ME-280: Fluid Mechanics
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Problem Set 4: Solutions

1 Surface reaction

A beaker is filled with a chemical liquid which starts to react strongly with oxygen at the surface (Figure 1). The reaction product diffuses from the surface into the liquid with a diffusion constant D. The concentration of the reaction product $C(z,t)$ depends only on height z and time t. At $t = 0$, the concentration in the liquid is zero: $C(z,t=0) = 0$. At the surface, the reaction creates a constant concentration of $C(z = H,t) = C_s$. Solve the time-dependent problem of the vertical concentration profile $C(z,t)$ by calculating first **the steady state solution (a)** and second **the time-dependent solution (b)**. Follow the steps below:

(a1) Boundary conditions: What is the boundary condition at $z = 0$ (bottom)?

(a2) Steady state: Calculate the steady state solution $\bar{C}(z)$.

(b1) Homogeneous problem: Consider the decomposition of the concentration profile into the steady state solution $\bar{C}(z)$ and the time-dependent deviations $\tilde{c}(z,t)$ around the equilibrium:

$$C(z,t) = \bar{C}(z) + \tilde{c}(z,t).$$

Insert this decomposition into the diffusion equation. State the partial differential equation (PDE) of the deviation $\tilde{c}(z,t)$ and define the boundary conditions for $\tilde{c}$ at $z = 0$ and $z = H$. Is the PDE linear and homogeneous? Discuss the difference between the boundary conditions of $C(z,t)$ and $\tilde{c}(z,t)$.

(b2) General solution: You want to find the time-dependent deviations from the steady state with the general solution ansatz

$$\tilde{c}(z,t) = \sum_{n=1}^{\infty} A_n \varphi_n(z,t).$$

Calculate the base solutions $\varphi_n(z,t) = Z_n(z)T_n(t)$ using the method of separation of variables. State the general solution for $C(z,t)$. (Hint: Choose your solution ansatz such that it does not become imaginary! Use a series of sines and cosines for the spatial problem.)

(b3) Solution satisfying initial condition: Having found a general solution for the problem, remember that, so far, it only satisfies the boundary conditions of the concentration profile. Determine the set of coefficients A_n for general initial conditions of the deviation $\tilde{c}(z,t=0) = \tilde{c}_0(z)$ by using the "Fourier-Trick". Insert the given initial conditions and evaluate the integrals to obtain the final expression for the time-dependent solution of the total concentration profile $C(z,t)$.

Solution: (a1) Boundary conditions: Since at the bottom of the beaker there is a wall and in the time scale of this problem nothing can diffuse into the wall, there is no flux at $z = 0$ which means:

$$j(z=0,t) = 0 \rightarrow -D\frac{\partial C}{\partial z}(z=0,t) = 0$$
$$\rightarrow \frac{\partial C}{\partial z}(z=0,t) = 0.$$

(a2) Steady state: The steady state solution of

$$\frac{\partial \bar{C}}{\partial t} = D\frac{\partial^2 \bar{C}}{\partial z^2},$$

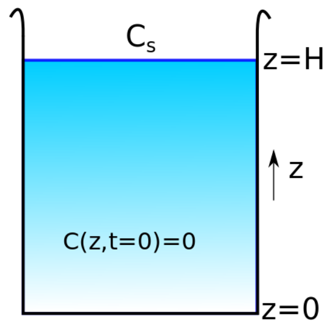

Figure 1: Beaker filled with a substance which reacts at the surface. See problem 1.

does not depend on time, so

$$0 = D\frac{\partial^2 \bar{C}}{\partial z^2} \to \frac{d^2\bar{C}}{dz^2} = 0 \to \bar{C} = \alpha_1 z + \alpha_2,$$

with the boundary conditions

$$\frac{d\bar{C}}{dz}(0) = 0 \to \alpha_1 = 0,$$
$$\bar{C}(H) = C_s \to \alpha_2 = C_s.$$

Inserting the coefficients into the general form, the steady state solution of this problem is

$$\bar{C}(z) = C_s.$$

This means that after damping unsteady parts of a solution the concentration will be constant and equal to C_s in the whole beaker.

