Problem Set 5: Solutions

1 Vaccination

Problem: A hypodermic syringe (see Figure 1) is used to apply a vaccine. If the plunger is moved forward at a steady rate of $20\,mm/s$ and if vaccine leaks past the plunger at 0.1 of the volume flow rate out of the needle opening, calculate the average velocity of the needle exit flow. The inside diameters of the syringe and the needle are $20\,mm$ and $0.7\,mm$.
Hint: You can use a deforming or a non-deforming control volume to solve this exercise.

Figure 1: See problem 1.

Solution: *Control volume:* If we choose a control volume (CV) which coincides with the entire fluid filled interior of the syringe (see Figure 2), the CV is deforming with the plunger.

Figure 2: See problem 1.

The Reynolds Transport Theorem together with mass conservation is

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{v}\cdot\hat{n}dA = 0.$$

The deformation of the CV gives

$$\rho\frac{\partial}{\partial t}\int_{CV} dV = -\rho\, v_p\, A_s,$$

with $v_p = 20\,mm/s$ being the velocity of the plunger and $A_s = (\pi/4)\,20^2\,mm^2$ being the cross-sectional area of the syringe. The mass flux through the control surface is

$$\int_{CS}\rho\vec{v}\cdot\hat{n}dA = \rho\, Q_{leak} + \rho\, Q_{out},$$

where we know that the leakage and the needle out flow rate relate as $Q_{leak} = 0.1\,Q_{out}$. Both fluxes are positive because the velocity points outward of the CV. Combined we have

$$v_p\, A_s = 0.1\, Q_{out} + Q_{out} = 1.1\, v_{out} A_n,$$

with the needle cross-section area $A_n = (\pi/4)\, 0.7^2\, mm^2$. The exit velocity is

$$v_{out} = \frac{v_p\, A_s}{1.1\, A_n} = \frac{\left(20\, \frac{mm}{s}\right)\left(\frac{\pi}{4} 20^2\, mm^2\right)}{1.1\left(\frac{\pi}{4} 0.7^2\, mm^2\right)}$$
$$= 14.8\, \frac{m}{s}.$$

If one chooses a non-deforming control volume which is not in contact with the plunger (for example the control volume indicated in Figure 3), then $\frac{\partial}{\partial t}\int_{CV} dV = 0$. However, the mass flux through the left surface inside the syringe becomes a combination of leakage and plunger movement

$$\int_{CS} \rho \vec{v} \cdot \hat{n} dA = -\rho\, v_p\, A_s + \rho\, Q_{leak} + \rho\, Q_{out}\ .$$

The following calculation is the same as above.

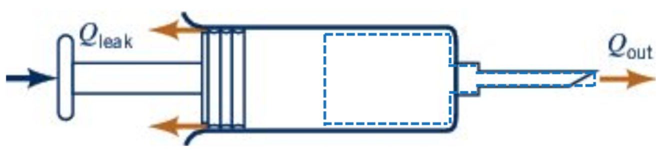

Figure 3: See problem 1.

2 How to anchor a pipe?

Problem: Water flows through a horizontal, 180° pipe bend as is illustrated in Figure 4. The flow cross-sectional area is constant at a value of $9000\, mm^2$. The flow velocity everywhere in the bend is $15\, m/s$. The absolute pressures at the entrance and the exit of the bend are $p_{in} = 210\, kPa$ and $p_{out} = 165\, kPa$, respectively. Calculate the horizontal (x and y) components of the anchoring force needed to hold the bended pipe in place. To obtain the force on the pipe you need to choose your control volume wisely.
Hint: Think about the role of atmospheric pressure $p_{atm} = 101\, kPa$.

