ME-280: Fluid Mechanics
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Problem Set 4: Diffusion and Fluid Kinematics

1 Surface reaction

A beaker is filled with a chemical liquid which starts to react strongly with oxygen at the surface (Figure 1). The reaction product diffuses from the surface into the liquid with a diffusion constant D. The concentration of the reaction product $C(z,t)$ depends only on height z and time t. At $t = 0$, the concentration in the liquid is zero: $C(z, t = 0) = 0$. At the surface, the reaction creates a constant concentration of $C(z = H, t) = C_s$. Solve the time-dependent problem of the vertical concentration profile $C(z,t)$ by calculating first **the steady state solution (a)** and second **the time-dependent solution (b)**. Follow the steps below:

(a1) Boundary conditions: What is the boundary condition at $z = 0$ (bottom)?

(a2) Steady state: Calculate the steady state solution $\bar{C}(z)$.

(b1) Homogeneous problem: Consider the decomposition of the concentration profile into the steady state solution $\bar{C}(z)$ and the time-dependent deviations $\tilde{c}(z,t)$ around the equilibrium:

$$C(z,t) = \bar{C}(z) + \tilde{c}(z,t).$$

Insert this decomposition into the diffusion equation. State the partial differential equation (PDE) of the deviation $\tilde{c}(z,t)$ and define the boundary conditions for $\tilde{c}$ at $z = 0$ and $z = H$. Is the PDE linear and homogeneous? Discuss the difference between the boundary conditions of $C(z,t)$ and $\tilde{c}(z,t)$.

(b2) General solution: You want to find the time-dependent deviations from the steady state with the general solution ansatz

$$\tilde{c}(z,t) = \sum_{n=1}^{\infty} A_n \varphi_n(z,t).$$

Calculate the base solutions $\varphi_n(z,t) = Z_n(z)T_n(t)$ using the method of separation of variables. State the general solution for $C(z,t)$. (Hint: Choose your solution ansatz such that it does not become imaginary! Use a series of sines and cosines for the spatial problem.)

(b3) Solution satisfying initial condition: Having found a general solution for the problem, remember that, so far, it only satisfies the boundary conditions of the concentration profile. Determine the set of coefficients A_n for general initial conditions of the deviation $\tilde{c}(z, t = 0) = \tilde{c}_0(z)$ by using the "Fourier-Trick". Insert the given initial conditions and evaluate the integrals to obtain the final expression for the time-dependent solution of the total concentration profile $C(z,t)$.

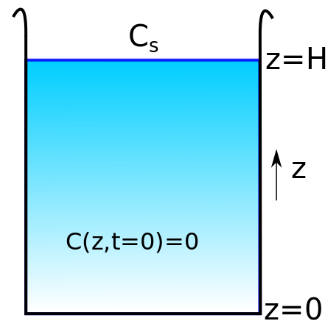

Figure 1: Beaker filled with a substance which reacts at the surface. See problem 1.

2 Field lines

Consider an unsteady planar flow field, $\mathbf{u} = (u, v)$, given by

$$
\begin{aligned}
u &= x, \\
v &= y \cdot (1 + 2t).
\end{aligned}
$$

(a) Calculate an expression for the **streamline** passing through the point (x_0, y_0) at time t. Your equation should be of the form $y = f(x, x_0, y_0, t)$.

(b) Calculate an expression for the **pathline** for a fluid element initially located at the position (x_0, y_0) at time t_0. Your equation for the pathline should be of the form $y = f(x, x_0, y_0, t_0)$.

(c) Calculate the **streakline** equation at time t for the family of fluid elements that pass through the point (x_0, y_0). Your equation for the streakline should be of the form $y = f(x, x_0, y_0, t)$.

(d) For $(x_0, y_0) = (1, 1)$ and $t = 0$, plot the streamline, pathline and streakline in the interval $(x, y) \in [0 : 10] \times [0 : 10]$. Animate the changing velocity field for $t \in [0 : 1]$.

3 Acceleration in a trough

Water flows through the slit at the bottom of a two-dimensional water trough as shown in Figure 2. Throughout most of the trough the flow is approximately radial (along rays from O) with a velocity of $V = c/r$, where r is the radial coordinate and c is a constant. If the velocity is $0.4\,m/s$ when $r = 0.1\,m$, determine the acceleration at points A and B.

4 Oil film

A layer of oil flows down a vertical plate as shown in Figure 3 with a velocity of $\mathbf{V} = (V_0/h^2)(2hx - x^2)\hat{\jmath}$ where V_0 and h are constants.

(a) Show that the fluid sticks to the plate and that the shear stress at the edge of the layer $x = h$ is zero.

(b) Determine the flow rate across the surface AB. Assume the width of the plate is b.

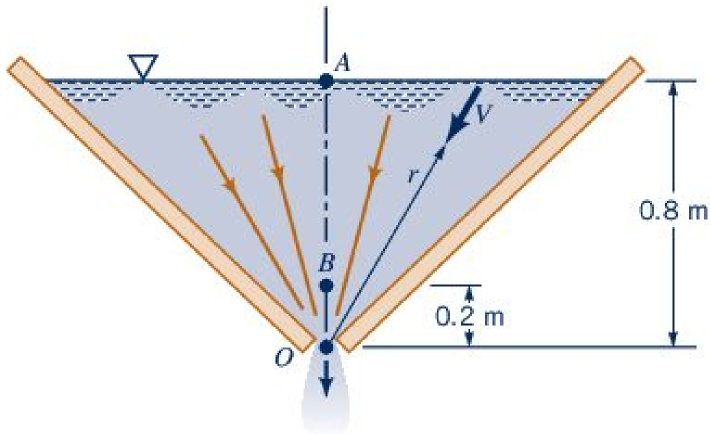

Figure 2: see problem 3

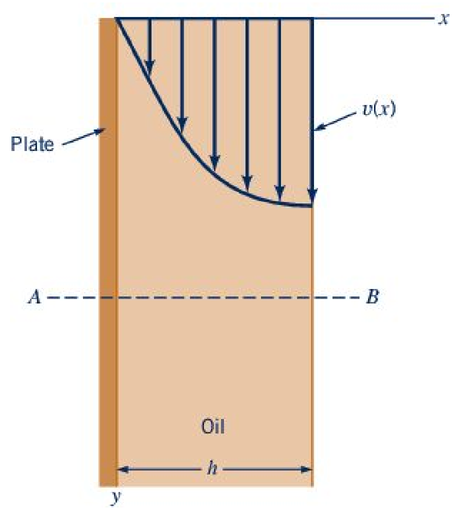

Figure 3: see problem 4