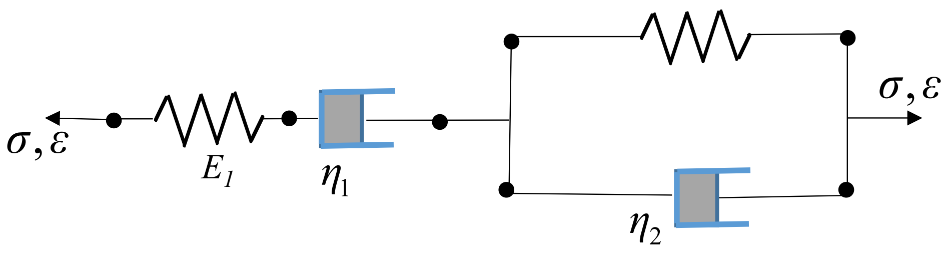
Creep of the four-parameter model
(Maxwell and Kelvin-Voigt in series)



asymptote $\hat{\varepsilon}(t) = \frac{\sigma_0}{\eta_1} t + \left(\frac{\sigma_0}{E_1} + \frac{\sigma_0}{E_2} \right)$

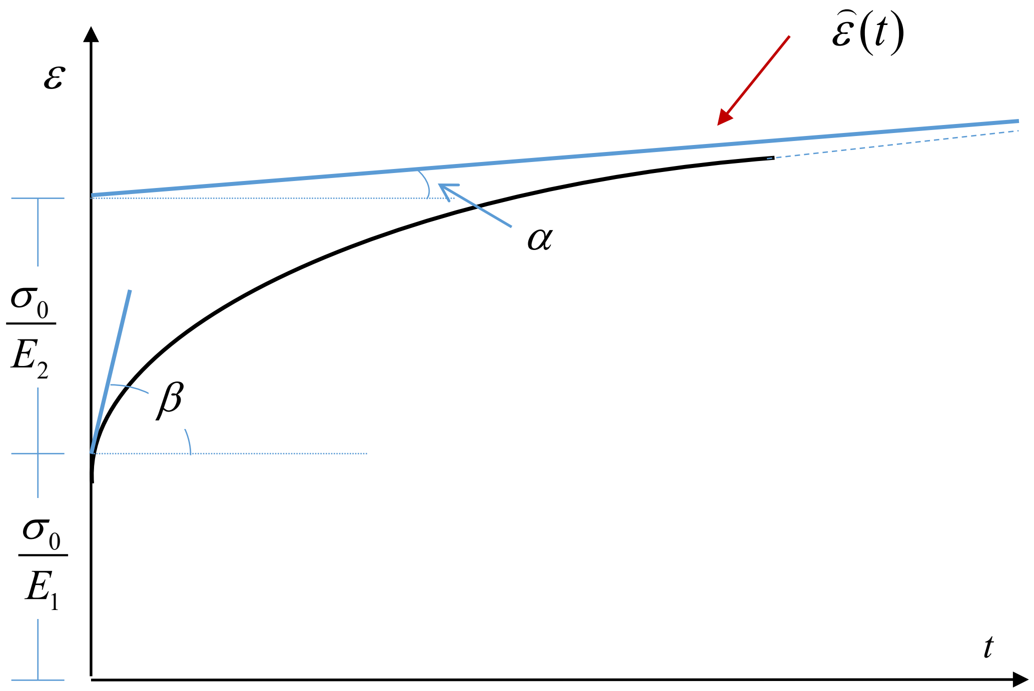
Slope: $\sigma_0 / \eta_1 = \tan \alpha$

Intercept: $\left(\frac{\sigma_0}{E_1} + \frac{\sigma_0}{E_2} \right)$

$$\varepsilon(t) = -\frac{\sigma_0}{E_2} e^{-\frac{E_2}{\eta_2} t} + \frac{\sigma_0}{E_2} + \frac{\sigma_0}{E_1} + \frac{\sigma_0}{\eta_1} t$$

Slope at the origin

$$\left.\frac{d\varepsilon}{dt}\right|_{t=0} = \frac{\sigma_0}{\eta_2} + \frac{\sigma_0}{\eta_1} = \tan \beta$$



