ME-280: Fluid Mechanics
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Problem Set 2: Solutions

1 Archimedes

Problem: The homogeneous timber AB in Figure 1 has a cross section of $0.15\,m$ by $0.35\,m$. Determine the specific weight of the timber and the tension in the rope.

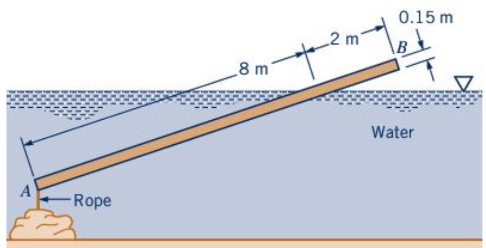

Figure 1: see problem 1

Solution:

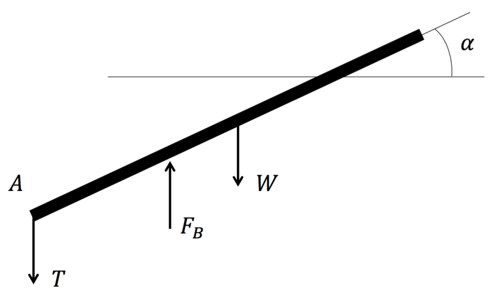

Figure 2: see problem 1

The timber has a length $L = 10\,m$ and volume

$$V = (0.15\,m \times 0.35\,m \times 10\,m) = 0.525\,m^3.$$

The submerged part has length $L_{sub} = 8\,m$ and volume

$$V_{sub} = (0.15\,m \times 0.35\,m \times 8\,m) = 0.420\,m^3.$$

The forces on the timber are the weight

$$W = \gamma V,$$

(where γ is the specific weight of the timber), the buoyancy force

$$F_B = \gamma_{H_2O} V_{sub},$$

(where $\gamma_{H_2O} = \rho_{H_2O} g = 9.80\ kN/m^3$ is the specific weight of water) and the tension force T.

The timber is at equilibrium and does not move. Consider first the torque balance of the timber. Since T is unknown, it is easiest to consider the torque around point A,

$$0 = \sum M_A = W \cos\alpha \times \frac{L}{2} - F_B \cos\alpha \times \frac{L_{sub}}{2},$$

where weight and buoyancy are projected onto the timber-normal direction and multiplied with the length of the lever. Canceling out the $\cos\alpha$-terms and using the expressions for W and F_B gives

$$\gamma = \gamma_{H_2O} \frac{V_{sub} L_{sub}}{VL} = 9.80\,\frac{kN}{m^3}\,\frac{(0.420\,m^3)(8\,m)}{(0.525\,m^3)(10\,m)} = 6.27\,\frac{kN}{m^3}.$$

Now that the specific weight of the timber is known, the tension force can be calculated from the force balance in the vertical direction, i.e. normal to the water surface,

$$\sum F_{vertical} = F_B - W - T = 0.$$

The tension in the rope is

$$T = F_B - W = \left(0.420\,m^3\right)\left(9.80\,\frac{kN}{m^3}\right) - \left(0.525\,m^3\right)\left(6.27\,\frac{kN}{m^3}\right) = 0.824\,kN = 824\,N.$$

2 Hydrostatic pressure

2.1 Blood pressure

Problem: (a) Determine the change in hydrostatic pressure in a giraffe's head as it lowers its head from eating leaves $6\,m$ above the ground to getting a drink of water at ground level (Figure 3). Assume that the absolute pressure in the giraffe's heart does not change, and the specific gravity of blood is $SG = 1$.

(b) Blood pressure is measured by two numbers in units of millimeters of mercury (mmHg): The first number, called *systolic* pressure, measures the blood pressure in the arteries when the heart beats; and the second number, called *diastolic* pressure, measures the pressure in the arteries when the heart rests between beats. What systolic blood pressure is considered high for our hearts? Express the pressure difference calculated in part (a) in units of mmHg, and compare it to a high blood pressure in human's heart.

