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Solutions: 28.05.2025

Problem Set 7: Navier-Stokes Equations

1 Uniform flow with a sink

1.1 Mathematics

Consider a flow that consists of a sink of strength M located at the origin and a uniform flow, as shown in Figure 1. Infinitely far away from the origin, the flow is parallel to the x-axis with a uniform velocity, U, and pressure, p_∞. The density of the fluid is ρ, which is constant.

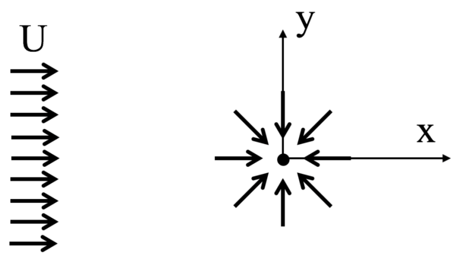

Figure 1: See problem 1.1.

(a) Using superposition, determine the stream function, $\psi(r, \theta)$, and velocity potential, $\phi(r, \theta)$, for this system.
(b) Calculate the velocity field for the flow.
(c) Identify the location of any stagnation point in the flow.
(d) Determine the equation for the stagnation streamline.
(e) Determine the pressure along the line $y = 0$ (i.e. calculate $p(x)$ along $y = 0$). Does anything unusual happen to the pressure at $x = 0$? Explain.
(f) Find the value of x at which the pressure along the line $y = 0$ is maximized.

1.2 Flow into a slit

Water flows over a flat surface at $1.2\,m/s$ as shown in Figure 2. A pump draws off water through a narrow slit at a volume rate of $0.01\,m^3/s$ per meter length of the slit. Assume that the fluid is incompressible and inviscid and can be represented by the combination of a uniform flow and a sink. How far above the surface, H, must the fluid be so that it does not get sucked into the slit? Use the results from 1.1.

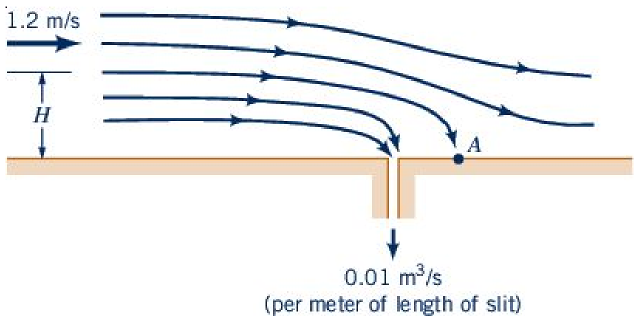

Figure 2: See problem 1.2.

2 Quonset hut

Wind at velocity U_∞ and pressure p_∞ flows past a Quonset hut which is a half-cylinder of radius a and length L (Figure 3). The internal pressure is p_i. Derive an expression for the upward force on the hut due to the difference between p_i and p_s.

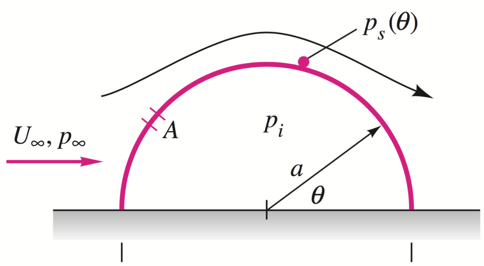

Figure 3: See problem 2.

3 Rotating shaft in a pipe

Consider the fully developed flow of an incompressible fluid with viscosity μ and density ρ enclosed between two concentric cylinders of radii R_1 and R_2, $(R_1 < R_2)$, as shown in Figure 4. The outer pipe is held stationary, while the inner pipe rotates slowly with a constant angular velocity ω. Due to the symmetry of the system, the pressure gradient only varies spatially in the radial direction; that is, $p = p(r)$.

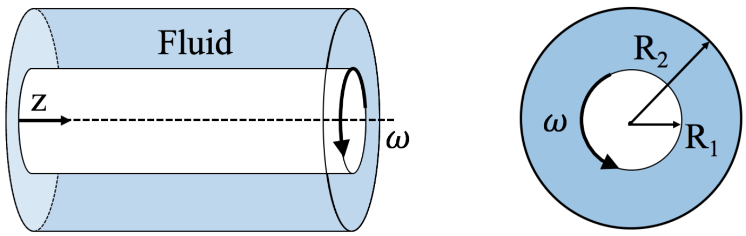

Figure 4: Rotating shaft in a pipe. See problem 3.

(a) Simplify the Navier-Stokes equations to derive the flow's equation of motion. State your assumptions.
(b) What are the boundary conditions for the flow?
(c) Calculate the velocity profile of the flow.
(d) If the pressure at $r = R_1$ is p_0, what is the pressure distribution in the fluid, $p(r)$?

Hint: The general form of the incompressible Navier-Stokes equations in cylindrical coordinates was given on the last exercise sheet.

4 Pulled shaft in a pipe

An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Figure 5. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. The pressure gradient in the axial directions is $-\Delta p/l$ where $\Delta p > 0$ is the magnitude of the pressure difference between to sections of distance l. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric and fully developed.

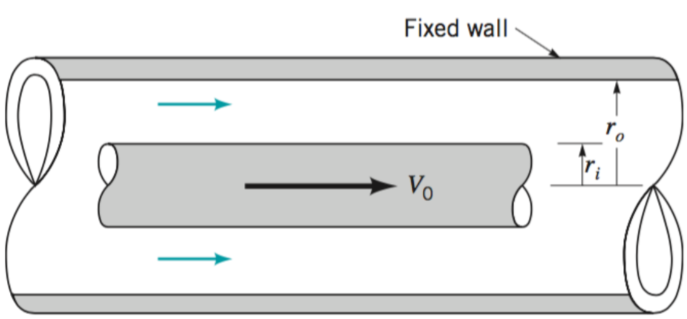

Figure 5: See problem 4.

5 Time-dependent channel flow

A section of a water channel is shown in Figure 6. It is $l = 0.5\,m$ long and $d = 1\,cm$ wide. The water is incompressible and has a dynamic viscosity of $\mu = 10^{-3}\,Ns/m^2$. The fluid is initially at rest but at time $t = 0$, a pump is switched on and the fluid becomes subject to a pressure difference of $\Delta p = p_{in} - p_{out} = 0.01\,kPa$ along the channel in x-direction. Due to *no slip* boundary conditions the velocity at the channel walls is zero: $\mathbf{U}(y = 0, t) = \mathbf{U}(y = d, t) = 0$.

(a) Simplify the Navier-Stokes Equations for the planar velocity $\mathbf{U}(\mathbf{x}, t) = U(y, t)\,\hat{\mathbf{i}}$.

Then, solve the time-dependent problem of the vertical velocity profile $U(y, t)$ by calculating

(b) the steady state solution,

(c) the time-dependent solution for the deviations from the steady state.

Hint: Follow the same solution strategy as in the diffusion exercises in problem set 3 and 4.

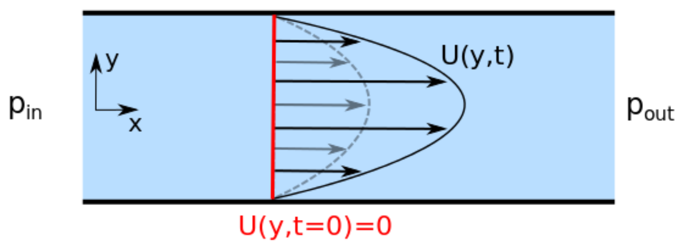

Figure 6: Pressure driven channel flow. See problem 5.