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Creep of the four-parameter model
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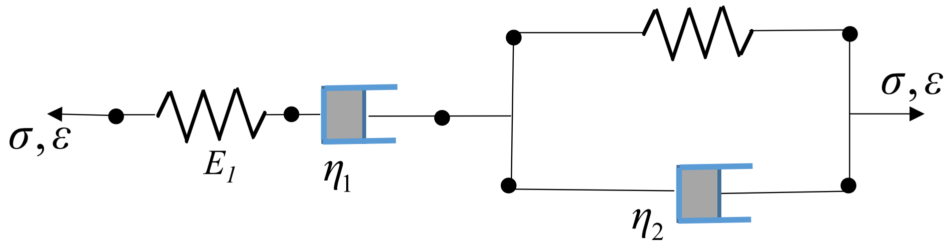
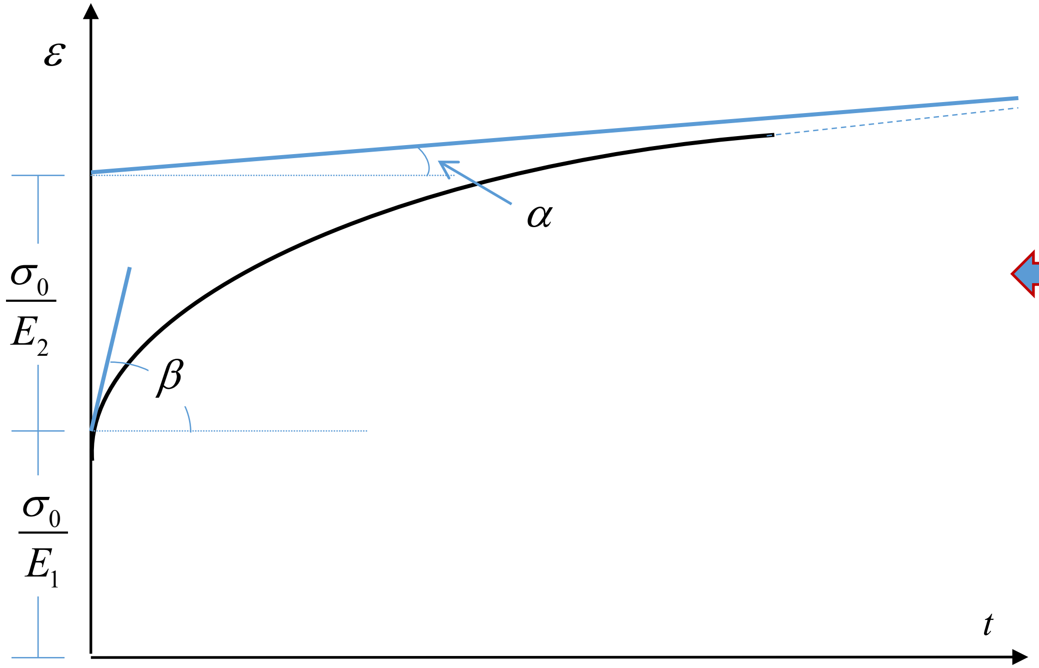
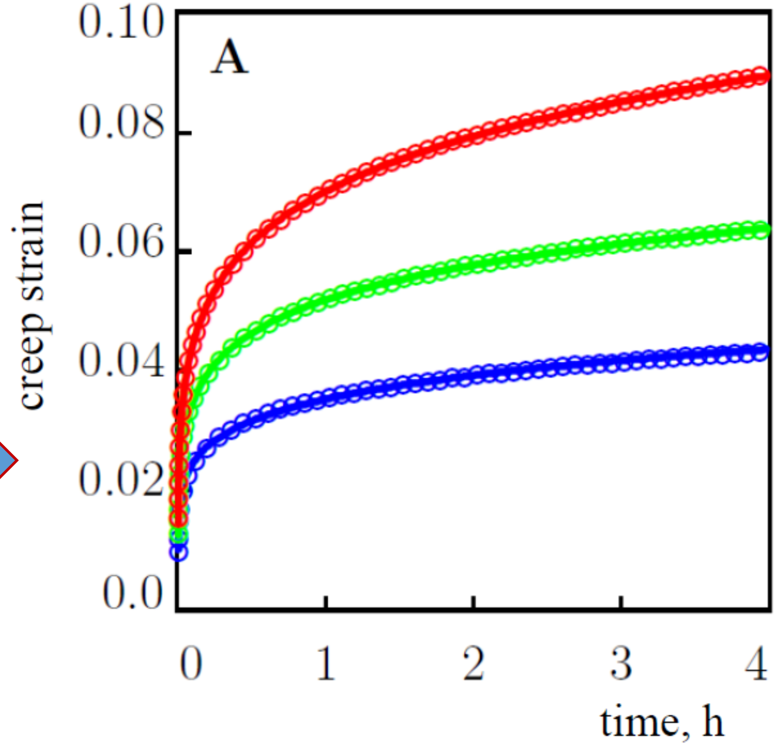