(b1) Homogeneous problem: Applying the decomposition in the diffusion equation gives

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial z^2}$$
$$\to \frac{\partial(\bar{C}+\tilde{c})}{\partial t} = D\frac{\partial^2(\bar{C}+\tilde{c})}{\partial z^2}$$
$$\to \frac{\partial \bar{C}}{\partial t} + \frac{\partial \tilde{c}}{\partial t} = D\frac{\partial^2 \bar{C}}{\partial z^2} + D\frac{\partial^2 \tilde{c}}{\partial z^2}.$$

As shown in **(a2)**, the steady state solution satisfies the diffusion equation ($\partial\bar{C}/\partial t = D\,\partial^2\bar{C}/\partial z^2$), so the terms related to this part can be canceled which gives

$$\frac{\partial \tilde{c}}{\partial t} = D\frac{\partial^2 \tilde{c}}{\partial z^2},$$

which is the PDE that should be solved to find deviation from the steady state. In other words, this equation governs the unsteady part of the total solution. To find the boundary conditions for the unsteady part we perform the same decomposition on the boundary conditions:

$$\frac{\partial(\bar{C}+\tilde{c})}{\partial z}(z=0,t) = 0 \to \frac{\partial \bar{C}}{\partial z}(z=0) + \frac{\partial \tilde{c}}{\partial z}(z=0,t) = 0,$$
$$(\bar{C}+\tilde{c})(z=H,t) = C_s \to \bar{C}(z=H) + \tilde{c}(z=H,t) = C_s.$$

From part (a) we know that the steady part is a constant, $\bar{C}(z) = C_s$, thus

$$\frac{\partial \bar{C}}{\partial z}(z=0) + \frac{\partial \tilde{c}}{\partial z}(z=0,t) = 0 \to \frac{\partial \tilde{c}}{\partial z}(z=0,t) = 0,$$
$$\bar{C}(z=H) + \tilde{c}(z=H,t) = C_s \to \tilde{c}(z=H,t) = 0.$$

Therefore, the unsteady part of the solution should also satisfy the diffusion equation that is a linear PDE. The boundary conditions for the unsteady part are such that the PDE to find this solution is homogeneous even though the initial PDE to find the total solution is not homogeneous. It means that by decomposition of the total concentration into the steady and unsteady parts, a linear and homogeneous PDE is obtained that can be solved by the method of separation of variables.

(b2) General solution: The PDE obtained in part (b1) is linear and homogeneous, and is subject to homogeneous boundary conditions. Hence, it is possible to express its solution as a linear combination of the so-called base functions that are themselves solutions of the same PDE with the same boundary conditions:

$$\frac{\partial \varphi_n}{\partial t} = D\frac{\partial^2 \varphi_n}{\partial z^2},$$
$$\frac{\partial \varphi_n}{\partial z}(z=0,t) = 0,$$
$$\varphi_n(z=H,t) = 0.$$

Using the method of separation of variables, we assume that every base function is a product of a function that only depends on z and a function that only depends on t (then we try to find such solutions; if we succeed it means that this assumption is true):

$$\varphi_n(z,t) = Z_n(z)T_n(t).$$

By putting this assumption into the diffusion equation:

$$\frac{\partial (Z_n(z)T_n(t))}{\partial t} = D\frac{\partial^2 (Z_n(z)T_n(t))}{\partial z^2}$$
$$\to Z_n\frac{dT_n}{dt} = DT_n\frac{d^2 Z_n}{dz^2}$$
$$\to Z_n T_n^{'} = DT_n Z_n^{''}$$
$$\to \frac{1}{D}\frac{T_n^{'}}{T_n} = \frac{Z_n^{''}}{Z_n}.$$

