ME-280: Fluid Mechanics
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Problem Set 7: Solutions

1 Uniform flow with a sink

1.1 Mathematics

Problem: Consider a flow that consists of a sink of strength M located at the origin and a uniform flow, as shown in Figure 1. Infinitely far away from the origin, the flow is parallel to the x-axis with a uniform velocity, U, and pressure, p_∞. The density of the fluid is ρ, which is constant.

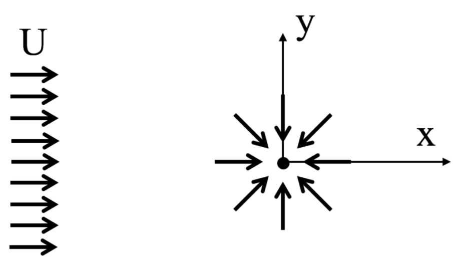

Figure 1: See problem 1.1.

(a) Using superposition, determine the stream function, $\psi(r,\theta)$, and velocity potential, $\phi(r,\theta)$, for this system.
(b) Calculate the velocity field for the flow.
(c) Identify the location of any stagnation point in the flow.
(d) Determine the equation for the stagnation streamline.
(e) Determine the pressure along the line $y = 0$ (i.e. calculate $p(x)$ along $y = 0$). Does anything unusual happen to the pressure at $x = 0$? Explain.
(f) Find the value of x at which the pressure along the line $y = 0$ is maximized.

Solution: **(a)** The potential flow system consists of a sink of strength M at the origin and a uniform flow of strength U in the positive x-direction.
As given in the lecture, the basic solution of a uniform flow is

$$\phi(x,y) = Ux \quad \text{or} \quad \phi(r,\theta) = Ur\cos(\theta),$$
$$\psi(x,y) = Uy \quad \text{or} \quad \psi(r,\theta) = Ur\sin(\theta).$$

A sink in cylindrical coordinates is

$$\phi(r,\theta) = -\frac{M}{2\pi}\ln(r),$$
$$\psi(r,\theta) = -\frac{M}{2\pi}\theta,$$

with $M > 0$. Since both ϕ and ψ satisfy the Laplace equation, $\Delta\phi = 0$ and $\Delta\psi = 0$, which is linear and homogeneous, we can construct velocity potential and stream function of the system as a superposition of the two basic flows:

$$\phi(r,\theta) = Ur\cos(\theta) - \frac{M}{2\pi}\ln(r),$$
$$\psi(r,\theta) = Ur\sin(\theta) - \frac{M}{2\pi}\theta.$$

(b) To calculate the velocity field we can use either ϕ or ψ. Here we calculate $\mathbf{u}$ through the gradient of ϕ in cylindrical coordinates:

$$\mathbf{u} = (u_r, u_\theta, 0) = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, 0\right).$$

$$\begin{cases} \dfrac{\partial\phi}{\partial r} = U\cos(\theta) - \dfrac{M}{2\pi r} \\ \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -U\sin(\theta) \end{cases} \quad \Rightarrow \quad \mathbf{u} = \left(U\,\cos(\theta) - \frac{M}{2\pi r}\right)\hat{\mathbf{e}}_r - U\,\sin(\theta)\hat{\mathbf{e}}_\theta.$$

(c) A stagnation point occurs in the flow at a point where $\mathbf{u} = \mathbf{0}$. Solving for the values of r and θ that cause this for u_r and u_θ, we find that

$$u_\theta = -U\,\sin(\theta) = 0 \quad \Rightarrow \quad \theta = 0,\ \pm\pi,\ \pm2\pi, ..., n\pi \quad \text{where } n \text{ is an integer,}$$
$$u_r = U\,\cos(\theta) - \left.\frac{M}{2\pi r}\right|_{\theta=n\pi} = 0 \quad \Rightarrow \quad r = \frac{M}{2\pi U(-1)^n}.$$

The radial component, r, cannot be negative which means n is even. Additionally, if n is an even integer and we consider that the polar coordinate is only defined for $0 \le \theta < 2\pi$, then $\theta = 0$. Thus, if we let $\theta = 0$, the position of the only stagnation point of the flow is

$$\theta = 0 \quad \text{and} \quad r = \frac{M}{2\pi U},$$
$$\text{or} \quad y = 0 \quad \text{and} \quad x = \frac{M}{2\pi U}.$$

(d) We evaluate the stream function ψ constructed in (a) at the stagnation point

$$\psi\left(r = \frac{M}{2\pi U}, \theta = 0\right) = U\frac{M}{2\pi U}\,\sin(0) - \frac{M}{2\pi}\cdot 0 = 0.$$

The stagnation streamline is defined by $\psi = 0$ which makes the streamline equation

$$r\,\sin(\theta) = \frac{M}{2\pi U}\theta.$$

(e) In order to calculate the pressure along the line $y = 0$, we must apply the Bernoulli Equation at a point on $y = 0$ relative to a point on $y = 0$ that is "infinitely" far away. In this case, we will neglect gravity. We are given that infinitely far away from the origin, the pressure is p_∞ and the velocity is the free stream velocity, $\mathbf{u} = (U, 0, 0)$:

$$p_\infty + \frac{1}{2}\rho|\mathbf{u}(x \to \infty, y = 0)|^2 = p(x) + \frac{1}{2}\rho|\mathbf{u}(x, y = 0)|^2$$
$$\Rightarrow \quad p_\infty + \frac{1}{2}\rho U^2 = p(x) + \frac{1}{2}\rho|\mathbf{u}(x, y = 0)|^2.$$

