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Problem Set 6: Solutions

1 Circulation

Problem: We want to analyze the circulation, Γ, for a flow in a two-dimensional channel. The velocity profile for the channel flow is

$$u(y) = u_m \left[1 - 4\left(\frac{y}{h}\right)^2\right],$$

where u_m is the maximum velocity, as shown in Figure 1 (note that $y = 0$ is at the center of the channel).

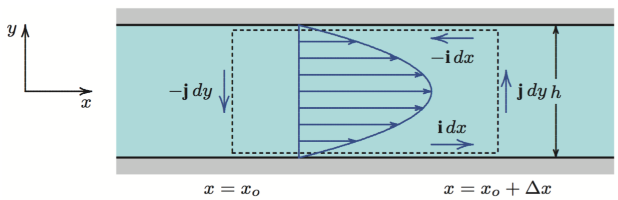

Figure 1: See problem 1.

(a) Calculate the circulation of the flow, Γ, along the dashed rectangular contour shown in Figure 1.
(b) Calculate the circulation of the flow, Γ, along a rectangular contour that extends from the bottom wall of the channel to the centerline (i.e. the bottom half of the dashed contour in Figure 1).
(c) Compute the vorticity ($\omega = \nabla \times \mathbf{u}$) and, noting that $\Gamma = \iint \omega \, dxdy$ where ω is the vorticity component normal to the xy-plane, explain the results of parts (a) and (b).

Solution: **(a)** We evaluate the line integral along the dashed line in the counterclockwise direction indicated in Figure 1:

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{s} = \int_{x_0}^{x_0+\Delta x} \mathbf{u}(x, -h/2) \cdot (\hat{\mathbf{i}} dx) + \int_{-h/2}^{h/2} \mathbf{u}(x_0 + \Delta x, y) \cdot (\hat{\mathbf{j}} dy)$$
$$+ \int_{x_0}^{x_0+\Delta x} \mathbf{u}(x, h/2) \cdot (-\hat{\mathbf{i}} dx) + \int_{-h/2}^{h/2} \mathbf{u}(x_0, y) \cdot (-\hat{\mathbf{j}} dy).$$

The piecewise integrals along the same dimension are combined to

$$\Gamma = \int_{x_0}^{x_0+\Delta x} \left[u(x, -h/2) - u(x, h/2)\right] dx$$
$$+ \int_{-h/2}^{h/2} \left[v(x_0 + \Delta x, y) - v(x_0, y)\right] dy.$$

The first integral is zero because $u(-h/2) = u(h/2) = 0$ and the second integral is zero because $v \equiv 0$. Thus, the circulation is zero:

$$\Gamma = 0.$$

(b) Now, let the closed contour exclude the upper half of the channel so that y ranges from $-h/2$ to 0. We integrate again in counterclockwise direction:

$$\begin{aligned}\Gamma = \oint \mathbf{u} \cdot d\mathbf{s} &= \int_{x_0}^{x_0+\Delta x} \mathbf{u}(x, -h/2) \cdot (\hat{\mathbf{i}} dx) + \int_{-h/2}^{0} \mathbf{u}(x_0 + \Delta x, y) \cdot (\hat{\mathbf{j}} dy) \\ &+ \int_{x_0}^{x_0+\Delta x} \mathbf{u}(x, 0) \cdot (-\hat{\mathbf{i}} dx) + \int_{-h/2}^{0} \mathbf{u}(x_0, y) \cdot (-\hat{\mathbf{j}} dy) \\ &= \int_{x_0}^{x_0+\Delta x} [u(x, -h/2) - u(x, 0)]\, dx + \int_{-h/2}^{0} [v(x_0 + \Delta x, y) - v(x_0, y)]\, dy \\ &= \int_{x_0}^{x_0+\Delta x} \left[u_m \left(1 - 4\frac{1}{4}\right) - u_m (1 - 0)\right] dx + 0 = [-u_m\, x]_{x_0}^{x_0+\Delta x}.\end{aligned}$$

In this case we find a non-zero circulation:

$$\Gamma = -u_m \Delta x.$$

(c) In general, the vorticity for planar flow in the xy-plane is

$$\omega = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \hat{\mathbf{k}} = -\frac{\partial}{\partial y}\left[u_m \left(1 - 4\frac{y^2}{h^2}\right)\right] \hat{\mathbf{k}} = 8\frac{u_m y}{h^2} \hat{\mathbf{k}}.$$

So, the vorticity is positive above the centerline ($\omega > 0$ for $y > 0$) and negative below ($\omega < 0$ for $y < 0$). If the circulation Γ has equal contributions from the upper and the lower part, they cancel. That is,

$$\Gamma = \iint_A \omega \cdot \hat{\mathbf{k}}\, dA = \int_{x_0}^{x_0+\Delta x} \int_{-h/2}^{h/2} 8\frac{u_m y}{h^2}\, dy\, dx = \frac{8 u_m \Delta x}{h^2} \left[\frac{y^2}{2}\right]_{-h/2}^{h/2} = 0.$$