(See also "Giraffe's blood pressure", Section 2.3.1. in Munson *et al.*)

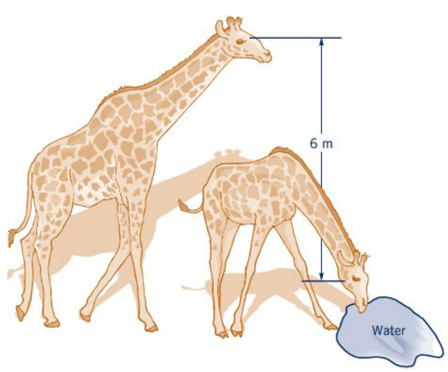

Figure 3: see problem 2.1

Solution: (a) Since we know the specific weight of water by heart ($\gamma_{H_2O} = 9.80\ kN/m^3$) and assume for blood $SG = 1$, the hydrostatic pressure change is simply given by the height difference

$$\Delta p = \gamma \Delta h = \left(9.80\,\frac{kN}{m^3}\right)(6\,m) = 58.8\,\frac{kN}{m^2} = 58.8\,kPa.$$

Note that we assumed both states of the giraffe, with the head up and the head down, belong to a single system, and then wrote the hydrostatic relation for this system between a point in the up head and a point in the down head. We are allowed to do this because the pressure in the giraffe's heart is assumed to be the same in both states of the animal, namely eating leaves and drinking water. As a result of this assumption, the giraffe's heart with constant pressure and a fixed height can be considered as a reference point in the system with respect to which the blood pressure can be determined in the head. Therefore, calculation of the pressure difference in the head between the two cases is really like calculation of the pressure difference between two heads in different positions in a single system.

(b) To compare with the pressure in a human heart, we convert the pressure from (a) to mmHg (millimeters of Mercury). Using the specific weight of mercury $\gamma_{\mathrm{Hg}} = 133\,kN/m^3$, the conversion of units is

$$58.8\,\frac{kN}{m^2} = \gamma_{\mathrm{Hg}} h_{\mathrm{Hg}} = \left(133\,\frac{kN}{m^3}\right) h_{\mathrm{Hg}},$$

giving an equivalent height of mercury column of

$$h_{\mathrm{Hg}} = 0.442\,m,$$

or pressure of $\Delta p = 442\,mm\mathrm{Hg}$.

Thus, the pressure change in the giraffe's head is $442\,mm$Hg. The normal blood pressure in human's heart ranges from $90/60\,mm$Hg to $120/80\,mm$Hg, and a blood pressure above $140/90\,mm$Hg is considered high. The $442\,mm$Hg change of blood pressure in giraffe's head is more than 3 times larger than a high systolic blood pressure in our arteries. How do giraffes' bodies stand such a high pressure?

2.2 Manometry

Problem: (a) A mercury manometer is connected to a large reservoir of water as shown in Figure 4. Determine the ratio, h_w/h_m, of the heights h_w and h_m indicated in the figure ($SG_m = 13.56$).

(b) Determine the elevation difference, Δh, between the water levels in the two open tanks shown in Figure 5.

(c) Why is the water not flowing from the left to the right tank? How can you change the setup to make the water flow?

Solution:

(a)

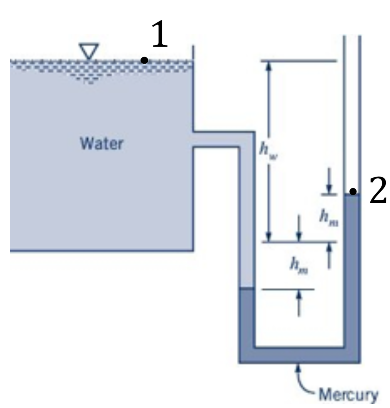

Figure 4: see problem 2.2a

The height ratio is calculated by summing the individual parts of the pressure head. Note the sign change when you change the direction along the tube. The pressure between point 1 and 2 (see Figure 4) relates as

$$P_1 + (h_w + h_m)\gamma_w - 2h_m\gamma_{\mathrm{Hg}} = P_2.$$

Since both fluid surfaces are at atmospheric pressure, we have $P_1 = P_2$ and thus

$$\frac{h_w + h_m}{h_m} = 2\frac{\gamma_{\mathrm{Hg}}}{\gamma_w} \rightarrow \frac{h_w}{h_m} + 1 = 2\frac{\gamma_{\mathrm{Hg}}}{\gamma_w}.$$

The specific gravity of mercury is given and we can substitute $\gamma_{\text{Hg}}/\gamma_w = SG_{\text{Hg}}$ to obtain the height ratio between water and mercury at atmospheric pressure

$$\frac{h_w}{h_m} = 2SG_{\text{Hg}} - 1 = 2 \times 13.56 - 1 = 26.12.$$

(b)

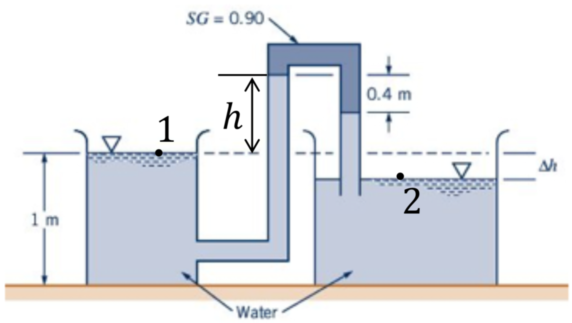

Figure 5: see problem 2.2b/c

Like in part (a) we sum up the individual elements of the pressure head (mind again the sign!)