The left hand side of the above relation is only a function of t while the right hand side is only a function of z; it is not possible unless both sides are constants. We call this constant $-\lambda_n$; the minus sign is only for convenience. Thus,

$$\frac{1}{D}\frac{T_n^{'}}{T_n} = \frac{Z_n^{''}}{Z_n} = -\lambda_n \to \begin{cases} T_n^{'} + D\lambda_n T_n = 0, \\ Z_n^{''} + \lambda_n Z_n = 0. \end{cases}$$

Thus, the separation ansatz converted one PDE into two ordinary differential equations (ODE) which can be solved for $Z_n(z)$ and $T_n(t)$ independently. The ODE for T_n, which is first order in time, is simply solved to get

$$T_n(t) = \exp(-D\lambda_n t).$$

The ODE for Z_n, which is second order in space, depends on the boundary conditions and the sign of λ_n. Putting the boundary conditions into the separation of variables assumption, $\varphi_n(z,t) = Z_n(z)T_n(t)$, we obtain

$$\frac{\partial (Z_n T_n)}{\partial z}(z=0,t) = 0 \to Z_n^{'}(0)T_n(t) = 0 \to Z_n^{'}(0) = 0,$$
$$(Z_n T_n)(z=H,t) = 0 \to Z_n(H)T_n(t) = 0 \to Z_n(H) = 0.$$

To solve the ODE for $Z_n(z)$ with above boundary conditions, we consider three cases:
Case i: $\lambda_n < 0$

$$\begin{aligned}
Z_n(z) &= \alpha_1 \cosh(\sqrt{-\lambda_n} z) + \alpha_2 \sinh(\sqrt{-\lambda_n} z) \\
Z_n'(z) &= \alpha_1 \sqrt{-\lambda_n} \sinh(\sqrt{-\lambda_n} z) + \alpha_2 \sqrt{-\lambda_n} \cosh(\sqrt{-\lambda_n} z) \\
Z_n'(0) &= 0 \rightarrow \alpha_1 \sqrt{-\lambda_n} \cdot (0) + \alpha_2 \sqrt{-\lambda_n} \cdot (1) = 0 \rightarrow \alpha_2 = 0 \\
Z_n(H) &= 0 \rightarrow \alpha_1 \cosh(\sqrt{-\lambda_n} H) = 0 \rightarrow \alpha_1 = 0 \\
&\rightarrow Z_n(z) = 0.
\end{aligned}$$

Thus in this case, no solution exists except the trivial solution, $\varphi_n = Z_n T_n = 0$.
Case ii: $\lambda_n = 0$

$$\begin{aligned}
Z_n(z) &= \alpha_1 z + \alpha_2 \\
Z_n'(z) &= \alpha_1 \\
Z_n'(0) &= 0 \rightarrow \alpha_1 = 0 \\
Z_n(H) &= 0 \rightarrow \alpha_2 = 0 \\
&\rightarrow Z_n(z) = 0.
\end{aligned}$$

In this case also only the trivial solution, $\varphi_n = Z_n T_n = 0$, exists.
Case iii: $\lambda_n > 0$

$$\begin{aligned}
Z_n(z) &= \alpha_1 \cos(\sqrt{\lambda_n} z) + \alpha_2 \sin(\sqrt{\lambda_n} z) \\
Z_n'(z) &= -\alpha_1 \sqrt{\lambda_n} \sin(\sqrt{\lambda_n} z) + \alpha_2 \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} z) \\
Z_n'(0) &= 0 \rightarrow -\alpha_1 \sqrt{\lambda_n} \cdot (0) + \alpha_2 \sqrt{\lambda_n} \cdot (1) = 0 \rightarrow \alpha_2 = 0 \\
Z_n(H) &= 0 \rightarrow \alpha_1 \cos(\sqrt{\lambda_n} H) = 0
\end{aligned}$$

For a nontrivial solution $\cos(\sqrt{\lambda_n} H)$ should be zero and not α_1, so

$$\begin{aligned}
\sqrt{\lambda_n} &= \frac{2n-1}{2H}\pi \quad n = 1, 2, 3, ... \\
\rightarrow Z_n &= \cos(\sqrt{\lambda_n} z), \quad \lambda_n = \left(\frac{2n-1}{2H}\pi\right)^2 \quad n = 1, 2, 3, ...
\end{aligned}$$