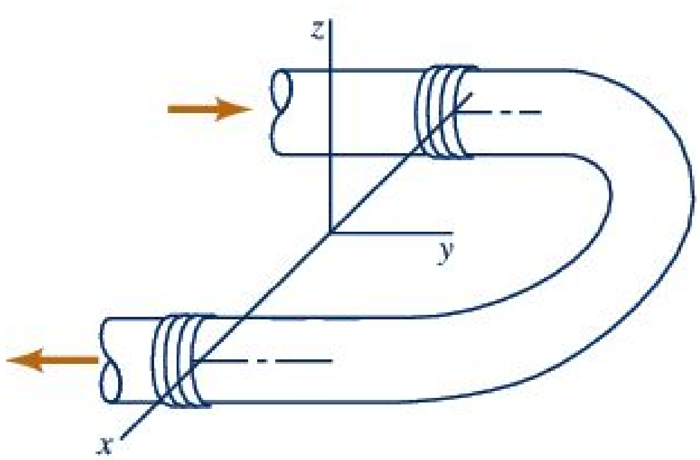

Figure 4: See problem 2.

Solution: This analysis is similar to the one of Example 5.12 of your text book.
Control volume: A fixed, non-deforming control volume that contains the water and the pipe between sections (1) and (2) at an instant is used. This control volume is chosen to find directly the anchoring force F_A. If we

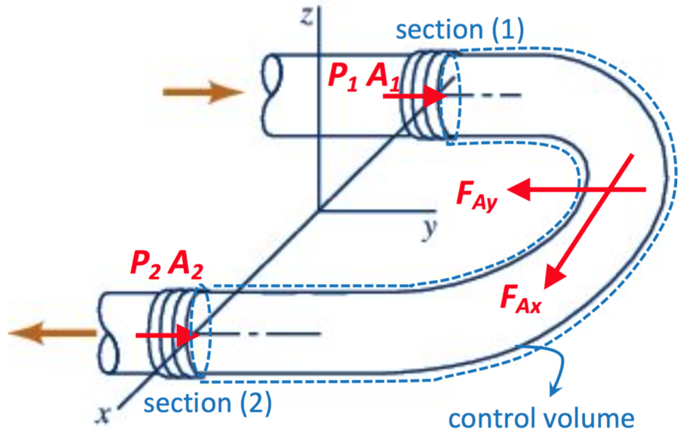

Figure 5: See problem 2.

choose a control volume that only includes the fluid and not the pipe, then the net forces on the fluid will be found, and in order to find the anchoring force we need to apply Newton's second law on the pipe separately. Therefore, using a bigger control volume that includes both the fluid and the pipe simplifies the analysis.

Forces: The horizontal forces acting on the contents of the control volume in the x and y directions are shown in Figure 5. Note that P_1 and P_2 refer to gauge pressures at the inlet and outlet and not to the absolute pressures given. Atmospheric pressure acts from all sides on the control surface so that the forces cancel. Thus, $P_1 = P_{in} - P_{atm}$ and $P_2 = P_{out} - P_{atm}$.

The x-direction component of the linear momentum equation for a steady state flow is

$$\int_{CS} u\rho\vec{V} \cdot \hat{n} dA = F_{Ax}.$$

At the sections (1) and (2) the flow is in the y direction and therefore $u = 0$ at both cross sections and

$$F_{Ax} = 0.$$

From the y-direction component of the linear momentum equation for a steady state flow, we get

$$\int_{CS} v\rho\vec{V} \cdot \hat{n} dA = -F_{Ay} + P_1 A_1 + P_2 A_2,$$

with the pressure forces discussed above.
For one-dimensional flow, the surface integral in the above equation is easy to evaluate and this equation becomes

$$-\dot{m}_1 v_1 + \dot{m}_2(-v_2) = P_1 A_1 - F_{Ay} + P_2 A_2$$
$$\rightarrow -v_1 \rho v_1 A_1 - v_2 \rho v_2 A_2 = P_1 A_1 - F_{Ay} + P_2 A_2.$$