$$P_1 - \gamma_{H_2O}h + (SG)\gamma_{H_2O}(0.4\,m) + \gamma_{H_2O}(h - 0.4\,m) + \gamma_{H_2O}(\Delta h) = P_2.$$

The variable height h is indicated in Figure 5. Again, both fluid surfaces are at atmospheric pressure, thus $P_1 = P_2$. With the given specific gravity of $SG = 0.9$ we find a height difference of

$$\Delta h = 0.4\,m - (0.9)(0.4\,m) = 0.040\,m.$$

(c) There is no movement of water because the pressure gradient along the connecting tube is compensated by the hydrostatic pressure difference caused by the other fluid. If this fluid is substituted with water, no equilibrium will be possible for different water levels in the two tanks, and the water will flow from the left tank to the right one until the free surfaces reach an equal level. This explains why you need to get the air out of a tube, in order to use the tube to move water from one bucket of water into another by exploiting only hydrostatic pressure.

2.3 Water dam

Problem: (a) The concrete dam in Figure 6 weighs $23.6\,kN/m^3$ and rests on a solid foundation. Determine the minimum coefficient of friction between the dam and the foundation required to keep the dam from sliding at the water depth shown. Assume no fluid uplift pressure along the base. Base your analysis on a unit length of the dam.
(b) Determine the horizontal hydrostatic force on the $2309\,m$ long *Three Gorges Dam* when the average depth of the water against it is $175\,m$. (See also "The Three Gorges Dam", Section 2.8. in Munson *et al.*)
(c) If all of the 6.4 billion people on Earth were to push horizontally against the *Three Gorges Dam*, could they generate enough force to hold it in place? Support your answer with appropriate calculations.
Solution: (a) In this problem we must consider two force balance equations. The first is a horizontal balance $\Sigma F_x = 0$ of the "pushing" force by the water load and the "resisting" friction force

$$F_R \sin\theta = F_f = \eta N,$$

where η is the friction coefficient. The second force balance is in the vertical direction, $\Sigma F_y = 0$, and defines the ground normal force N

$$N = W + F_R \cos\theta.$$

Here, $W = (\gamma_{concrete})V$ is the weight of the dam and $V = 20\,m^3$ is the volume of the analyzed dam segment. The angle θ is obtained from the geometry of the dam

$$\tan\theta = \frac{5\,m}{4\,m} \rightarrow \theta = 51.3°.$$

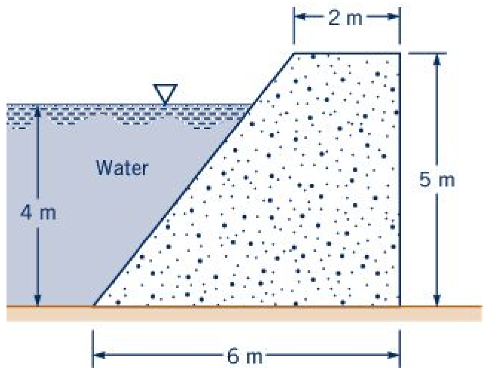

Figure 6: see problem 2.3

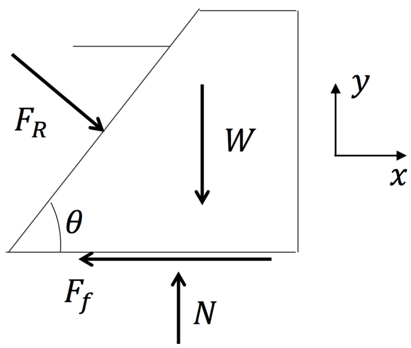

Figure 7: see problem 2.3

Both force balance equations depend on F_R which is the wall-normal pressure force due to the water behind the dam. It is

$$F_R = \gamma h_c A,$$

where $A = (4\,m\,/\sin 51.3°)\,(1m)$ is the area and $h_c = 4\,m/2$ is the height of the centroid of the submerged surface. This gives F_R as

$$F_R = \left(9.80\,\frac{kN}{m^3}\right)\left(\frac{4\,m}{2}\right)\left(\frac{4\,m}{\sin 51.3°}\right)(1\,m) = 100\,kN.$$