Evaluating the speed of the flow on the right side of the Bernoulli Equation, we find that

$$\begin{aligned}
|\mathbf{u}(x, y = 0)|^2 &= (u_r^2 + u_\theta^2)\Big|_{y=0} \\
&= \left(U^2\cos^2(\theta) - \frac{M}{\pi r}U\,\cos(\theta) + \left(\frac{M}{2\pi r}\right)^2 + U^2\sin^2(\theta)\right)\Bigg|_{y=0} \\
&= \left(U^2\underbrace{\left(\sin^2(\theta) + \cos^2(\theta)\right)}_{=1} - \frac{M}{\pi r}U\,\cos(\theta) + \left(\frac{M}{2\pi r}\right)^2\right)\Bigg|_{y=0} \\
&= \left(U^2 - \frac{M}{\pi r}U\,\cos(\theta) + \left(\frac{M}{2\pi r}\right)^2\right)\Bigg|_{y=0}.
\end{aligned}$$

Since we are evaluating the flow along the line $y = 0$, the radial coordinate is $r = |x|$. Along $y = 0$, when $x < 0$, the polar coordinate is $\theta = \pi$ and we have $r = -x$. Conversely, when $x > 0$, the polar coordinate is $\theta = 0$ and we have $r = x$. Applying this to the equation above, we can see that the expression for the speed of the flow squared is the same when $x < 0$ and when $x > 0$. Thus, we do not need to examine the pressure separately for these two regions along $y = 0$. The speed squared for the flow can therefore be rewritten as:

$$|\mathbf{u}(x, y = 0)|^2 = U^2 - \frac{MU}{\pi x} + \left(\frac{M}{2\pi x}\right)^2 .$$

Substituting back into the Bernoulli Equation, we find

$$\begin{aligned} p(x) &= p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho U^2 + \frac{1}{2}\rho\left[\frac{MU}{\pi x} - \left(\frac{M}{2\pi x}\right)^2\right] \\ &= \frac{\rho M}{2\pi x}\left(U - \frac{M}{4\pi x}\right) + p_\infty . \end{aligned}$$

We can see that when we evaluate the pressure at $x = 0$, the pressure diverges to $-\infty$. This makes sense because we have idealized the origin as a sink, and a pressure gradient of negative infinity demonstrates how the fluid is driven into the sink. Therefore, the sink acts as a singularity at $(x = 0, y = 0)$. We can avoid this problem by saying that the pressure, $p(x)$, is not defined at $x = 0$.

(f) The Bernoulli Equation along $y = 0$ implies that the pressure is highest where the U^2 is lowest. Consequently, we know that the maximum pressure occurs at the stagnation point

$$x = \frac{M}{2\pi U} .$$

The maximum pressure value can be obtained formally from first and second derivative of the function $p(x)$:

$$\begin{aligned} \frac{\partial p}{\partial x} = \frac{1}{2}\rho\left(-\frac{MU}{\pi x^2} + \frac{M^2}{2\pi^2 x^3}\right) = 0 \quad &\Leftrightarrow \quad \frac{M^2}{2\pi^2 x^3} = \frac{MU}{\pi x^2} \\ &\Leftrightarrow \quad x = \frac{M}{2\pi U}. \end{aligned}$$

We can now apply the second derivative test to check that this critical point is, indeed, a maximum.

$$\begin{aligned} \left.\frac{\partial^2 p}{\partial x^2}\right|_{x=\frac{M}{2\pi U}} &= \frac{1}{2}\rho\left(\frac{2MU}{\pi x^3} - \frac{3M^2}{2\pi^2 x^4}\right)\Bigg|_{x=\frac{M}{2\pi U}} = \frac{1}{2}\rho\left[\frac{2MU}{\pi\left(\frac{M}{2\pi U}\right)^3} - \frac{3M^2}{2\pi^2\left(\frac{M}{2\pi U}\right)^4}\right] \\ &= \frac{16\pi^2 U^4}{M^2} - \frac{24\pi^2 U^2}{M^2} = -\frac{8\pi^2 U^2}{M^2} \\ \Rightarrow \quad \left.\frac{\partial^2 p}{\partial x^2}\right|_{x=\frac{M}{2\pi U}} &< 0. \end{aligned}$$

Since the second derivative of $p(x)$ evaluated at the critical point is always negative, the pressure is maximized along $y = 0$ when

$$x = \frac{M}{2\pi U}.$$

1.2 Flow into a slit

Problem: Water flows over a flat surface at $1.2\,m/s$ as shown in Figure 2. A pump draws off water through a narrow slit at a volume rate of $0.01\,m^3/s$ per meter length of the slit. Assume that the fluid is incompressible and inviscid and can be represented by the combination of a uniform flow and a sink. How far above the surface, H, must the fluid be so that it does not get sucked into the slit? Use the results from 1.1.

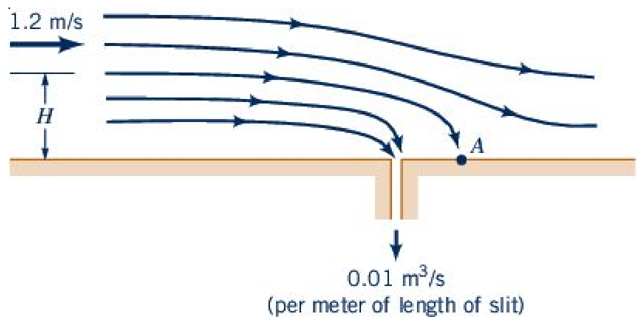

Figure 2: See problem 1.2.