From mass conservation we know that $\dot{m}_1 = \dot{m}_2$ or $A_1 v_1 = A_2 v_2$, so

$$F_{Ay} = \rho A_1 v_1 (v_1 + v_2) + P_1 A_1 + P_2 A_2$$

Thus,

$$F_{Ay} = \left(999\,\frac{kg}{m^3}\right)\frac{(9000\,mm^2)}{\left(1000\,\dfrac{mm}{m}\right)^2}\left(15\,\frac{m}{s}\right)\left(15\,\frac{m}{s}+15\,\frac{m}{s}\right)\left(1\,\frac{N}{kg\cdot m/s^2}\right)$$
$$+\frac{(210\,kPa-101\,kPa)(9000\,mm^2)}{\left(1000\,\dfrac{mm}{m}\right)^2\left(\dfrac{1}{1000}\,\dfrac{N}{m^2\cdot kPa}\right)}+\frac{(165\,kPa-101\,kPa)(9000\,mm^2)}{\left(1000\,\dfrac{mm}{m}\right)^2\left(\dfrac{1}{1000}\,\dfrac{N}{m^2\cdot kPa}\right)}$$
$$= 5603\,N.$$

3 How to anchor a plate?

Problem: A horizontal circular jet of air strikes a stationary flat plate as indicated in Figure 6. The jet velocity is $40\,m/s$ and the jet diameter is $30\,mm$. If the air velocity magnitude remains constant as the air flows over the plate surface in the directions shown, determine:

(a) the magnitude of F_A, which is the anchoring force required to hold the plate stationary.

(b) the fraction of mass flow along the plate surface in each of the two directions shown.

(c) the magnitude of F_A, the anchoring force required to allow the plate to move to the right at a constant speed of $c = 10\,m/s$.

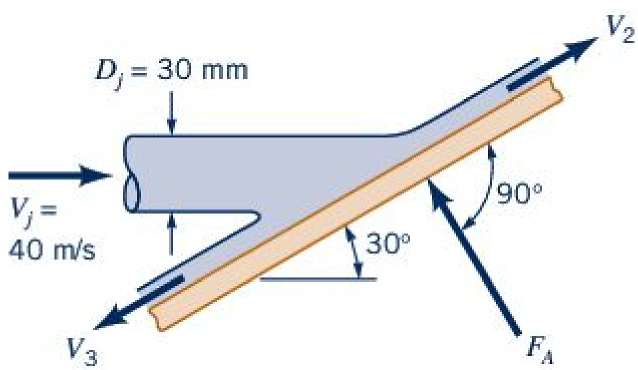

Figure 6: See problem 3.

Solution: *Control volume:* The non-deforming control volume shown in Figure 7 is used.

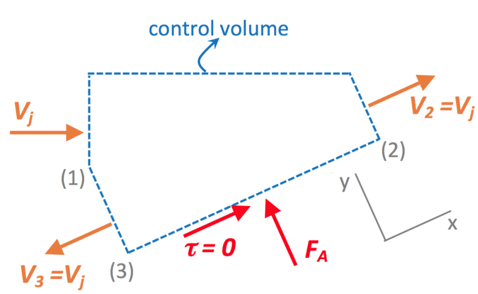

Figure 7: See problem 3.

Forces: The forces on the control volume are shown in the sketch. The system is steady and the energy of the inflow is equal to the energy of the outflow, because the inflow and the outflow have the same velocities, pressures and heights. Therefore, there is no energy loss in the system, or in other words, the flow is assumed to be inviscid which makes the wall friction force zero.

(a) To determine the magnitude of F_A we apply the component of the linear momentum equation along the direction of F_A, $\int_{CS} v\rho\vec{V}\cdot\hat{n}dA = \sum F_y$. At section (1), $v = V_j \sin 30°$ and at sections (2) and (3) velocity along y is zero, so the surface integral gives a non-zero result only at section (1). Thus, the momentum equation in y direction is simplified to

$$F_A = \dot{m}_j V_j \sin 30° = \rho A_j V_j V_j \sin 30° = \rho \frac{\pi D_j^2}{4} V_j^2 \sin 30°$$
$$= \left(1.23\,\frac{kg}{m^3}\right) \frac{\pi(0.030\,m)^2}{4} \left(40\,\frac{m}{s}\right)^2 (\sin 30°) \left(1 \frac{N}{kg\cdot m/s^2}\right) = 0.696\,N.$$

Note that a choice of coordinates along the inflow would have made the calculation more complicated.