Thus, the base functions are

$$\varphi_n(z,t) = Z_n(z) T_n(t) = \cos(\sqrt{\lambda_n} z) \exp(-D\lambda_n t), \quad \lambda_n = \left(\frac{2n-1}{2H}\pi\right)^2 \quad n = 1, 2, 3, ...$$

and the unsteady solution, that is a linear combination of the base functions, becomes

$$\tilde{c}(z,t) = \sum_{n=1}^{\infty} A_n \varphi_n(z,t) = \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n} z) \exp(-D\lambda_n t), \quad \lambda_n = \left(\frac{2n-1}{2H}\pi\right)^2$$

(b3) Solution satisfying initial condition: First of all we need to find the initial condition of the unsteady part, $\tilde{c}$:

$$\begin{aligned}
C(z,0) = 0 \rightarrow \bar{C}(z) + \tilde{c}(z,0) = 0, \\
\bar{C}(z) = C_s \rightarrow \tilde{c}(z,0) = -C_s.
\end{aligned}$$

Setting $t = 0$ in the solution ansatz obtained in part (b2) and using the initial condition, we get

$$\tilde{c}(z,0)=\sum_{n=1}^{\infty}A_n\cos(\sqrt{\lambda_n}z)\exp(-D\lambda_n\cdot 0)=-C_s\rightarrow\sum_{n=1}^{\infty}A_n\cos(\sqrt{\lambda_n}z)=-C_s.$$

To obtain A_n, we multiply the whole expression by $\cos(\sqrt{\lambda_m}z)$ and integrate from 0 to H:

$$\int_0^H\cos(\sqrt{\lambda_m}z)\sum_{n=1}^{\infty}A_n\cos(\sqrt{\lambda_n}z)dz=\int_0^H\cos(\sqrt{\lambda_m}z)(-C_s)dz.$$

Since the summation is over n, the integral and the sum are commutable:

$$\sum_{n=1}^{\infty}A_n\int_0^H\cos(\sqrt{\lambda_m}z)\cos(\sqrt{\lambda_n}z)dz=\int_0^H\cos(\sqrt{\lambda_m}z)(-C_s)dz.$$

Because of the special property of this method (Fourier trick), the integral is 0 if $n\neq m$, and it is not zero only if $n=m$. So, the summation will collapse to only one term:

$$A_n\int_0^H\cos(\sqrt{\lambda_n}z)\cos(\sqrt{\lambda_n}z)dz=\int_0^H\cos(\sqrt{\lambda_n}z)(-C_s)dz$$
$$\rightarrow A_n=-C_s\frac{\displaystyle\int_0^H\cos(\sqrt{\lambda_n}z)dz}{\displaystyle\int_0^H\cos^2(\sqrt{\lambda_n}z)dz}.$$

So we need to evaluate the integrals:

$$\int_0^H\cos(\sqrt{\lambda_n}z)dz=\frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}z)\Big|_0^H=\frac{1}{\sqrt{\lambda_n}}\sin\left(\frac{(2n-1)\pi}{2H}H\right)=\frac{1}{\sqrt{\lambda_n}}\sin\left(\frac{(2n-1)\pi}{2}\right)=-\frac{(-1)^n}{\sqrt{\lambda_n}},$$
$$\int_0^H\cos^2(\sqrt{\lambda_n}z)dz=\int_0^H\frac{1}{2}\left[\cos(2\sqrt{\lambda_n}z)+1\right]dz=\frac{1}{2}\left[\frac{1}{2\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n}z)+z\right]_0^H$$
$$=\frac{1}{2}\left\{\frac{1}{2\sqrt{\lambda_n}}\left[\sin\left(\frac{(2n-1)\pi}{2H}2H\right)-0\right]+(H-0)\right\}=\frac{1}{2}(0+H)=\frac{H}{2}.$$