With this information we first solve for the vertical force N:

$$N = \left(23.6\,\frac{kN}{m^3}\right)(20\,m^3) + (100\,kN)\cos 51.3° = 534\,kN,$$

which can then be used in the horizontal force balance to calculate the friction coefficient η

$$\eta = \frac{F_R \sin\theta}{N} = \frac{(100\,kN)\sin 51.3°}{534\,kN} = 0.146.$$

(b) Having done part (a), this calculation is straightforward:

$$F_{R,horizontal} = \gamma h_c A_{projected} = \left(9.80\,\frac{kN}{m^3}\right)\left(\frac{175\,m}{2}\right)(2309\,m)(175\,m) \approx 3.5 \times 10^8\,kN.$$

(c) Each person should cover a force equal to

$$\frac{3.5 \times 10^8\,kN}{6.4 \times 10^9\,person} = 0.055\,\frac{kN}{person} = 55\,\frac{N}{person}.$$

$55\,N$ is equal to the weight of a $5.6\,kg$ object. Therefore, the world population can generate sufficient force to hold the dam in place.

3 Water supply

Problem: Water flows from the faucet on the first floor of the building shown in Figure 8 with a maximum velocity of $6\,m/s$. For steady inviscid flow, determine the maximum water velocity from the basement faucet and from the faucet on the second floor when all three faucets are open (assume each floor is $3.6\,m$ in height).

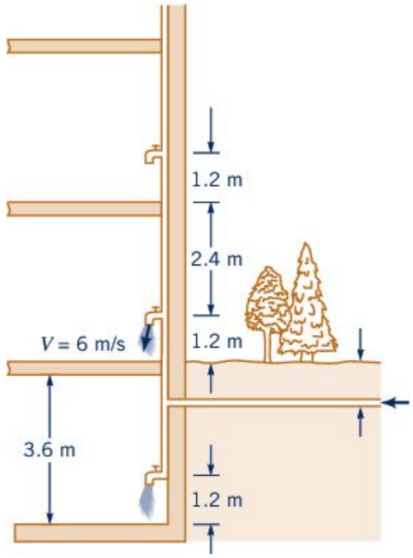

Figure 8: see problem 3

Solution: The problem assumes a steady and inviscid flow through pipes. The slow flow rate makes the assumption of incompressibility reasonable. If we trace a streamline along the pipes, we can use Bernoulli, which is here divided by the specific weight compared to the form given in the lecture:

$$\frac{p}{\gamma} + \frac{v^2}{2g} + z = constant,$$

where z is the elevation above ground. We compare the faucet velocities v_1, v_2, v_3 on the three floors with the velocity v_0 and pressure p_0 in the inlet pipe. A direct comparison between the faucets is not possible because there is no flow between them and thus, no streamline. All faucets are open and the inlet pipe feeds all three faucets with constant velocity and pressure. The inlet conditions are the same for the streamlines leading to each faucet. Setting up Bernoulli for each faucet gives

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 = \frac{p_0}{\gamma} + \frac{v_0^2}{2g} + z_0,$$
$$\frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2 = \frac{p_0}{\gamma} + \frac{v_0^2}{2g} + z_0,$$
$$\frac{p_3}{\gamma} + \frac{v_3^2}{2g} + z_3 = \frac{p_0}{\gamma} + \frac{v_0^2}{2g} + z_0,$$

with $z_1 = -2.4\,m$, $z_2 = 1.2\,m$, $z_3 = 4.8\,m$, $v_2 = 6\,m/s$, and $p_1 = p_2 = p_3 = 0$ (free jet). The values in the inlet pipe remain unknown, but since they are the same for each streamline, the velocities on the different floors can be related. Inserting the numerical values and rearranging the equation gives the unknown maximum velocity in the basement

$$\frac{(6\,m/s)^2}{2\,(9.81\,m/s^2)} + 1.2\,m = \frac{v_1^2}{2\,(9.81\,m/s^2)} + (-2.4\,m) \rightarrow v_1 = 10.3\,\frac{m}{s}.$$

For the 2nd floor, we again compare with the 1st floor (could also compare with the basement):

$$\frac{(6\,m/s)^2}{2\,(9.81\,m/s^2)} + 1.2\,m = \frac{v_3^2}{2\,(9.81\,m/s^2)} + 4.8\,m \rightarrow v_3 = \sqrt{6^2 - 2(9.81)(3.6)}\,\frac{m}{s} = \sqrt{-34.6}\,\frac{m}{s}.$$

The value under the square root is negative, meaning that no (real) velocity can be found that produces flow through the 2nd floor faucet. Notice that by writing the Bernoulli equation we implicitly assumed that a streamline exists between the inlet pipe and the 2nd floor faucet and thus water does reach the 2nd floor. The calculation showed that this assumption is not valid, and the inlet pressure is not high enough for the water to reach the 2nd floor. Now that water does not reach the 2nd floor, can we calculate where the free surface of the water column is?