(b) To determine the fraction of mass flow along the plate surface in each of the two directions shown in the sketch above, we apply the component of the linear momentum equation parallel to the surface of the plate, $\int_{CS} u\rho\vec{V}\cdot\hat{n}dA = \sum F_x$, to obtain

$$R_{\text{along plate surface}} = \dot{m}_2 V_2 - \dot{m}_3 V_3 - \dot{m}_j V_j \cos 30°.$$

$R_{\text{along plate surface}}$ is zero, because the flow is inviscid. Thus, we obtain

$$\dot{m}_3 V_3 = \dot{m}_2 V_2 - \dot{m}_j V_j \cos 30°.$$

Since $V_3 = V_2 = V_j$, the above equation becomes

$$\dot{m}_3 = \dot{m}_2 - \dot{m}_j \cos 30°.$$

From conservation of mass we conclude that $\dot{m}_j = \dot{m}_2 + \dot{m}_3$, so

$$\dot{m}_3 = \dot{m}_j - \dot{m}_3 - \dot{m}_j \cos 30°$$
$$\rightarrow \dot{m}_3 = \dot{m}_j \frac{1-\cos 30°}{2} = \dot{m}_j(0.0670),$$

and

$$\dot{m}_2 = \dot{m}_j - \dot{m}_3 = \dot{m}_j(1-0.0670) = \dot{m}_j(0.933)$$

Thus, $\dot{m}_2$ contains 93.3% of $\dot{m}_j$, and $\dot{m}_3$ contains 6.7% of $\dot{m}_j$.

(c) To determine the magnitude of F_A required to allow the plate to move to the right at a constant speed of $10\,m/s$, we use a non-deforming control volume like the one in the sketch above that moves to the right with a speed of $10\,m/s$. The translation velocity of the control volume enters the linear momentum equation for a steady flow as $\int_{CS} W_y \rho \mathbf{W}\cdot\hat{\mathbf{n}}dA = \sum F_y$ where $\mathbf{W}$ is the velocity field relative to the control volume. This equation leads to

$$F_A = \dot{m}_1 w_1 \sin 30° = \left(\rho \frac{\pi D_j^2}{4} w_1\right) w_1 \sin 30° = \rho \frac{\pi D_j^2}{4} w_1^2 \sin 30°$$
$$= \left(1.23\,\frac{kg}{m^3}\right) \frac{\pi(0.030\,m)^2}{4} \left(40\,\frac{m}{s} - 10\,\frac{m}{s}\right)^2 (\sin 30°) \left(1\,\frac{N}{kg\cdot m/s^2}\right) = 0.391\,N.$$

4 Jet ski

Problem: The thrust developed to propel the jet ski shown in Figure 8 is a result of water pumped through the vehicle and exiting as a high-speed water jet. For the conditions shown in the figure, what flow rate is needed to produce a $1.3\,kN$ thrust? Assume the inlet and outlet jets of water are free jets.

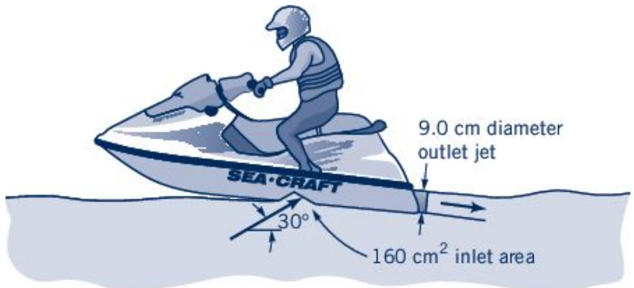

Figure 8: See problem 4.

Solution: *Control volume:* We use the control volume shown in Figure 9, that has only one inlet and one outlet. The flow is steady.

