Problem Set 1 : Solutions

1 Vector fields

1.1 Gradient and Laplacian

Problem:
Calculate the gradient and the Laplacian of the following scalar fields.

$$\begin{aligned} &(a) \quad f(x,y) = x^3 + y^2(3x-y) \\ &(b) \quad g(x,y,z) = 4x\sqrt{y} + 2y^2 z \\ &(c) \quad h(x,y,z) = \frac{y}{x} + \sin(y^2) - \ln\left(2y + e^{z^2}\right) \end{aligned}$$

Recall that for a scalar field, $f(x,y,z)$, gradient and Laplacian are defined as:

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

Solution:

(a) $f(x,y) = x^3 + y^2(3x-y)$

$$\begin{aligned} \nabla f(x,y) &= \left(3x^2 + 3y^2, 6xy - 3y^2, 0\right) \\ \nabla^2 f(x,y) &= 6x + (6x - 6y) + 0 = 12x - 6y \end{aligned}$$

(b) $g(x,y,z) = 4x\sqrt{y} + 2y^2 z$

$$\begin{aligned} \nabla g(x,y,z) &= \left(4\sqrt{y}, 2xy^{-1/2} + 4yz, 2y^2\right) \\ \nabla^2 g(x,y,z) &= 0 + \left(-xy^{-3/2} + 4z\right) + 0 = -xy^{-3/2} + 4z \end{aligned}$$

(c) $h(x,y,z) = \frac{y}{x} + \sin(y^2) - \ln\left(2y + e^{z^2}\right)$

$$\begin{aligned} \nabla h(x,y,z) &= \left(-yx^{-2}, \frac{1}{x} + 2y\cos(y^2) - \frac{2}{2y + e^{z^2}}, -\frac{2ze^{z^2}}{2y + e^{z^2}}\right) \\ \nabla^2 h(x,y,z) &= 2yx^{-3} + \left[2\cos(y^2) - 4y^2\sin(y^2) + \frac{4}{\left(2y + e^{z^2}\right)^2}\right] - 2\left[\frac{e^{z^2}(2z^2+1)\left(2y + e^{z^2}\right) - 2z^2 e^{2z^2}}{\left(2y + e^{z^2}\right)^2}\right] \\ &= 2yx^{-3} + 2\cos(y^2) - 4y^2\sin(y^2) - \frac{2e^{z^2}(2z^2+1)}{\left(2y + e^{z^2}\right)} + \frac{4\left(z^2 e^{2z^2} + 1\right)}{\left(2y + e^{z^2}\right)^2} \end{aligned}$$

1.2 Divergence and curl

Problem:
Calculate the divergence and the curl of the following vector fields.

$$
\begin{aligned}
&(a) \qquad \mathbf{V} = (x^2 + z)\hat{\mathbf{i}} + (yz - 2x)\hat{\mathbf{j}} + \left(\frac{xy^2}{z^2}\right)\hat{\mathbf{k}} \\
&(b) \qquad \mathbf{V} = \left(\frac{y^3 - z^2}{x}\right)\hat{\mathbf{i}} + \left(xz + \frac{1}{2}\right)\hat{\mathbf{j}} + (2xyz)\,\hat{\mathbf{k}} \\
&(c) \qquad \mathbf{V} = \left(e^{x^2 y}\right)\hat{\mathbf{i}} + \left[\ln\left(\cos\left(\frac{xz}{y}\right)\right)\right]\hat{\mathbf{j}}
\end{aligned}
$$

Recall that for a vector field, $\mathbf{V}(x, y, z)$, divergence and curl are defined as:

$$
\begin{aligned}
\nabla \cdot \mathbf{V} &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \\
\nabla \times \mathbf{V} &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)
\end{aligned}
$$

Solution:

$(a) \quad \mathbf{V} = (x^2 + z)\hat{\mathbf{i}} + (yz - 2x)\hat{\mathbf{j}} + \left(\frac{xy^2}{z^2}\right)\hat{\mathbf{k}}$

$$\nabla \cdot \mathbf{V} = 2x + z - 2xy^2 z^{-3}$$

$$\nabla \times \mathbf{V} = \left(2xyz^{-2} - y, 1 - \left(\frac{y}{z}\right)^2, -2\right)$$

$(b) \quad \mathbf{V} = \left(\frac{y^3 - z^2}{x}\right)\hat{\mathbf{i}} + \left(xz - \frac{1}{2}\right)\hat{\mathbf{j}} + (2xyz)\,\hat{\mathbf{k}}$

$$\nabla \cdot \mathbf{V} = -(y^3 - z^2)x^{-2} + 2xy$$

$$\nabla \times \mathbf{V} = \left(2xz - x, -\frac{2z}{x} - 2yz, z - \frac{3y^2}{x}\right)$$

$(c) \quad \mathbf{V} = \left(e^{x^2 y}\right)\hat{\mathbf{i}} + \left[\ln\left(\cos\left(\frac{xz}{y}\right)\right)\right]\hat{\mathbf{j}}$

$$\nabla \cdot \mathbf{V} = 2xye^{x^2 y} + \frac{xz}{y^2}\tan\left(\frac{xz}{y}\right)$$

$$\nabla \times \mathbf{V} = \left(\frac{x}{y}\tan\left(\frac{xz}{y}\right), 0, -\frac{z}{y}\tan\left(\frac{xz}{y}\right) - x^2 e^{x^2 y}\right)$$

1.3 Cylindrical coordinates

Problem:
Consider the scalar field $f(x, y, z) = x + 5zy^2$, which is given in Cartesian coordinates. Calculate the gradient of the scalar field in cylindrical coordinates by
(a) first doing the transformation $f(x, y, z) \to f(r, \phi, z)$ and then calculating $\mathbf{V}(r, \phi, z) = \nabla f(r, \phi, z)$,
(b) first calculating $\mathbf{V}(x, y, z) = \nabla f(x, y, z)$ and then doing the transformation $\mathbf{V}(x, y, z) \to \mathbf{V}(r, \phi, z)$.