Thus,

$$A_n=-C_s\frac{\displaystyle\int_0^H\cos(\sqrt{\lambda_n}z)dz}{\displaystyle\int_0^H\cos^2(\sqrt{\lambda_n}z)dz}=-C_s\frac{-\frac{(-1)^n}{\sqrt{\lambda_n}}}{\frac{H}{2}}=\frac{2C_s(-1)^n}{H\sqrt{\lambda_n}},$$

and,

$$\tilde{c}(z,t)=\sum_{n=1}^{\infty}A_n\varphi_n(z,t)=\sum_{n=1}^{\infty}\frac{2C_s(-1)^n}{H\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}z)\exp(-D\lambda_n t),\quad\lambda_n=\left(\frac{2n-1}{2H}\pi\right)^2$$

and,

$$C(z,t)=\bar{C}+\tilde{c}(z,t)=C_s+\sum_{n=1}^{\infty}\frac{2C_s(-1)^n}{H\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}z)\exp(-D\lambda_n t),\quad\lambda_n=\left(\frac{2n-1}{2H}\pi\right)^2.$$

This is the final time-dependent solution of this diffusion problem.

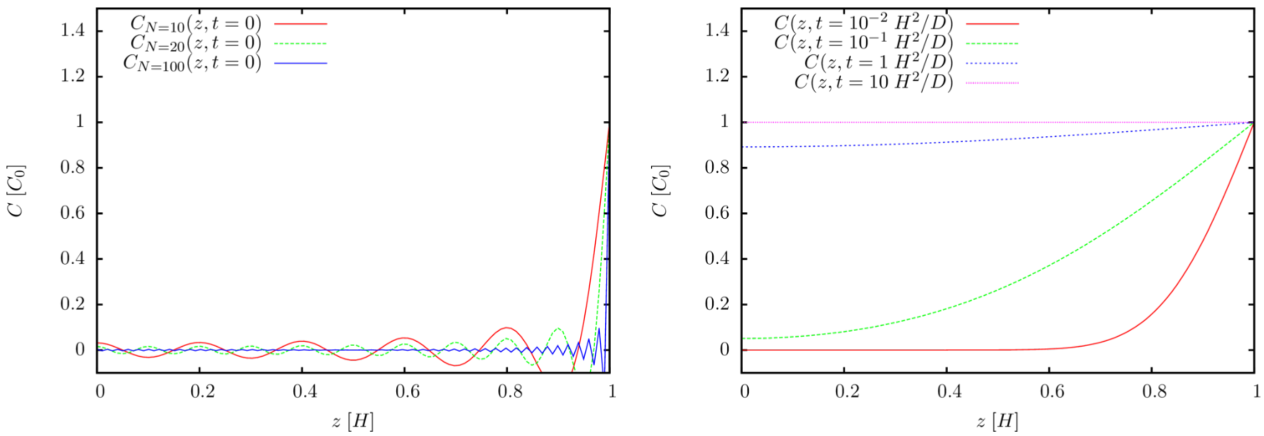

Figure 2: Concentration profile $C(z,t)$, at $t = 0$ and at later times, for a limited number N of terms φ_n (the infinite series is hard to plot). The initial condition cannot be well-represented by a small number of cosine-functions due to the unnatural jump in concentration at $z = H$ (Gibbs phenomenon), so it is not easy to see that the infinite series fulfills the initial and boundary conditions. At later times, the concentration is more smooth and can be properly represented by the series of cosine-functions: The solution fulfills the boundary conditions at all times, and approaches the steady-state solution for long times.

2 Field lines

Problem: Consider an unsteady planar flow field, $\mathbf{u} = (u, v)$, given by

$$u = x,$$
$$v = y \cdot (1 + 2t).$$

(a) Calculate an expression for the **streamline** passing through the point (x_0, y_0) at time t. Your equation should be of the form $y = f(x, x_0, y_0, t)$.

(b) Calculate an expression for the **pathline** for a fluid element initially located at the position (x_0, y_0) at time t_0. Your equation for the pathline should be of the form $y = f(x, x_0, y_0, t_0)$.