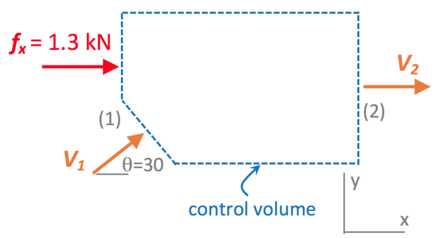

Figure 9: See problem 4.

Force: The only force on the control volume is the force that should be balanced with the thrust.

For this control volume, the x-component of the momentum equation $\int_{CS} u\rho\vec{V}\cdot\hat{n}dA = \sum F_x$ becomes

$$(V_1\cos 30°)\rho(-V_1)A_1 + V_2\rho(+V_2)A_2 = f_x,$$

where we have assumed that $P = 0$ on the entire control volume (*free jet* assumption) and that the exiting water jet is horizontal. With $\dot{m} = \rho A_1 V_1 = \rho A_2 V_2$, the above equation becomes

$$f_x = \dot{m}(V_2 - V_1\cos 30°) = \rho V_1 A_1(V_2 - V_1\cos 30°). \tag{1}$$

Also, due to the conservation of mass $A_1V_1 = A_2V_2$, so

$$V_2 = \frac{A_1}{A_2}V_1 = \frac{0.016\,m^2}{\frac{\pi}{4}(0.09\,m)^2}V_1 = 2.5V_1. \tag{2}$$

By combining equations 1 and 2

$$f_x = \rho V_1^2 A_1(2.5 - \cos 30°),$$

$$\rightarrow V_1 = \left[\frac{1.3\,kN}{\left(1001.21\,\frac{kg}{m^3}\right)(0.016\,m^2)(2.5-\cos 30°)}\right]^{1/2} = 7\,\frac{m}{s}.$$

Thus,

$$Q = A_1V_1 = (0.016\,m^2)\left(7\,\frac{m}{s}\right) = 0.112\,\frac{m^3}{s}.$$

5 Snowplow

Problem: A snowplow mounted on a truck clears a path of $3.6\,m$ width through heavy wet snow, as shown in Figure 10. The snow is $20\,cm$ deep and its density is $160\,kg/m^3$. The truck travels at $48\,km/h$. The snow is discharged from the plow at an angle of 45° above the horizontal, as shown in Figure 10. Estimate the force required to push the plow. Note that the problem is two-dimensional.

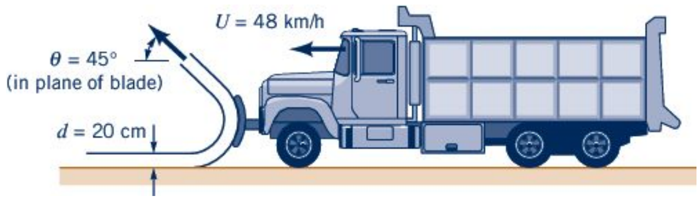

Figure 10: See problem 5.

Solution: *Control volume:* The control volume is chosen such that it includes the snowplow and moves with it. It has only one inlet and one outlet and the relative velocity of the wet snow is normal to the inlet and outlet surfaces (see Figure 11).

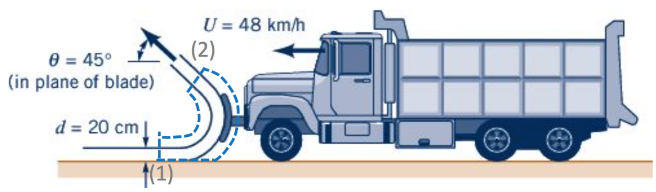

Figure 11: See problem 5.

Force: Force in this problem is unknown.