4 Siphon

Problem: Water is siphoned from a large tank and discharges into the atmosphere through a $5.08\,cm$ diameter tube as shown in Figure 9. The end of the tube is $0.9\,m$ below the tank bottom, and viscous effects are negligible.
(a) Determine the volume flow rate from the tank.
(b) Determine the maximum height, H, over which the water can be siphoned without cavitation occurring. Atmospheric pressure is $101.3\,kPa$, and the water vapor pressure is $1.8\,kPa$.

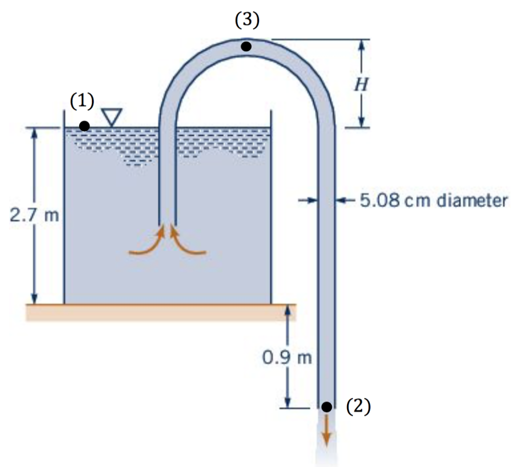

Figure 9: see problem 4

Solution: (a) To find the flowrate, we compare the two points of the setup which are both at atmospheric pressure and apply the Bernoulli equation. The two points are the surface of the tank (1) and the outlet of the siphon (2):

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2.$$

We choose $p_1 = p_2 = 0$ (free surface/jet) and also $v_1 = 0$. The second assumption is justified because the tank is very large compared to the amount of fluid in the siphon. Mass conservation implies that the height of the fluid surface, with a large area $A_1 \gg A_2$, changes with velocity $v_1 = (A_2/A_1)v_2 \ll v_2$. Thus, we can set $v_1 = 0$ which makes this problem quasi-steady because the fluid level remains approximately at height $2.7\,m$. Only steady problems can be solved with Bernoulli! The remaining terms become

$$z_1 = \frac{v_2^2}{2g} + z_2,$$

or

$$v_2 = \sqrt{2g(z_1 - z_2)} = \sqrt{2(9.81\,m/s^2)(2.7\,m + 0.9\,m)} = 8.4\,m/s.$$

Hence, the volume flow rate is

$$Q = A_2 v_2 = \frac{\pi}{4}(5.08\,cm)^2 \left(\frac{1\,m}{100\,cm}\right)^2 (8.4\,m/s) = 0.017\,m^3/s.$$

(b) Cavitation happens when the fluid pressure is below the vapour pressure of the fluid. Bubbles will form - the fluid is boiling. This effect is undesired in many engineering applications. In this example, the engineer must take care that the height H remains below a threshold which (s)he obtains again from the Bernoulli equation (Here, between point 2 and 3. Comparing 1 and 3 is also possible.)

$$\frac{p_3}{\gamma} + \frac{v_3^2}{2g} + z_3 = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2,$$

where $v_2 = v_3$ because mass conservation tells us $Q = A_2 v_2 = A_3 v_3$ and $A_2 = A_3$. Inserting the height difference $z_3 - z_2 = H + 3.6\,m$ (see Figure 9) and the specific weight of water $\gamma = 9.8\,kN/m^3$ into the remaining balance

$$p_3 + \gamma(z_3 - z_2) = p_2,$$

we get with the given pressure values $p_2 = 101.3\,kPa$ and $p_3 = 1.8\,kPa$:

$$\left(9.8\,\frac{kN}{m^3}\right)(H + 3.6\,m) = 101.3\,kPa - 1.8\,kPa,$$

or

$$H = 6.6\,m.$$

The height of the siphon must not exceed $H = 6.6\,m$. Note that this part of the problem uses **absolute pressure** (relative to perfect vacuum), while in part (a) we chose to use **gauge pressure** (zero-referenced against ambient air pressure). Make sure that you never mix the two definitions within one problem.

5 Soda bottle

Problem: Soda (with the same properties as water) flows from a $10.16\,cm$ diameter soda container that contains three holes as shown in Figure 10. The diameter of each fluid stream is $0.38\,cm$ and the distance between holes is $5.08\,cm$. If viscous effects are negligible and quasi-steady conditions are assumed, determine the time at which the soda stops draining from the top hole. Assume the soda surface is $5.08\,cm$ above the top hole when $t = 0$. (The final integral can be evaluated numerically.)