(c) Calculate the **streakline** equation at time t for the family of fluid elements that pass through the point (x_0, y_0). Your equation for the streakline should be of the form $y = f(x, x_0, y_0, t)$.

(d) For $(x_0, y_0) = (1, 1)$ and $t = 0$, plot the streamline, pathline and streakline in the interval $(x, y) \in [0 : 10] \times [0 : 10]$. Animate the changing velocity field for $t \in [0 : 1]$.

Solution: (a) An explicit description for the streamline can be derived via a separation of variables and integration from the initial position (x_0, y_0) to some final location (x, y):

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{v}{u} = (1+2t)\frac{y}{x} \\
\Leftrightarrow \quad \int_{y_0}^{y} \frac{1}{\tilde{y}}\,\mathrm{d}\tilde{y} &= (1+2t)\int_{x_0}^{x} \frac{1}{\tilde{x}}\,\mathrm{d}\tilde{x} \\
\Leftrightarrow \quad \ln\left(\frac{y}{y_0}\right) &= (1+2t)\ln\left(\frac{x}{x_0}\right) \\
\Leftrightarrow \quad y &= y_0\left(\frac{x}{x_0}\right)^{(1+2t)}.
\end{aligned}$$

(b) The pathline can be determined by separation of variables, separately for x and y:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= u = x \\ \Leftrightarrow \quad \int_{x_0}^{x} \frac{1}{\tilde{x}}\,\mathrm{d}\tilde{x} &= \int_{t_0}^{t} \mathrm{d}\tilde{t} \\ \Leftrightarrow \quad \ln\left(\frac{x}{x_0}\right) &= t - t_0 \\ \Leftrightarrow \quad t &= t_0 + \ln\left(\frac{x}{x_0}\right). \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= v = y \cdot (1+2t) \\ \Leftrightarrow \quad \int_{y_0}^{y} \frac{1}{\tilde{y}}\,\mathrm{d}\tilde{y} &= \int_{t_0}^{t} (1+2\tilde{t})\mathrm{d}\tilde{t} \\ \Leftrightarrow \quad \ln\left(\frac{y}{y_0}\right) &= (t+t^2) - (t_0 + t_0^2) \\ \Leftrightarrow \quad y &= y_0 \exp\left(t + t^2 - t_0 - t_0^2\right). \end{aligned}$$

Entering the equation for t into the relation for y yields

$$y = y_0 \exp\left((1+2t_0)\ln\left(\frac{x}{x_0}\right) + \left(\ln\left(\frac{x}{x_0}\right)\right)^2\right),$$

where t_0 is a time at which the particle passes the point (x_0, y_0).

(c) The streakline is determined like the pathline with one essential difference. Since we are searching for the locations of particles at time t that have been released at various initial times, t_0 is the variable integration limit (not t like in (b)). To find the streakline in the xy-plane, we use the x and y relations, obtained in (b), and eliminate t_0 between them. In other words, we should find t_0 in the relation for x, and enter it in the relation for y to find a relation for y in which x is the only variable:

$$\ln\left(\frac{x}{x_0}\right) = t - t_0 \Leftrightarrow t_0 = t - \ln\left(\frac{x}{x_0}\right), \quad y = y_0 \exp\left(t + t^2 - t_0 - t_0^2\right)$$

$$\Leftrightarrow \quad y = y_0 \exp\left((1+2t)\ln\left(\frac{x}{x_0}\right) - \left(\ln\left(\frac{x}{x_0}\right)\right)^2\right).$$

Comparing the solutions of (b) and (c) makes you realize that they only differ in one sign of the exponent. This essential difference becomes vilsible in the plot in Figure 3.

(d) The three field lines are plotted in Figure 3. Keep in mind that the velocity vector field is a snapshot from $t = 0$.

3 Acceleration in a trough

Problem: Water flows through the slit at the bottom of a two-dimensional water trough as shown in Figure 4. Throughout most of the trough the flow is approximately radial (along rays from O) with a velocity of $V = c/r$, where r is the radial coordinate and c is a constant. If the velocity is $0.4\,m/s$ when $r = 0.1\,m$, determine the acceleration at points A and B.