To estimate the force required to push the snowplow we use the momentum equation for an inertial, moving control volume that involves steady flow:

$$\int_{CS} \mathbf{W}\rho\mathbf{W}\cdot\hat{\mathbf{n}}dA = \sum F_{\text{contents of the control volume}}.$$

We neglect the friction force between the plow and the road surface. We also neglect any force associated with the plow deflecting air. We only consider how much force is required to turn wet snow 135°.
Since there is only one inlet and one outlet and the velocity is uniform across the flow, we get from the momentum equation:

$$F_x = \dot{m}(w_1 + w_2\cos 45°).$$

Since,

$$\dot{m} = \rho A w,$$

we assume $w_2 = w_1$, and get

$$F_x = \rho A w_1^2(1+\cos 45°),$$

then

$$F_x = \left(160\,\frac{kg}{m^3}\right)(0.20\,m)(3.6\,m)\left(48\,\frac{km}{h}\times\frac{1000\,m}{1\,km}\times\frac{1\,h}{3600\,s}\right)^2(1+0.707) = 34959\,N.$$

6 Water power

Problem: Water is to be moved from one large reservoir to another at a higher elevation as indicated in Figure 12. The loss of available energy associated with $Q = 0.07\,m^3/s$ pumped from section (1) to (2) is loss $l = \left(61\,\bar{v}^2/2\right)\,m^2/s^2$, where $\bar{v}$ is the average velocity of water in the 20 cm inside diameter piping involved.

(a) Select a control volume which allows you to simplify the general energy equation (RTT with first law of thermodynamics) to a sum over two surfaces which is valid for the *steady-in-the-mean flow* of this exercise. State all assumptions which you apply to arrive at a 1-D equation.

(b) Determine the amount of shaft power in the pump required to pump the water from (1) to (2).

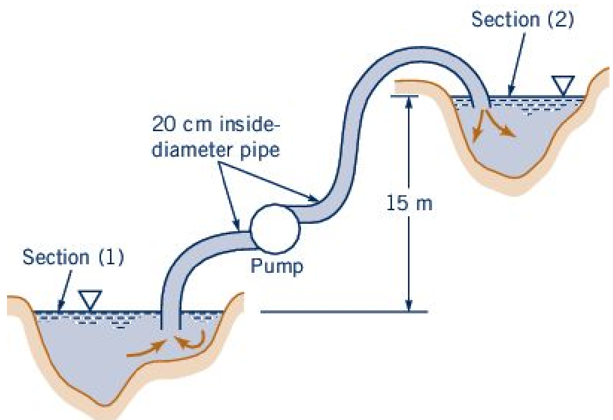

Figure 12: See problem 6.

Solution: (a) The control volume is chosen such that it contains all the fluid including the water in the reservoirs, the tube and the pump. This control volume contains one inlet, which is the water level of the lower reservoir, and one outlet, which is the water level of the upper reservoir. By the following assumptions, 1D energy equation can be used to solve this problem:

- The flow is steady-in-the-mean (since there is a pump in the system the flow cannot be steady but it can be cyclical and steady when averaged over a period, and therefore it is called a steady-in-the-mean flow). Although the water is continuously being pumped from the lower reservoir to the upper one and it makes the water levels change in both the reservoirs, but this change is negligible, because reservoirs are large (quasi-steady assumption).
- Flow is one dimensional; it means that pressure, velocity, and height are uniformly distributed along the inlet or the outlet of the control volume which seems to be the case, because the inlet and the outlet of the control volume are the water levels of the large reservoirs where pressure is atmospheric, velocity is assumed to be zero and height is essentially constant.

(b) The 1D energy equation is:

$$\frac{p_{out}}{\rho} + \frac{V_{out}^2}{2} + gz_{out} = \frac{p_{in}}{\rho} + \frac{V_{in}^2}{2} + gz_{in} + w_{\text{shaft net in}} - l.$$

For the flow from section (1) to (2) the above equation leads to:

$$\begin{aligned} 0 + 0 + gz_2 &= 0 + 0 + gz_1 + w_{\text{shaft net in}} - l \\ \rightarrow w_{\text{shaft net in}} &= g(z_2 - z_1) + l, \end{aligned}$$

and

$$\dot{W}_{\text{shaft net in}} = \dot{m}w_{\text{shaft net in}} = \rho Q\left[g(z_2 - z_1) + l\right] = \rho Q\left[g(z_2 - z_1) + \frac{61\,\bar{v}^2}{2}\right].$$