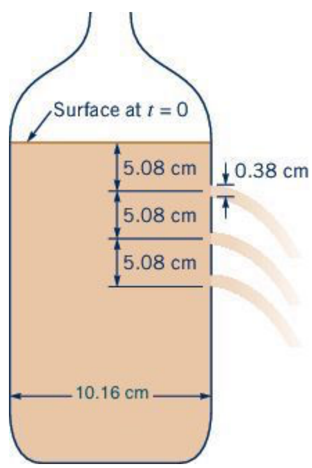

Figure 10: see problem 5

Solution: Write the Bernoulli equation for streamlines that connect the surface s with one of the holes i

$$\frac{p_s}{\rho} + \frac{v_s^2}{2} + gz_s = \frac{p_i}{\rho} + \frac{v_i^2}{2} + gz_i.$$

We assume free surface/jet and set $p_s = p_i = 0$. The surface at time $t = 0$ is chosen to be at $z_s = 0$. Now we make, like in exercise 4, a quasi-steady assumption which is $v_s \ll v_i$. This assumption is again crucial because it makes the left hand side of the above equation zero and allows us to obtain the velocities of the three jets $v_i = \sqrt{2gh_i}$ with $i = 1, 2, 3$ from the Bernoulli equation. These velocities can be used in the conservation of mass balance

$$Q = Q_1 + Q_2 + Q_3 = -A_s \frac{dh}{dt},$$

where we now allow the height $h(t)$ to vary in time. The volume flow rates are $Q_i = v_i A_i = \sqrt{2gh_i} A_i \quad (i = 1, 2, 3)$ with

$$A_1 = A_2 = A_3 = \frac{\pi}{4}(0.0038\, m)^2 = 1.13 \times 10^{-5}\, m^2,$$
$$A_s = \frac{\pi}{4}(0.1016\, m)^2 = 8.1 \times 10^{-3}\, m^2.$$

We simplify

$$\sqrt{2g} A_1 \left[\sqrt{h_1} + \sqrt{h_2} + \sqrt{h_3}\right] = -A_s \frac{dh}{dt},$$

where we now need to relate the heights h_i to the surface at $z_s = 0$. We set $h_1 = h$, $h_2 = h + L$, $h_3 = h + 2L$ with $L = 0.0508\, m$. We now have an ordinary differential equation (ODE) for $h(t)$ which can be solved by *separation of variables*:

$$-\frac{\sqrt{2g} A_1}{A_s} \int_0^T dt = \int_L^0 \frac{dh}{\left(\sqrt{h} + \sqrt{h+L} + \sqrt{h+2L}\right)},$$

where T is the time it takes for the free surface to reach the upper hole ($h = 0$):

$$\begin{aligned} T &= \frac{A_s}{\sqrt{2g} A_1} \int_0^L \frac{dh}{\left(\sqrt{h} + \sqrt{h+L} + \sqrt{h+2L}\right)} \\ &= \frac{8.1 \times 10^{-3}\, m^2}{\left[2(9.81\, m/s^2)\right]^{1/2} (1.13 \times 10^{-5}\, m^2)} \int_0^L \frac{dh}{(\sqrt{h} + \sqrt{h+L} + \sqrt{h+2L})}. \end{aligned}$$

We are left with

$$T = \left(162 \frac{s}{m^{1/2}}\right) \int_0^L \frac{dh}{\left(\sqrt{h} + \sqrt{h+L} + \sqrt{h+2L}\right)},$$

where $L = 0.0508\, m$. Note that with L in meters, this equation gives T in seconds. This integral should be evaluated numerically. For example you can ask www.wolframalpha.com what is
`162*Integrate[1/(sqrt[h]+sqrt[h+0.0508]+sqrt[h+2*0.0508]),{h,0,0.0508}]`

The numerical value of the integral is 0.06622 with physical unit of $m^{1/2}$. Therefore, the time T is

$$T = \left(162 \frac{s}{m^{1/2}}\right) \left(0.06622\, m^{1/2}\right) = 10.7\, s.$$