2 Tornado

Problem: A tornado may be modeled as the circulating flow shown in Figure 2, with $v_r = v_z = 0$ and $v_\theta(r)$ such that

$$v_\theta = \begin{cases} \omega r & r \leq R \\ \dfrac{\omega R^2}{r} & r > R \end{cases}$$

Determine whether this flow pattern is irrotational in either the inner or the outer region. Using the r-momentum equation (see the last page of this exercise sheet), determine the pressure distribution $p(r)$ in the tornado, assuming $p = p_\infty$ as $r \to \infty$. Find the location and magnitude of the lowest pressure.

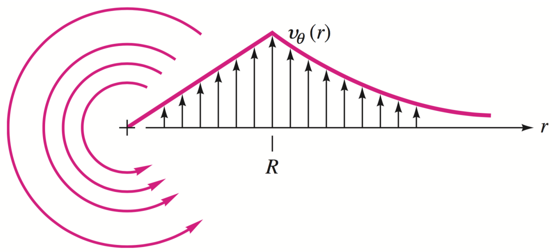

Figure 2: See problem 2.

Solution: We compute the vorticity $\vec{\xi}$ of the given flow using its definition in cylindrical coordinates (see the last page), where we use the short notation $\vec{\xi} = (\xi_r, \xi_\theta, \xi_z) = \xi_r\,\hat{\mathbf{e}}_r + \xi_\theta\,\hat{\mathbf{e}}_\theta + \xi_z\,\hat{\mathbf{e}}_z$:

$$\vec{\xi}(r,\theta,z) = \left(-\frac{\partial v_\theta}{\partial z}, 0, \frac{1}{r}\frac{\partial}{\partial r}(r\,v_\theta)\right).$$

The azimuthal velocity only depends on r and we find for inner and outer region:

$$r \leq R: \qquad \xi_z = \frac{1}{r}\frac{\partial}{\partial r}\omega r^2 = 2\omega = \text{const.}$$
$$r > R: \qquad \xi_z = \frac{1}{r}\frac{\partial}{\partial r}\omega R^2 = 0.$$

Thus, the inner region is **rotational** (*solid body rotation*) and the outer region is **irrotational** (*free vortex*). The pressure is found by integrating the r-momentum equation

$$-\frac{1}{r}v_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{or} \qquad \frac{\partial p}{\partial r} = \frac{\rho\,v_\theta^2}{r}.$$

For the two regions we find

$$r \leq R: \quad p_{inner}(r) = \rho\omega^2 \int r\,dr = \rho\omega^2\frac{r^2}{2} + C_{inner},$$
$$r > R: \quad p_{outer}(r) = \rho\omega^2 \int \frac{R^4}{r^3}\,dr = -\rho\frac{\omega^2 R^4}{2r^2} + C_{outer}.$$

The boundary conditions

$$\text{BC1:} \qquad p_{outer}(r=\infty) = p_\infty,$$
$$\text{BC2:} \qquad p_{inner}(r=R) = p_{outer}(r=R),$$

determine the integration constants

$$C_{outer} = p_\infty,$$
$$C_{inner} = p_\infty - \rho\omega^2 R^2.$$

We find the pressure fields

$$r \leq R: \qquad p_{inner}(r) = p_\infty + \frac{\rho\omega^2}{2}(r^2 - 2R^2),$$

$$r > R: \qquad p_{outer}(r) = p_\infty - \rho\omega^2 \frac{R^4}{2r^2},$$

with a minimum pressure at the origin $r = 0$:

$$\min(p(r)) = \min(p_{inner}(r)) = p(r = 0) = p_\infty - \rho\omega^2 R^2.$$

The pressure minimum at the center makes the "eye" of a tornado, that is a potential sink which sucks in the surrounding air. The strength of this sink depends on the rotation rate ω and the size of the core R.

3 Velocity potential and stream functions

Problem: The velocity field for a 2D flow is given by:

$$\mathbf{u} = C\left[(x^2 - y^2)\,\hat{\mathbf{i}} - 2xy\,\hat{\mathbf{j}}\right],$$

where C is a constant.
(a) Calculate the velocity potential for the flow, $\phi(x, y)$, given the boundary condition $\phi(x = 0, y = 1) = 0$. Is the flow irrotational?
(b) Calculate the stream function for the flow, $\psi(x, y)$, given the boundary condition $\psi(x = 1, y = 0) = 0$. Is the flow incompressible?
(c) If $C = 1$, plot the streamlines and equipotential lines for the flow in the following region: $-10 \le x \le 10$ and $-10 \le y \le 10$.