Solution: Fluid gets sucked into the slit if it lies below the stagnation streamline which terminates at point A. Therefore, the height H defines the location of the stagnation streamline in the uniform flow. The equation of the stagnation streamline was calculated in 1.1d:

$$r\,\sin(\theta) = \frac{M}{2\pi U}\theta.$$

Since $y = r\,\sin(\theta)$, we can express the equation in terms of the vertical coordinate

$$y = \frac{M}{2\pi U}\theta.$$

To obtain the height $y = H$ in the uniform flow, we take the limit $\theta \to \pi$ so that

$$H = \frac{M}{2U}.$$

Note that the given volume rate of drainage through the slit refers only to the space above the flat surface, while the assumed point sink covers the full space. Consequently, the strength of the sink is twice the given numerical value

$$M = \frac{0.02\,\frac{m^3}{s}}{1\,m} = 0.02\,\frac{m^2}{s}$$

$$\Rightarrow \quad H = \frac{0.02\,\frac{m^2}{s}}{2 \times 1.2\,\frac{m}{s}} = 0.008\,m.$$

Compare this exercise with the boundary layer exercise of the previous problem set. It is also possible to compute H using a control volume approach with mass conservation.

2 Quonset hut

Problem: Wind at velocity U_∞ and pressure p_∞ flows past a Quonset hut which is a half-cylinder of radius a and length L (Figure 3). The internal pressure is p_i. Derive an expression for the upward force on the hut due to the difference between p_i and p_s.

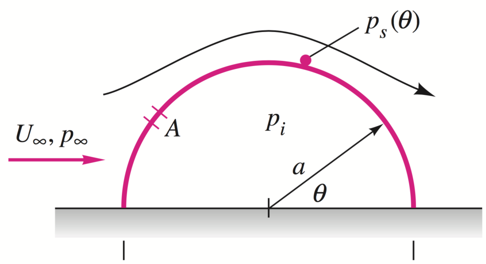

Figure 3: See problem 2.

Solution: The upward force (i.e. the *lift*) depends on the surface pressure p_s on the Quonset hut relative to the inside pressure p_i. Like it was done in the lecture for the flow around a full cylinder, the surface pressure on the Quonset hut is obtained from the Bernoulli equation once the surface velocity is known. The hut is a half-cylinder and can be constructed by a superposition of a uniform flow and a doublet. We know that the given flow geometry has two symmetric stagnation points at $\theta = 0$ and $\theta = \pi$. Therefore, we do not need a free vortex to create circulation and change the location of the stagnation points. The streamfunction is

$$\begin{aligned}\psi(r,\theta) = \psi_{uniform} + \psi_{doublet} &= U_\infty r\sin(\theta) - U_\infty \frac{a^2\sin(\theta)}{r} \\ &= U_\infty r\left(1 - \frac{a^2}{r^2}\right)\sin(\theta),\end{aligned}$$

where we consider only the symmetric upper half. The velocity field is given by

$$\begin{aligned}v_r &= \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty\left(1 - \frac{a^2}{r^2}\right)\cos(\theta), \\ v_\theta &= -\frac{\partial\psi}{\partial r} = -U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin(\theta).\end{aligned}$$

On the surface where $r = a$ we have

$$v_{rs} = 0\ , \quad v_{\theta s} = -2U_\infty\sin(\theta).$$

We obtain the surface pressure p_s from the Bernoulli equation which relates to the pressure in the p_∞ in the uniform flow:

$$p_\infty + \frac{1}{2}\rho U_\infty^2 = p_s + \frac{1}{2}\rho\left(v_{rs}^2 + v_{\theta s}^2\right) = p_s + \frac{1}{2}\rho v_{\theta s}^2,$$

where we neglect gravity effects of the elevation. Inserting $v_{\theta s}$ we find

$$p_s = p_\infty + \frac{1}{2}\rho U_\infty^2\left(1 - 4\sin^2(\theta)\right).$$

The lift per unit length L is the integrated gauge pressure $p_s - p_i$:

$$\begin{aligned}\frac{F_{lift}}{L} &= -\int_0^\pi (p_s - p_i)\sin(\theta)a\,d\theta \\ &= -\int_0^\pi \left[p_\infty + \frac{1}{2}\rho U_\infty^2(1 - 4\sin^2(\theta)) - p_i\right]\sin(\theta)a\,d\theta \\ &= -(p_\infty - p_i)a\int_0^\pi \sin(\theta)\,d\theta - \frac{1}{2}a\rho U_\infty^2\int_0^\pi \sin(\theta)\,d\theta + 2a\rho U_\infty^2\int_0^\pi \sin^3(\theta)\,d\theta,\end{aligned}$$

with

$$\int_0^\pi \sin(\theta)\, d\theta = [-\cos(\theta)]_0^\pi = 2,$$
$$\int_0^\pi \sin^3(\theta)\, d\theta = \frac{1}{12}\left[\cos(3\theta) - 9\cos(\theta)\right]_0^\pi = \frac{4}{3}.$$

The lift force is

$$F_{lift} = 2aL(p_i - p_\infty) - aL\rho U_\infty^2 + \frac{8}{3}aL\rho U_\infty^2$$
$$= 2aL(p_i - p_\infty) + \frac{5}{3}aL\rho U_\infty^2.$$

3 Rotating shaft in a pipe

Problem: Consider the fully developed flow of an incompressible fluid with viscosity μ and density ρ enclosed between two concentric cylinders of radii R_1 and R_2, ($R_1 < R_2$), as shown in Figure 4. The outer pipe is held stationary, while the inner pipe rotates slowly with a constant angular velocity ω. Due to the symmetry of the system, the pressure gradient only varies spatially in the radial direction; that is, $p = p(r)$.