From the volume flow rate we obtain

$$\bar{v} = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}D^2} = \frac{0.07\,\frac{m^3}{s}}{\frac{\pi}{4}(0.2\,m)^2} = 2.2\,\frac{m}{s}.$$

Thus,

$$\dot{W}_{\text{shaft net in}} = \left(999\,\frac{kg}{m^3}\right)\left(0.07\,\frac{m^3}{s}\right)\left[\left(9.81\,\frac{m}{s^2}\right)(15\,m) + \frac{61}{2}\left(2.2\,\frac{m}{s}\right)^2\right] = 20613.3\,\frac{N\cdot m}{s} = {\color{red}20.6\,kW}.$$

7 Boundary layer flow

Problem: The flow of a viscous fluid over a plate results in the development of a region of reduced velocity adjacent to the wetted flat surface as depicted in Figure 13. The region of reduced flow velocity is called *boundary layer*. Inside the boundary layer the viscous force is significant but outside of it the flow remains inviscid. At the leading edge of the plate, the velocity profile may be considered uniformly distributed with a value U_∞. All along the outer edge of the boundary layer, the fluid velocity component parallel to the plate is also U_∞.

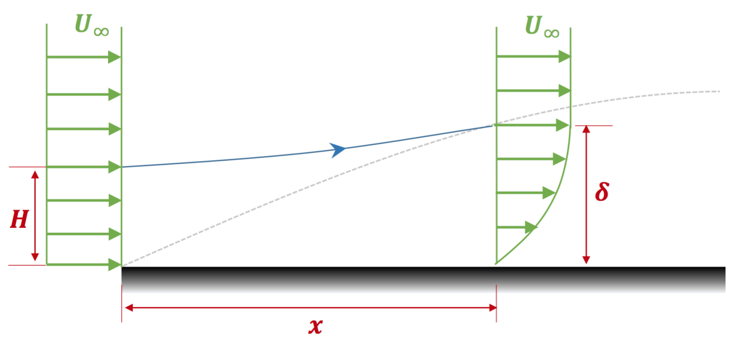

Figure 13: See problem 7.

(a) Approximate the x-direction velocity profile $u(y, \delta(x))$ inside the boundary layer by a second order polynomial. Here, y is the plate normal direction. Determine the polynomial by considering the following constraints:

- The velocity at the wall is zero.
- The velocity at $y = \delta$ is U_∞.
- The shear at $y = \delta$ is zero.

(b) A streamline that is initially located at height H, after a length x, intersects the boundary layer tip at height δ. Using mass conservation for an appropriate control volume of unit length along the cross stream direction z, find a relation between H and $\delta(x)$.

(c) Assuming that the only force acting on the flow is wall friction (we assume p = const.), write the momentum equation for the same control volume to find an integral equation for $\delta(x)$.
Hint: Use the relation of (b) to eliminate H.

(d) Differentiate the equation to find an ODE of the form $\delta' = f(x, \delta)$, and solve it by using an appropriate boundary condition.

Solution: (a) The second order polynomial to approximate the velocity profile is $u(y, \delta) = ay^2 + by + c$ with three unknowns that should be found by the constraints. Thus,

$$\begin{aligned} u(0) &= 0 \to c = 0, \\ u(\delta) &= U_\infty \to a(\delta)^2 + b(\delta) = U_\infty, \\ u'(\delta) &= 0 \to 2a(\delta) + b = 0. \end{aligned}$$

Solution of this system of equations gives

$$\begin{aligned} a &= -\frac{U_\infty}{\delta^2}, \\ b &= 2\frac{U_\infty}{\delta}, \\ c &= 0. \end{aligned}$$

Thus,

$$u(y, \delta) = U_\infty \left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \right).$$

(b) We use the control volume indicated in Figure 14 to solve this problem. The upper edge of the control volume is along the streamline, so there is no flux through this edge.