Solution: **(a)** The flow is irrotational if $curl(\mathbf{u}) = \mathbf{0}$:

$$\nabla \times \mathbf{u} = C\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 - y^2)\right)\hat{\mathbf{k}} = C(-2y + 2y)\hat{\mathbf{k}} = \mathbf{0}.$$

Therefore, we can find a velocity potential ϕ. Recall that $\mathbf{u} = \nabla\phi$ which means

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y},$$

We integrate along x and introduce a y-dependent integration constant

$$\phi(x, y) = \int u\, dx = \int C(x^2 - y^2)\, dx = C\left(\frac{1}{3}x^3 - xy^2\right) + f(y).$$

The function $f(y)$ is determined by differentiating ϕ with respect to y and setting it equal to v:

$$\begin{aligned} \frac{\partial \phi}{\partial y} = v \quad &\Rightarrow \quad -C2xy + f'(y) = -C2xy \\ &\Rightarrow \quad f'(y) = 0 \\ &\Rightarrow \quad f(y) = A, \end{aligned}$$

where A is a constant along x and y and can be determined by the boundary condition

$$\phi(0, 1) = A = 0 \ \Rightarrow \ \phi(x, y) = C\left(\frac{1}{3}x^3 - xy^2\right).$$

(b) The flow is incompressible if $\nabla \cdot \mathbf{u} = 0$:

$$\nabla \cdot \mathbf{u} = C\left(\frac{\partial}{\partial x}(x^2 - y^2) - \frac{\partial}{\partial y}(2xy)\right) = C(2x - 2x) = 0.$$

Therefore, we can find a stream function ψ. Recall that the stream function defines the velocity fields as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

We integrate along y and introduce an x-dependent integration constant

$$\psi(x, y) = \int u\, dy = \int C(x^2 - y^2)\, dy = C\left(x^2 y - \frac{1}{3}y^3\right) + f(x)$$

The function $f(x)$ is determined by differentiating ψ with respect to x and setting it equal to $-v$:

$$\begin{aligned} \frac{\partial \psi}{\partial x} = -v \quad &\Rightarrow \quad C2xy + f'(x) = C2xy \\ &\Rightarrow \quad f'(x) = 0 \\ &\Rightarrow \quad f(x) = B, \end{aligned}$$

where B is a constant along x and y and can be determined by the boundary condition

$$\psi(1,0) = B = 0 \;\Rightarrow\; \psi(x,y) = C\left(x^2 y - \frac{1}{3}y^3\right).$$

(c) Equipotential lines and stream lines are plotted for $C = 1$ in Figure 3. We use the MATLAB function `fcontour()` here, but you can choose other plotting methods. The plot illustrates that the contour lines of ϕ and ψ are always orthogonal on each other.

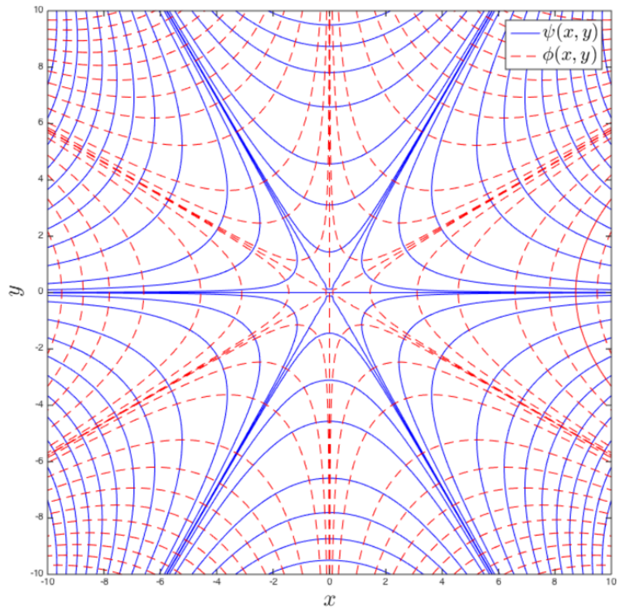

Figure 3: Equipotential lines (constant ϕ) and streamlines (constant ψ). See problem 3c.

Equations of motion in cylindrical coordinates:

The equations of motion of an incompressible Newtonian fluid are given here in cylindrical coordinates:

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

The r-momentum equation:

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{1}{r}v_\theta^2 =$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

The θ-momentum equation:

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{1}{r}v_r v_\theta =$$
$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)$$

The z-momentum equation:

$$\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} =$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

Vorticity:
Using the short notation $\vec{\xi} = (\xi_r, \xi_\theta, \xi_z) = \xi_r\,\hat{\mathbf{e}}_r + \xi_\theta\,\hat{\mathbf{e}}_\theta + \xi_z\,\hat{\mathbf{e}}_z$:

$$\xi(r,\theta,z) = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$$