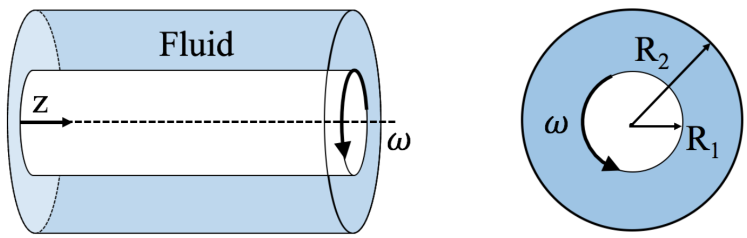

Figure 4: Rotating shaft in a pipe. See problem 3.

(a) Simplify the Navier-Stokes equations to derive the flow's equation of motion. State your assumptions.
(b) What are the boundary conditions for the flow?
(c) Calculate the velocity profile of the flow.
(d) If the pressure at $r = R_1$ is p_0, what is the pressure distribution in the fluid, $p(r)$?

Hint: The general form of the incompressible Navier-Stokes equations in cylindrical coordinates was given on the last exercise sheet.

Solution: **(a)** The Navier-Stokes equations in cylindrical coordinate are
Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0,$$

The r-momentum equation:

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{1}{r}v_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right),$$

The θ-momentum equation:

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{1}{r}v_r v_\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right),$$

The z-momentum equation:

$$\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right).$$

We can simplify the equations given the following assumptions. There is no gravity as it is neither mentioned in the problem description nor indicated in the schematic, so all of those gravity terms vanish. The flow is

steady, so all of the $\partial()/\partial t$ terms vanish. Clearly because the only source of motion is the rotation of the inner cylinder, the flow is purely in the θ-direction, and by the symmetry, we can see that the velocity does not vary in the θ-direction, thus $\partial()/\partial\theta$ terms vanish. Additionally, the channel is assumed to be very long such that all of the $\partial()/\partial z$ terms vanish. Thus, we know that $\vec{V} = v_\theta(r)\hat{e_\theta}$ (i.e. all of the terms with v_r and v_z vanish). Accordingly, the Navier-Stokes Equations reduce to:
Continuity:

$$0 = 0,$$

The r-momentum equation:

$$\frac{\rho}{r}v_\theta^2 = \frac{\partial p}{\partial r},$$

The θ-momentum equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0,$$

The z-momentum equation:

$$0 = 0.$$

(b) We can see from the θ-component of the simplified Navier-Stokes equations that the equation defining v_θ is second order, meaning that we will need to implement two boundary conditions to solve for the flow. The boundary conditions are the no-slip conditions for the flow at the rotating inner cylinder and the stationary outer cylinder, that is

$$v_\theta(r = R_1) = \omega R_1,$$
$$v_\theta(r = R_2) = 0.$$

(c) We can solve the equation for v_θ by finding a simplified form, integrating twice, and solving for the integration constants by applying the boundary conditions. Doing so, we see that the equation for v_θ in part (a) can be simplified to a gradient form:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right] = 0.$$

Integrating twice, we find that:

$$v_\theta(r) = C_1 r + \frac{C_2}{r}.$$

Alternatively, one can expand the equation to the following form:

$$r^2\frac{\partial^2 v_\theta}{\partial r^2} + r\frac{\partial v_\theta}{\partial r} - v_\theta = 0,$$

which is a second order differential equation with non-constant coefficients. This type of ODE is called Cauchy-Euler equation that allows for solutions of the form Cr^n. Replacing v_θ by Cr^n in the equation we find $n = 1$ or $n = -1$. So the general solution to this differential equation is $v_\theta(r) = C_1 r + C_2/r$.
In this solution, C_1 and C_2 are constants. When we apply boundary conditions, we find that:

$$v_\theta(R_1) = \omega R_1 \rightarrow \omega R_1 = C_1 R_1 + \frac{C_2}{R_1},$$
$$v_\theta(R_2) = 0 \rightarrow 0 = C_1 R_2 + \frac{C_2}{R_2}.$$

Solving this system of equations, we find that

$$C_1 = -\frac{\omega\left(\frac{R_1}{R_2}\right)^2}{1-\left(\frac{R_1}{R_2}\right)^2},$$
$$C_2 = \frac{\omega R_1^2}{1-\left(\frac{R_1}{R_2}\right)^2}.$$

Thus, the velocity profile for the the fluid is

$$v_\theta(r) = \frac{\omega R_1^2}{r\left(1-\left(\frac{R_1}{R_2}\right)^2\right)}\left[1-\left(\frac{r}{R_2}\right)^2\right].$$

(d) We can solve for the pressure distribution in the fluid, $p(r)$, by substituting our expression for $v_\theta(r)$ into the r-component of the Navier Stokes-Equations from part (a):

$$\frac{\rho}{r}v_\theta^2 = \frac{\partial p}{\partial r}.$$

Substituting, we find that

$$\frac{\partial p}{\partial r} = \frac{\rho\omega^2 R_1^4}{r^3\left(1-\left(\frac{R_1}{R_2}\right)^2\right)^2}\left[1-\left(\frac{r}{R_2}\right)^2\right]^2.$$

Integrating this equation, we find that

$$p(r) = A\left(-\frac{1}{2r^2}-\frac{2}{R_2^2}\ln(r)+\frac{r^2}{2R_2^4}\right)+C_0,$$

where $A = \rho\omega^2 R_1^4/\left(1-(R_1/R_2)^2\right)^2$. In this equation, C_0 is an integration constant. We can solve for C_0 since we are given that $p(r=R_1)=p_0$.

$$C_0 = A\left(\frac{1}{2R_1^2}+\frac{2}{R_2^2}\ln(R_1)-\frac{R_1^2}{2R_2^4}\right)+p_0.$$