Solution: The problem has only radial dependence from the origin O and we seek solutions for the accelerations $\mathbf{a}(r = r_B)$ and $\mathbf{a}(r = r_A)$. If we choose to solve the problem in Cartesian coordinates (see Munson, Chapter 4.2.4, for Streamline coordinates), the acceleration is

$$\mathbf{a} = \frac{D\mathbf{V}(x(t), y(t), t)}{Dt},$$

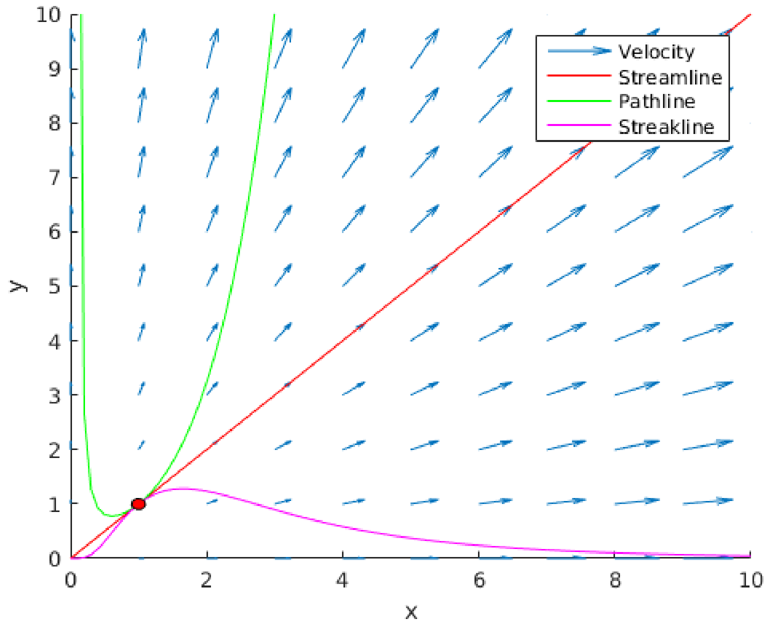

Figure 3: Different field lines through $(x_0, y_0) = (1, 1)$ over the velocity field given in problem 4.

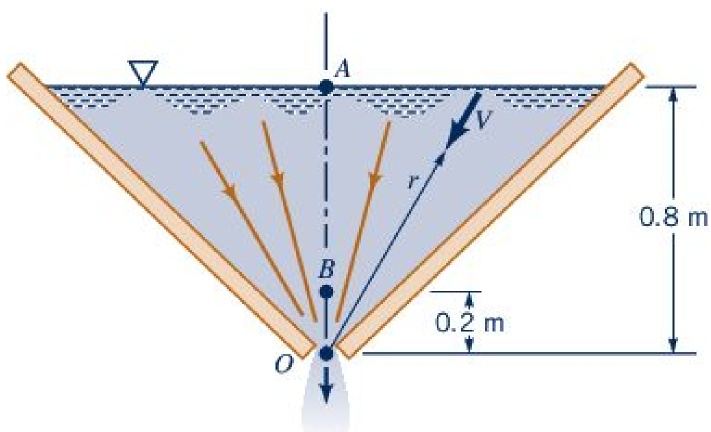

Figure 4: see problem 3

with x being horizontal and y being vertical. The points of interest, A and B, lie on the central line of symmetry where we can express the velocity as $\mathbf{V} = -V\hat{\mathbf{y}}$ which makes the material derivative

$$\frac{D\mathbf{V}}{Dt} = -\frac{D(V\hat{\mathbf{y}})}{Dt} = -\hat{\mathbf{y}}\frac{DV(y(t),t)}{Dt} - V\frac{d\hat{\mathbf{y}}}{dt},$$

with $d\hat{\mathbf{y}}/dt = 0$ because the unit vector of the coordinate system is time-independent (Note: Here it is obvious that the unit vector is time independent, but in general it can depend on time). The material derivative of the velocity field is