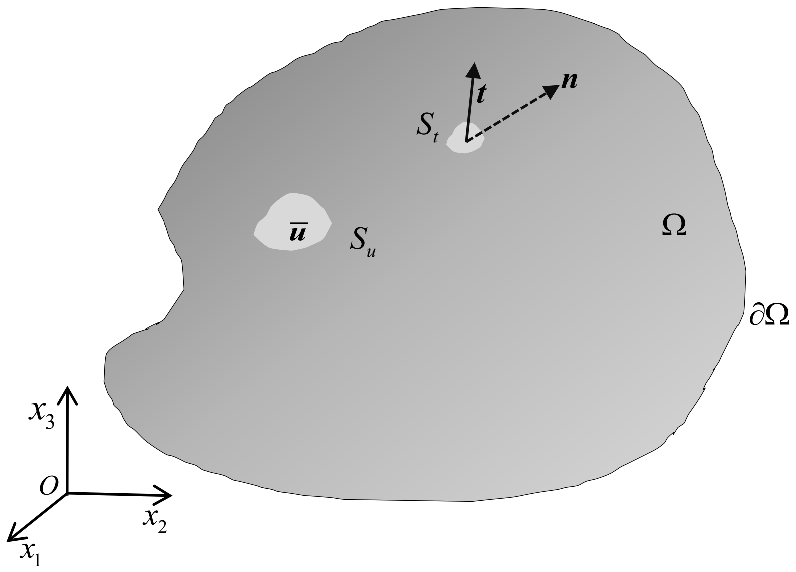
$$\varepsilon(t) = -\frac{\sigma_0}{E_2} e^{-\frac{E_2}{\eta_2}t} + \frac{\sigma_0}{E_2} + \frac{\sigma_0}{E_1} + \frac{\sigma_0}{\eta_1} t$$



FROM: A. D. Drozdov, et al, *Polymers* **2023**, 15, 334.

Solid Mechanics: Elements of linear viscoelasticity

VISCOELASTIC STRESS ANALYSIS



Boundary conditions:

on S_t → $\sigma_{ij}(x_k,t)n_i(x_k) = t_i(x_k,t)$

on S_u → $u_i(x_k,t) = \bar{u}_i(x_k,t)$

For a body in equilibrium, the governing equations are:

Equilibrium: → $\sigma_{ij,j}(x_k,t) + f_i = 0$

Strain-rate-velocity equations: → $\dot{\varepsilon}_{ij}(x_k,t) = \frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right)$

Constitutive Equations : →

$$b_o\sigma_{ij}(x_k,t) + b_1\frac{d\sigma_{ij}(x_k,t)}{dt} + b_2\frac{d^2\sigma_{ij}(x_k,t)}{dt^2} + \ldots$$

$$= a_o\varepsilon_{ij}(t) + a_1\frac{d\varepsilon_{ij}(x_k,t)}{dt} + a_2\frac{d^2\varepsilon_{ij}(t)}{dt^2} + \ldots$$

Initial conditions: → $\dot{u}_i(x_k,0) = \dot{u}_0$

For simple geometries and loading conditions, and when the constitutive relation is represented by a simple equation, the field equations above can be integrated to obtain the solution.

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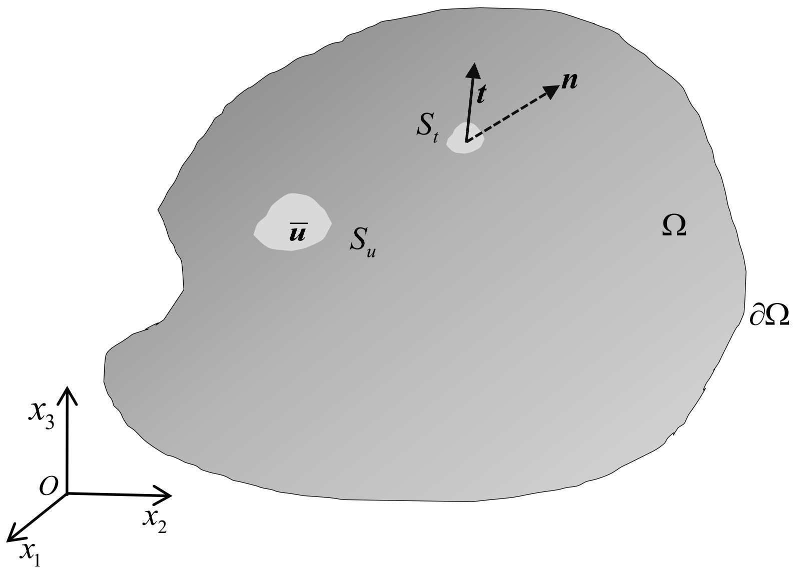
VISCOELASTIC STRESS ANALYSIS



For general conditions, it is not easy to integrate the field equations (with a time variable).

In such cases we can use the *correspondence principle*:

The field equations are transformed to corresponding equations using the *Laplace Transform* (with a complex variable).

The resulting equations are simpler to solve.

We apply the inverse Laplace transform to obtain the solution on the original space (time variable).