Substituting this expression into the pressure distribution, we find that:

$$p(r) = A\left(\left(\frac{r^2-R_1^2}{2r^2R_1^2}\right)+\frac{2}{R_2^2}\ln\left(\frac{R_1}{r}\right)+\frac{r^2-R_1^2}{2R_2^4}\right)+p_0,$$

or

$$p(r) = \frac{\rho\omega^2 R_1^4}{\left(1-\left(\frac{R_1}{R_2}\right)^2\right)^2}\left(\left(\frac{r^2-R_1^2}{2r^2R_1^2}\right)+\frac{2}{R_2^2}\ln\left(\frac{R_1}{r}\right)+\frac{r^2-R_1^2}{2R_2^4}\right)+p_0.$$

4 Pulled shaft in a pipe

Problem: An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Figure 5. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. The pressure gradient in the axial directions is $-\Delta p/l$ where $\Delta p > 0$ is the magnitude of the pressure difference between to sections of distance l. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric and fully developed.

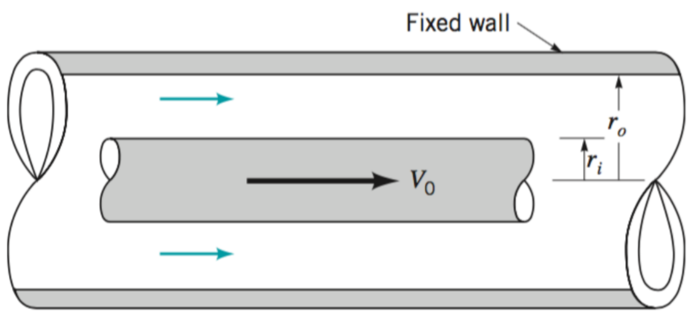

Figure 5: See problem 4.

Solution: The Navier-Stokes equations in cylindrical coordinate are written in the last exercise. In this problem we can simplify the equations given the following assumptions. There is no gravity. The flow is steady, so all of the $\partial()/\partial t$ terms vanish. The flow is purely in the $\hat{z}$-direction, and by the symmetry, we can see that the velocity does not vary in the θ-direction, hence all $\partial()/\partial\theta$ terms vanish. Additionally, the pipe is assumed to be very long such that all of the $\partial()/\partial z$ terms vanish as well. Thus, we know that $\vec{V} = v_z(r)\hat{e_z}$ (i.e. all of the terms with v_r and v_θ vanish). Accordingly, the Navier Stokes Equations reduce to:

Continuity:

$$0 = 0,$$

The r-momentum equation:

$$0 = \frac{\partial p}{\partial r},$$

The θ-momentum equation:

$$0 = 0,$$

The z-momentum equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}.$$

Two equations are automatically satisfied and r-momentum only shows that pressure does not depend on r. The last equation should be solved to find velocity profile:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) &= \frac{1}{\mu}\frac{\partial p}{\partial z} \\
\to \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) &= \frac{r}{\mu}\frac{\partial p}{\partial z} \\
\to r\frac{\partial v_z}{\partial r} &= \frac{r^2}{2\mu}\frac{\partial p}{\partial z} + C_1 \\
\to \frac{\partial v_z}{\partial r} &= \frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{C_1}{r}
\end{aligned}$$

$$\rightarrow v_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + C_1 \ln(r) + C_2, \tag{1}$$

with boundary conditions, $v_z(r = r_o) = 0$ and $v_z(r = r_i) = V_0$, it follows that:

$$0 = \frac{r_o^2}{4\mu}\frac{\partial p}{\partial z} + C_1 \ln(r_o) + C_2, \tag{2}$$

$$V_0 = \frac{r_i^2}{4\mu}\frac{\partial p}{\partial z} + C_1 \ln(r_i) + C_2. \tag{3}$$

We subtract equation 2 from 3 to obtain

$$V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2) + C_1 \ln\left(\frac{r_i}{r_o}\right),$$

so that

$$C_1 = \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2)}{\ln\left(\frac{r_i}{r_o}\right)}.$$

The drag on the inner cylinder will be zero if

$$(\tau_{rz})_{r=r_i} = 0.$$

Since $\tau_{rz} = \mu\left(\partial v_r/\partial z + \partial v_z/\partial r\right)$, with $v_r = 0$ it follows that

$$\tau_{rz} = \mu\frac{\partial v_z}{\partial r}.$$

Differentiate equation 1 with respect to r to obtain

$$\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right) r + \frac{C_1}{r},$$

so that at $r = r_i$

$$(\tau_{rz})_{r=r_i} = \mu\left[\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right) r_i + \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2)}{r_i \ln\left(\frac{r_i}{r_o}\right)}\right].$$

Thus, in order for the drag to be zero,

$$\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right) r_i + \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2)}{r_i \ln\left(\frac{r_i}{r_o}\right)} = 0,$$

or

$$V_0 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[2r_i^2 \ln\left(\frac{r_i}{r_o}\right) - (r_i^2 - r_o^2)\right].$$

5 Time-dependent channel flow

Problem: A section of a water channel is shown in Figure 6. It is $l = 0.5\,m$ long and $d = 1\,cm$ wide. The water is incompressible and has a dynamic viscosity of $\mu = 10^{-3}\,Ns/m^2$. The fluid is initially at rest but at time $t = 0$, a pump is switched on and the fluid becomes subject to a pressure difference of $\Delta p = p_{in} - p_{out} = 0.01\,kPa$ along the channel in x-direction. Due to *no slip* boundary conditions the velocity at the channel walls is zero: $\mathbf{U}(y = 0, t) = \mathbf{U}(y = d, t) = 0$.
(a) Simplify the Navier-Stokes Equations for the planar velocity $\mathbf{U}(\mathbf{x}, t) = U(y, t)\,\hat{\mathbf{i}}$.
Then, solve the time-dependent problem of the vertical velocity profile $U(y, t)$ by calculating
(b) the steady state solution,
(c) the time-dependent solution for the deviations from the steady state.