6 Normal to a Streamline

Problem: Water flows around the vertical two-dimensional bend with circular streamlines and constant velocity as shown in Figure 11. If the gauge pressure is $40\, kPa$ at point (1), determine the pressure at points (2) and (3). Assume that the velocity profile is uniform as indicated.
Solution: In this problem, we consider streamlines with varying radius of curvature. For the same assumptions as for the Bernoulli equation, we have an equation of motion along the normal direction of streamlines

$$-\gamma \frac{dz}{dn} - \frac{dp}{dn} = \frac{\rho v^2}{R},$$

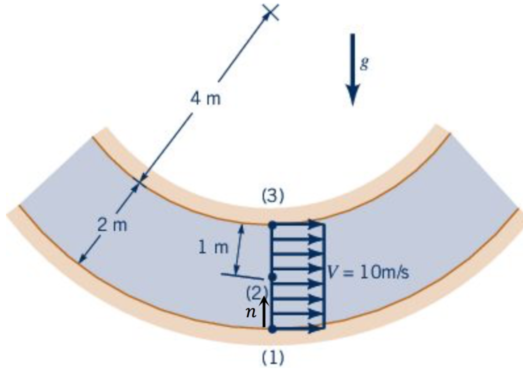

Figure 11: see problem 6

where the streamline normal direction $\vec{n}$ is indicated in Figure 11. At point (1) we find $\vec{n}$ aligned with the z-direction, i.e. $dz/dn = 1$. The velocity is constant $v = 10\,m/s$ and the radius is $R(n) = R_1 - n$, where $R_1 = 6\,m$ is the radius at (1). Rearranging the equation for the pressure change, we find

$$\frac{dp}{dn} = -\gamma - \frac{\rho v^2}{R_1 - n}.$$

To determine the pressure values, we need to integrate this equation along n from $n = 0$ to the point n^* we are interested in:

$$\int_{n=0}^{n^*} \frac{dp}{dn} dn = -\int_{n=0}^{n^*} \gamma dn - \int_{n=0}^{n^*} \frac{\rho v^2 dn}{R_1 - n}$$

Since the specific weight γ and the velocity v are constants, we can take them out of the integral and have

$$p - p_1 = -\gamma n^* - \rho v^2 \int_{n=0}^{n^*} \frac{dn}{R_1 - n}.$$

The integral has a closed-form solution, which gives

$$p = p_1 - \gamma n^* - \rho v^2 \ln\left(\frac{R_1}{R_1 - n^*}\right).$$

Now, with the properties of water $\gamma = 9.80\,kN/m^3$ and $\rho = 10^3\,kg/m^3$, we can insert the pressure at point (1), $p_1 = 40\,kPa$, to find the pressure at point (2) located at $n^* = n_2 = 1\,m$:

$$p_2 = 40\,kPa - \left(9.80\,\frac{kN}{m^3}\right)(1\,m) - \left(10^3\,\frac{kg}{m^3}\right)\left(10\,\frac{m}{s}\right)^2 \ln\left(\frac{6}{5}\right)\left(\frac{1\,kPa}{1000\,Pa}\right) = 12\,kPa.$$

The pressure at point (3) located at $n^* = n_3 = 2\,m$ is

$$p_3 = 40\,kPa - \left(9.80\,\frac{kN}{m^3}\right)(2\,m) - \left(10^3\,\frac{kg}{m^3}\right)\left(10\,\frac{m}{s}\right)^2 \ln\left(\frac{6}{4}\right)\left(\frac{1\,kPa}{1000\,Pa}\right) = -20.1kPa$$

Remember that pressure here will usually be given as gauge pressure (relative to atmospheric pressure), where negative values of a few kPa don't represent a problem.

7 Vaporizer

Problem: Air flows through the device shown in Figure 12. If the flow rate is large enough, the pressure within the constriction will be low enough to draw the water up into the tube. Determine the flow rate, Q, and the pressure needed at section (1) to draw the water into section (2). Neglect compressibility and viscous effects.

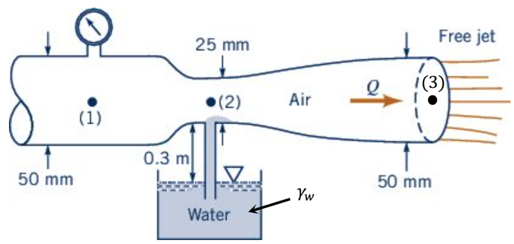

Figure 12: see problem 7

Solution: We want to know velocity and pressure at point (1). The velocity of the flow is unknown in the entire tube but we know the pressure at the outlet (point (3) in Figure 12) where we have a free jet with $p_3 = 0$. Therefore, we compare a steady flow along a streamline between point (2) and (3)

$$\frac{p_2}{\gamma_a} + \frac{v_2^2}{2g} + z_2 = \frac{p_3}{\gamma_a} + \frac{v_3^2}{2g} + z_3,$$

where we use the specific weight of air as $\gamma_a = 12\,N/m^3$. We find $z_2 = z_3$ and from mass conservation $A_2 v_2 = A_3 v_3$. Thus, the velocities relate through the ratio of squared diameters given in the problem:

$$v_2 = \left(\frac{D_3}{D_2}\right)^2 v_3 = \left(\frac{50\,mm}{25\,mm}\right)^2 v_3 = 4v_3.$$