$$\frac{DV(y(t),t)}{Dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial y}\frac{dy}{dt} = -V\frac{\partial V}{\partial y},$$

because the flow is stationary ($\partial V/\partial t = 0$) and $dy/dt = -V$. The acceleration along $\hat{\mathbf{y}}$ is then simply

$$a = V\frac{\partial V}{\partial y} = V\frac{\partial V}{\partial r},$$

where we use the fact that the radial coordinate vectors $\hat{\mathbf{r}}$ are aligned with $\hat{\mathbf{y}}$. Since the velocity is given as $V = c/r$, we have

$$a = \frac{c}{r}\left(-\frac{c}{r^2}\right) = -\frac{c^2}{r^3}.$$

The constant c is defined by the given velocity at $r = 0.1\,m$: $c = Vr = \left(0.4\,ms^{-1}\right)(0.1\,m) = 4 \cdot 10^{-2}\,m^2 s^{-1}$. Therefore, the accelerations at points A and B are

$$a(r = r_A) = -\frac{\left(4 \cdot 10^{-2}\,\dfrac{m^2}{s}\right)^2}{(0.8\,m)^3} = -3.13 \cdot 10^{-3}\,\frac{m}{s^2},$$

$$a(r = r_B) = -\frac{\left(4 \cdot 10^{-2}\,\dfrac{m^2}{s}\right)^2}{(0.2\,m)^3} = -2.00 \cdot 10^{-1}\,\frac{m}{s^2}.$$

4 Oil film

Problem: A layer of oil flows down a vertical plate as shown in Figure 5 with a velocity of $\mathbf{V} = (V_0/h^2)(2hx - x^2)\hat{j}$ where V_0 and h are constants.

(a) Show that the fluid sticks to the plate and that the shear stress at the edge of the layer $x = h$ is zero.

(b) Determine the flow rate across the surface AB. Assume the width of the plate is b.

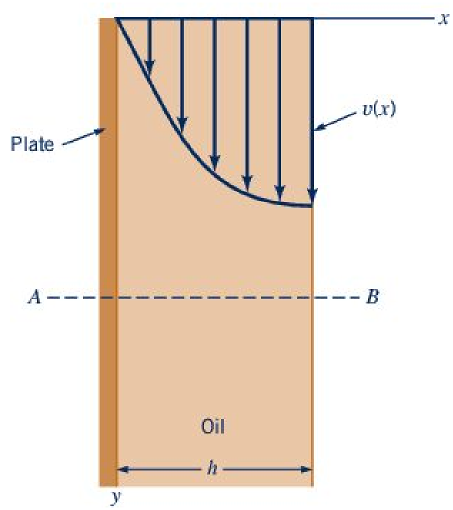

Figure 5: see problem 4

Solution: **(a)** Evaluate the velocity profile at the boundaries

$$v(x = 0) = \frac{V_0}{h^2}(0 - 0) = 0,$$

$$\tau(x = h) = \mu \frac{dv}{dx}(x = h) = \mu \frac{V_0}{h^2}\left[2h - 2x\right]_{x=h} = 0.$$

Hence, the fluid sticks to the plate (*no slip BC*) and there is no shear stress at the free surface (*free slip BC*).

(b) The flow rate is the integrated velocity across the plane AB with the normal vector n:

$$\begin{aligned} Q_{AB} &= \int \mathbf{v}(x)\,\mathbf{dA} = \int \mathbf{v}(x)\cdot\mathbf{n}\,dA = \int v(x)\,dA \\ &= \int_0^h v(x)b\,dx = \int_0^h \frac{V_0}{h^2}(2hx - x^2)b\,dx \\ &= \frac{V_0 b}{h^2}\left[hx^2 - \frac{1}{3}x^3\right]_0^h = \frac{2}{3}V_0 hb. \end{aligned}$$

Note: By revolving the velocity profile shown in Figure 5 around the line $x = h$, we obtain the velocity profile for the laminar flow in a pipe with radius h. Can you explain why $\tau = 0$ along the centerline of the pipe?