Hint: Follow the same solution strategy as in the diffusion exercises in problem set 3 and 4.

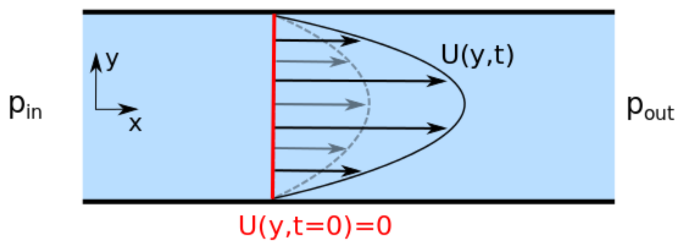

Figure 6: Pressure driven channel flow. See problem 5.

Solution: **(a)** Navier-Stokes equations are

$$\nabla \cdot \vec{V} = 0,$$
$$\rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}.$$

For a general velocity field (U, V, W) in Cartesian coordinates, they become

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0,$$
$$\rho \left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right),$$
$$\rho \left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} \right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right),$$
$$\rho \left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} \right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right).$$

In parallel channel flow, $\vec{V} = U(y)\hat{\mathbf{i}}$ which means $V = W = \partial U/\partial x = \partial U/\partial z = 0$. Assuming that gravity applies in the y-direction implies that $g_x = g_z = 0$. So these equations can be simplified to

$$0 = 0,$$
$$\rho \frac{\partial U}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 U}{\partial y^2},$$
$$0 = -\frac{\partial p}{\partial y} + \rho g_y,$$
$$0 = -\frac{\partial p}{\partial z}.$$

The continuity equation is automatically satisfied, and the last two equations only lead to hydrostatic pressure distribution in the y-direction. The only equation left is the momentum equation in x-direction, thus we need to solve

$$\rho \frac{\partial U}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 U}{\partial y^2}.$$

(b) The steady solution is denoted by $U_s(y)$. We know that $\partial U_s/\partial t = 0$ and U_s only depends on y, thus

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 U_s}{dy^2},$$

or

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 U_s}{dy^2}.$$

The right hand side is only a function of y while the left hand side does not have any dependency on y so both sides should be constants; it means that pressure gradient is constant along the channel. Integration of this equation twice with respect to y gives

$$U_s(y) = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_1 y + C_2.$$

Applying the boundary conditions leads to

$$\begin{aligned} U_s(0) = 0 &\rightarrow 0 = 0 + 0 + C_2 \rightarrow C_2 = 0, \\ U_s(d) = 0 &\rightarrow 0 = \frac{1}{2\mu} \frac{\partial p}{\partial x} d^2 + C_1 d \rightarrow C_1 = -\frac{1}{2\mu} \frac{\partial p}{\partial x} d. \end{aligned}$$

Thus,

$$U_s(y) = \frac{1}{2\mu} \frac{\partial p}{\partial x} y(y - d),$$

or

$$U_s(y) = -\frac{d^2}{2\mu} \frac{\partial p}{\partial x} \frac{y}{d} \left(1 - \frac{y}{d}\right).$$

(c) $U(y,t)$ can be decomposed into the steady state solution and the deviation from steady state as

$$U(y,t) = U_s(y) + u(y,t).$$

Inserting the decomposition into the Navier-Stokes equation gives

$$\rho \frac{\partial U}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 U}{\partial y^2} \;\rightarrow\; \rho \frac{\partial U_s}{\partial t} + \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 U_s}{\partial y^2} + \mu \frac{\partial^2 u}{\partial y^2}.$$

We know that $\partial U_s/\partial t = 0$, and also the steady solution satisfies $-\partial p/\partial x + \mu \left(\partial^2 U_s/\partial y^2\right) = 0$. So

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

Therefore, Navier-Stokes equations for deviation from steady states have been simplified to the diffusion equation without a source term which makes the problem homogeneous. Initial condition for velocity deviation can be derived by decomposition of initial condition for U as

$$U(y,0) = 0 \rightarrow u(y,0) + U_s(y) = 0 \rightarrow u(y,0) = -U_s(y) = \frac{d^2}{2\mu} \frac{\partial p}{\partial x} \frac{y}{d} \left(1 - \frac{y}{d}\right).$$

The boundary conditions are

$$\begin{aligned} u(0,t) &= U(0,t) - U_s(0) = 0, \\ u(d,t) &= U(d,t) - U_s(d) = 0. \end{aligned}$$

The diffusion equation with the boundary conditions are linear and homogeneous, so it is possible to express its solution as a linear combination of the so called base functions that are themselves solutions to the same PDE with the same boundary conditions:

$$\frac{\partial \varphi_n}{\partial t} = \nu \frac{\partial^2 \varphi_n}{\partial y^2},$$
$$\varphi_n(y=0,t) = 0,$$
$$\varphi_n(y=d,t) = 0.$$