Since $p_3 = 0$ we get a balance between p_2 and one of the velocities, e.g. v_3:

$$\frac{p_2}{\gamma_a} = \frac{v_3^2 - v_2^2}{2g} = \frac{-15v_3^2}{2g}.$$

The pressure p_2 defines the height of the water column in the vertical tube below point (2). From hydrostatics (see the Manometry exercises) we know

$$p_2 = p_0 - \gamma_w h = -\gamma_w h,$$

where $p_0 = 0$ is the atmospheric pressure, $\gamma_w = 9.80 \times 10^3\,N/m^3$ is the specific weight of water and $h = 0.3\,m$ is the required height for the water to enter the tube (see Figure 12). Inserting the hydrostatic pressure p_2 into the Bernoulli balance, we can solve for the velocity

$$\begin{aligned} v_3^2 &= \frac{2}{15} gh \frac{\gamma_w}{\gamma_a} \\ &= \frac{2}{15}\left(9.81\,\frac{m}{s^2}\right)(0.3\,m)\frac{9.80 \times 10^3\,\dfrac{N}{m^3}}{12\,\dfrac{N}{m^3}} \\ &= 320.46\,\frac{m^2}{s^2}. \end{aligned}$$

Thus, the outlet velocity is

$$v_3 = 17.9\,\frac{m}{s}.$$

The flow rate must be the same along the duct (from mass conservation). Therefore,

$$Q_1 = Q_3 = A_3 v_3 = \frac{\pi}{4}(0.050\,m)^2(17.9\,\frac{m}{s}) = 0.0351\,\frac{m^3}{s}.$$

To obtain the pressure at (1), we use again Bernoulli along a streamline

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 = \frac{p_3}{\gamma} + \frac{v_3^2}{2g} + z_3,$$

where $v_1 = (A_3/A_1)v_3 = v_3$ and $z_1 = z_3$ which implies $p_1 = p_3$, and consequently

$$p_1 = 0\,Pa;$$

pressure at (1) matches the atmospheric pressure at (3). This is, of course, possible under the assumption that the fluid is inviscid. (Why equal pressure at the inlet and outlet of the device could not be possible if the fluid had a non-negligible viscosity?)

8 Ball in a funnel

Problem: Observations show that it is not possible to blow the table tennis ball from the funnel shown in Figure 13a. In fact, the ball can be kept in an inverted funnel, Figure 13b, by blowing through it. The harder one blows through the funnel, the harder the ball is held within the funnel. Explain this phenomenon (no explicit calculation is required).

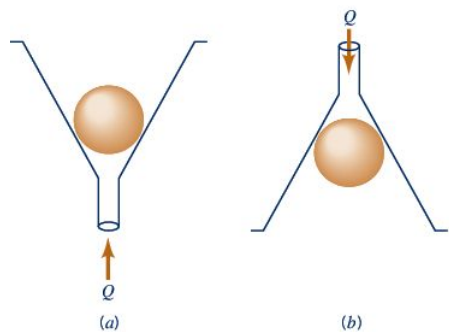

Figure 13: see problem 8

Solution: The ball is held in the funnel by the pressure difference between the gap and the surrounding atmosphere. In the Bernoulli equation along a streamline we can neglect the effect of gravity because of the little weight of air (which is a common assumption in gas flows). We, therefore, have

$$p + \frac{1}{2}\rho v^2 = constant.$$

Consequently, large v means small p. The flow field around the ball looks like the figure below:

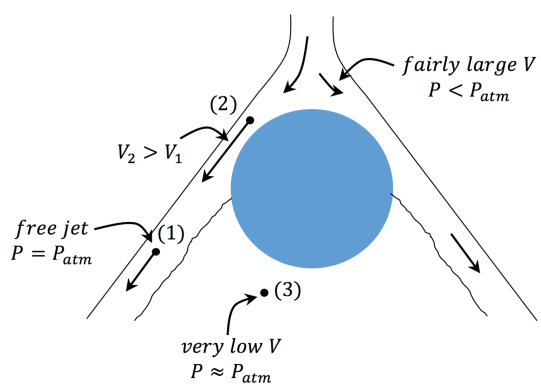

Figure 14: Streamlines past the ball in a funnel. See problem 8.

The net effect is a pressure force upward that can balance the weight of the ball. Note that we can apply the Bernoulli equation along streamlines between (1) and (2), but not across the streamlines from (1) to (3). The reason is the detachment of the flow from the ball which happens at large flow velocities.