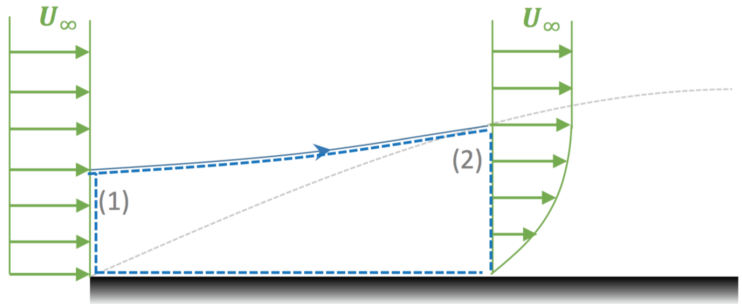

Figure 14: See problem 7.

The mass conservation equation for a non-deforming stationary control volume is

$$\frac{\partial}{\partial t} \int_{CV} \rho \, dV + \int_{CS} \rho \vec{v} \cdot \hat{n} dA = 0.$$

The flow is steady state so the first integral becomes zero. The second integral becomes zero at the wall and the upper edge of the control volume. So the above equation is simplified to two simple integrals at the sections (1) and (2):

$$\begin{aligned} &-U_\infty H + \int_0^\delta u(y) dy = 0 \\ &\to -U_\infty H + \int_0^\delta U_\infty \left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \right) dy = 0 \\ &\to -U_\infty H + \frac{2}{3} U_\infty \delta = 0 \\ &\to H = \frac{2}{3}\delta. \end{aligned}$$

(c) The linear momentum equation for a stationary control volume in direction of the main flow, x, is

$$\frac{\partial}{\partial t}\int_{CV}\rho u dV + \int_{CS}\rho u \vec{v}\cdot\hat{n} dA = \sum F_x.$$

There is no time dependence in the problem and the surface integral gives non-zero results only at the sections (1) and (2). The only force acting on the control volume is the wall shear force that is a function of x; so the total force is the integral of the wall shear along the wall. Thus, the above equation leads to

$$-\rho U_\infty^2 H + \int_0^\delta \rho u^2 dy = -\int_0^x \tau_w dx = -\int_0^x \mu \left.\frac{du}{dy}\right|_{y=0} dx.$$

Inserting the results of part (a) and (b) into the above equation gives

$$\begin{aligned}
-\frac{2}{3}\rho U_\infty^2 \delta + \int_0^\delta \rho U_\infty^2 \left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)^2 dy &= -\int_0^x \mu \frac{d}{dy}\left[U_\infty \left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\right]_{y=0} dx \\
\rightarrow -\frac{2}{3}\rho U_\infty^2 \delta + \frac{8}{15}\rho U_\infty^2 \delta &= -\int_0^x \frac{2\mu U_\infty}{\delta} dx \\
\rightarrow \frac{2}{15}\rho U_\infty^2 \delta &= \int_0^x \frac{2\mu U_\infty}{\delta} dx.
\end{aligned}$$

After simplification and reordering we obtain

$$\delta = \frac{15\mu}{\rho U_\infty}\int_0^x \frac{1}{\delta} dx.$$

(d) Differentiation of the integral equation, obtained in part (c), with respect to x gives the ordinary differential equation for the boundary layer thickness as a function of x:

$$\begin{aligned}
\frac{d\delta}{dx} &= \frac{15\mu}{\rho U_\infty}\frac{1}{\delta} \\
\rightarrow \delta\frac{d\delta}{dx} &= \frac{15\mu}{\rho U_\infty}.
\end{aligned}$$

This ode is solved in the following way

$$\begin{aligned}
\delta d\delta &= \frac{15\mu}{\rho U_\infty} dx \\
\rightarrow \int \delta d\delta &= \int \frac{15\mu}{\rho U_\infty} dx \\
\rightarrow \frac{1}{2}\delta^2 &= \frac{15\mu}{\rho U_\infty} x + C.
\end{aligned}$$

The height of the boundary layer is zero at $x = 0$. Hence,

$$\delta(x = 0) = 0 \rightarrow C = 0.$$

and

$$\delta(x) = \sqrt{\frac{30\mu x}{\rho U_\infty}}.$$

Therefore, the boundary layer thickness grows with $\sqrt{x}$.