Using the method of *separation of variables*, we assume that every base function is a product of a function that only depends on y and a function that only depends on t (then we try to find such solutions; if we succeed it means that this assumption is true):

$$\varphi_n(y,t) = Y_n(y)T_n(t).$$

By putting this assumption into the diffusion equation:

$$\frac{\partial (Y_n(y)T_n(t))}{\partial t} = \nu \frac{\partial^2 (Y_n(y)T_n(t))}{\partial y^2}$$
$$\rightarrow Y_n \frac{dT_n}{dt} = \nu T_n \frac{d^2 Y_n}{dy^2}$$
$$\rightarrow Y_n T_n^{'} = \nu T_n Y_n^{''}$$
$$\rightarrow \frac{1}{\nu}\frac{T_n^{'}}{T_n} = \frac{Y_n^{''}}{Y_n}.$$

The left hand side of the above relation is only a function of t while the right hand side is only a function of y; the equality is only possible if both sides are constants. We call this constant $-\lambda_n$; the minus sign is only for convenience. Thus,

$$\frac{1}{\nu}\frac{T_n^{'}}{T_n} = \frac{Y_n^{''}}{Y} = -\lambda_n \rightarrow \begin{cases} T_n^{'} + \nu\lambda_n T_n = 0, \\ Y_n^{''} + \lambda_n Y_n = 0. \end{cases}$$

Thus, the separation ansatz converted one PDE into two ordinary differential equations (ODE) which can be solved for $Y_n(y)$ and $T_n(t)$ independently. The ODE for T_n, which is first order in time, is simply solved to get:

$$T_n(t) = \exp(-\nu\lambda_n t),$$

but the ODE for Y_n, which is second order in space, depends on the boundary conditions and the sign of λ_n. Putting the boundary conditions into the separation of variables assumption, $\varphi_n(y,t) = Y_n(y)T_n(t)$, we obtain:

$$Y_n(0)T_n(t) = 0 \rightarrow Y_n(0) = 0,$$
$$Y_n(d)T_n(t) = 0 \rightarrow Y_n(d) = 0.$$

To solve the ODE for $Y_n(y)$ with above boundary conditions, we consider three cases:

Case i: $\lambda_n < 0$

$$Y_n(y) = \alpha_1 \cosh(\sqrt{-\lambda_n} y) + \alpha_2 \sinh(\sqrt{-\lambda_n} y),$$

$$\begin{cases} Y_n(0) = 0 \rightarrow \alpha_1 \cdot (1) + \alpha_2 \cdot (0) = 0 \rightarrow \alpha_1 = 0 \\ Y_n(d) = 0 \rightarrow \alpha_2 \sinh(\sqrt{-\lambda_n} d) = 0 \rightarrow \alpha_2 = 0 \end{cases} \rightarrow Y_n(y) = 0.$$

Thus in this case, no solution exists except the trivial solution, $\varphi_n = Y_n T_n = 0$.

Case ii: $\lambda_n = 0$

$$Y_n(y) = \alpha_1 y + \alpha_2,$$

$$\begin{cases} Y_n(0) = 0 \rightarrow \alpha_2 = 0 \\ Y_n(d) = 0 \rightarrow \alpha_1 = 0 \end{cases} \rightarrow Y_n(y) = 0.$$

In this case also only trivial solution, $\varphi_n = Y_n T_n = 0$, exists.

Case iii: $\lambda_n > 0$

$$Y_n(y) = \alpha_1 \cos(\sqrt{\lambda_n} y) + \alpha_2 \sin(\sqrt{\lambda_n} y),$$

$$\begin{cases} Y_n(0) = 0 \to \alpha_1 \cdot (1) + \alpha_2 \cdot (0) = 0 \to \alpha_1 = 0, \\ Y_n(d) = 0 \to \alpha_2 \sin(\sqrt{\lambda_n} d) = 0. \end{cases}$$

For a nontrivial solution $\sin(\sqrt{\lambda_n} d)$ should be zero and not α_2, so

$$\sqrt{\lambda_n} = \frac{n\pi}{d} \quad n = 1, 2, 3, ...$$

$$\to Y_n = \sin(\sqrt{\lambda_n} y), \quad \lambda_n = \left(\frac{n\pi}{d}\right)^2 \quad n = 1, 2, 3, ...$$

Thus, the base functions are

$$\varphi_n(y,t) = Y_n(y) T_n(t) = \sin(\sqrt{\lambda_n} y) \exp(-\nu \lambda_n t), \quad \lambda_n = \left(\frac{n\pi}{d}\right)^2 \quad n = 1, 2, 3, ...$$

and the unsteady solution, that is a linear combination of base functions, becomes

$$u(y,t) = \sum_{n=1}^{\infty} A_n \varphi_n(y,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n} y) \exp(-\nu \lambda_n t), \quad \lambda_n = \left(\frac{n\pi}{d}\right)^2.$$

Setting $t = 0$ in the solution ansatz and using the initial condition, we get

$$u(y,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{d} y\right) \exp(-\nu \lambda_n \cdot 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{d} y\right) = \frac{d^2}{2\mu} \frac{\partial p}{\partial x} \frac{y}{d} \left(1 - \frac{y}{d}\right).$$

To obtain A_n, we multiply the whole expression by $\sin(\sqrt{\lambda_m} y)$ and integrate from 0 to d:

$$\int_0^d \sin\left(\frac{m\pi}{d} y\right) \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{d} y\right) dy = \int_0^d \sin\left(\frac{m\pi}{d} y\right) \left(\frac{d^2}{2\mu} \frac{\partial p}{\partial x} \frac{y}{d} \left(1 - \frac{y}{d}\right)\right) dy.$$

Since the summation is over n, the integral and sum are commutable:

$$\sum_{n=1}^{\infty} A_n \int_0^d \sin\left(\frac{m\pi}{d} y\right) \sin\left(\frac{n\pi}{d} y\right) dy = \int_0^d \sin\left(\frac{m\pi}{d} y\right) \left(\frac{d^2}{2\mu} \frac{\partial p}{\partial x} \frac{y}{d} \left(1 - \frac{y}{d}\right)\right) dy.$$

If $n \neq m$ the integral at the left hand side is 0 and only if $n = m$ it is not zero. So the summation will collapse to only one term:

$$A_n \int_0^d \sin\left(\frac{n\pi}{d} y\right) \sin\left(\frac{n\pi}{d} y\right) dy = \int_0^d \sin\left(\frac{n\pi}{d} y\right) \left(\frac{d^2}{2\mu} \frac{\partial p}{\partial x} \frac{y}{d} \left(1 - \frac{y}{d}\right)\right) dy$$

$$\to A_n = \left(\frac{-1}{2\mu} \frac{\partial p}{\partial x}\right) \frac{\displaystyle\int_0^d \sin\left(\frac{n\pi}{d} y\right) (y^2 - yd) dy}{\displaystyle\int_0^d \sin^2\left(\frac{n\pi}{d} y\right) dy}.$$

So we need to evaluate the integrals:

$$\int_0^d \sin\left(\frac{n\pi}{d}y\right)(y^2-yd)dy = \frac{-d}{n\pi}(y^2-yd)\cos\left(\frac{n\pi}{d}y\right)\Big|_0^d - \frac{-d}{n\pi}\int_0^d \cos\left(\frac{n\pi}{d}y\right)(2y-d)dy$$

$$= 0 + \frac{d}{n\pi}\left[\frac{d}{n\pi}(2y-d)\sin\left(\frac{n\pi}{d}y\right)\Big|_0^d - \frac{d}{n\pi}\int_0^d 2\sin\left(\frac{n\pi}{d}y\right)dy\right]$$

$$= 0 + \frac{d}{n\pi}\left[0 + 2\left(\frac{d}{n\pi}\right)^2\cos\left(\frac{n\pi}{d}y\right)\Big|_0^d\right] = 2\left(\frac{d}{n\pi}\right)^3(\cos(n\pi)-1)$$

$$= 2\left(\frac{d}{n\pi}\right)^3((-1)^n-1),$$

$$\int_0^d \sin^2\left(\frac{n\pi}{d}y\right)dy = \int_0^d \frac{1}{2}\left[1-\cos\left(2\frac{n\pi}{d}y\right)\right]dy = \frac{1}{2}\left[y - \frac{1}{2\sqrt{\lambda_n}}\sin\left(2\frac{n\pi}{d}y\right)\right]_0^d$$

$$= \frac{1}{2}\left[(d-0) - \frac{1}{2\frac{n\pi}{d}}\left(\sin\left(\frac{n\pi}{d}2d\right)-0\right)\right] = \frac{1}{2}[d-0] = \frac{d}{2}.$$

Thus,

$$A_n = \left(\frac{-1}{2\mu}\frac{\partial p}{\partial x}\right)\frac{\int_0^d \sin\left(\frac{n\pi}{d}y\right)(y^2-yd)dy}{\int_0^d \sin^2\left(\frac{n\pi}{d}y\right)dy} = \left(\frac{-1}{2\mu}\frac{\partial p}{\partial x}\right)\frac{2\left(\frac{d}{n\pi}\right)^3((-1)^n-1)}{\frac{d}{2}}$$

$$= \frac{2d^2(1-(-1)^n)}{(n\pi)^3\mu}\frac{\partial p}{\partial x},$$

and,

$$u(y,t) = \sum_{n=1}^{\infty} A_n\varphi_n(y,t) = \sum_{n=1}^{\infty}\frac{2d^2(1-(-1)^n)}{(n\pi)^3\mu}\frac{\partial p}{\partial x}\sin\left(\frac{n\pi}{d}y\right)\exp\left(-\nu\left(\frac{n\pi}{d}\right)^2 t\right),$$

and,

$$U(y,t) = U_s + u(y,t) = -\frac{d^2}{2\mu}\frac{\partial p}{\partial x}\frac{y}{d}\left(1-\frac{y}{d}\right) + \sum_{n=1}^{\infty}\frac{2d^2(1-(-1)^n)}{(n\pi)^3\mu}\frac{\partial p}{\partial x}\sin\left(\frac{n\pi}{d}y\right)\exp\left(-\nu\left(\frac{n\pi}{d}\right)^2 t\right).$$

By putting the given numbers into this formula and using $\partial p/\partial x = \Delta p/\Delta x$ ("homogeneous in x") we obtain

$$U(y,t) = -\frac{(0.01\,m)^2}{2\cdot 0.001\,Pa\cdot s}\frac{10\,Pa}{0.5\,m}\frac{y}{0.01\,m}\left(1-\frac{y}{0.01\,m}\right)$$
$$+\sum_{n=1}^{\infty}\frac{2(0.01\,m)^2(1-(-1)^n)}{(n\pi)^3\cdot 0.001\,Pa\cdot s}\frac{10\,Pa}{0.5\,m}\sin\left(\frac{n\pi}{0.01\,m}y\right)\exp\left(-\frac{0.001\,Pa\cdot s}{1000\,kg/m^3}\left(\frac{n\pi}{0.01\,m}\right)^2 t\right),$$

or

$$U(y,t) = -100\left(\frac{1}{s}\right)y\left(1-100\left(\frac{1}{m}\right)y\right)$$
$$+\sum_{n=1}^{\infty}\frac{4(1-(-1)^n)}{(n\pi)^3}\left(\frac{m}{s}\right)\sin\left(100\left(\frac{1}{m}\right)n\pi y\right)\exp\left(-0.01\left(\frac{1}{s}\right)(n\pi)^2 t\right).$$