Recall the coordinate transformation from cylindrical to Cartesian coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\,\cos(\phi) \\ r\,\sin(\phi) \\ z \end{pmatrix}_{(x,y,z)}$$

and the transformation matrix

$$(V_r, V_\phi, V_z) = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} .$$

The gradient in cylindrical coordinates is

$$\nabla f(r,\phi,z) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z} \right)_{(r,\phi,z)} .$$

Solution:
A field vector $\mathbf{V}$ at point P is represented by $\mathbf{V} = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}}$ in the standard Cartesian coordinates, and by $\mathbf{V} = V_r\hat{\mathbf{r}} + V_\phi\hat{\boldsymbol{\phi}} + V_z\hat{\mathbf{z}}$ in cylindrical coordinates. Figure 1 shows the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{z}}$, and allows us to verify how the given transformation matrix maps the components matrix (V_x, V_y, V_z) to (V_r, V_ϕ, V_z).

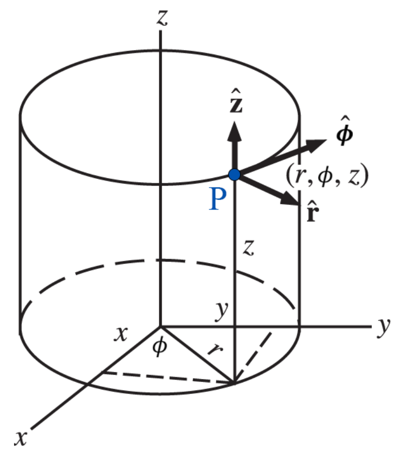

Figure 1: see problem 1.3

(a) The substitution of the coordinate transformation $(x = r\cos\phi, y = r\sin\phi, z = z)$ into $f(x,y,z)$ gives

$$\begin{aligned} f(r,\phi,z) &= r\cos\phi + 5zr^2\sin^2\phi \\ \mathbf{V}(r,\phi,z) = \nabla f(r,\phi,z) &= \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \phi}, \frac{\partial}{\partial z} \right) (r\cos(\phi) + 5zr^2\sin^2(\phi)) \\ &= \begin{pmatrix} \cos(\phi) + 10zr\sin^2(\phi) \\ -\sin(\phi) + 10zr\sin(\phi)\cos(\phi) \\ 5r^2\sin^2(\phi) \end{pmatrix} \end{aligned}$$

(b) The gradient of $f(x,y,z)$ in Cartesian coordinates gives a vector field which is then multiplied with a rotation matrix:

$$\begin{aligned} \mathbf{V}(x,y,z) = \nabla f(x,y,z) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)(x + 5zy^2) = (1, 10zy, 5y^2) \\ \mathbf{V}(r,\phi,z) &= \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 10zy \\ 5y^2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi) + 10zy\sin(\phi) \\ -\sin(\phi) + 10zy\cos(\phi) \\ 5y^2 \end{pmatrix} = \begin{pmatrix} \cos(\phi) + 10zr\sin^2(\phi) \\ -\sin(\phi) + 10zr\sin(\phi)\cos(\phi) \\ 5r^2\sin^2(\phi) \end{pmatrix} \end{aligned}$$

The substitution of the coordinate transformation is the final step in this approach.

1.4 Potential

Problem:
A vector field $\mathbf{V} = (V_x, V_y, V_z)$ is given by

$$V_x = 4z^3 + 6xy^4 + 3x^2y^2z \ ,$$
$$V_y = 12x^2y^3 + 2x^3yz \ ,$$
$$V_z = 12xz^2 + x^3y^2 + 8z.$$

Integrate the components of the vector field to find the potential function from which it is derived. (Find f such that $\nabla f = \mathbf{V}$.)
Solution:
We must integrate each of the components of the vector field in order to obtain the potential function, since we know that:

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \nabla f(x,y,z) = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

First, we will integrate V_x, with respect to x.

$$f(x,y,z) = \int V_x dx = \int (4z^3 + 6xy^4 + 3x^2y^2z)dx = 4xz^3 + 3x^2y^4 + x^3y^2z + h(y,z)$$

Here, $h(y,z)$ is some function of y and z that we will be evaluated by differentiating $f(x,y,z)$ with respect to y and z and then comparing with V_y and V_z, respectively.

$$\frac{\partial f(x,y,z)}{\partial y} = 12x^2y^3 + 2x^3yz + \frac{\partial h(y,z)}{\partial y} = V_y \quad \text{(which is given)}$$
$$12x^2y^3 + 2x^3yz + \frac{\partial h(y,z)}{\partial y} = 12x^2y^3 + 2x^3yz$$

After simplifying, we find that:

$$\frac{\partial h(y,z)}{\partial y} = 0$$

This means that $h(y,z) = g(z)$, where $g(z)$ is an arbitrary function of z only. Therefore, we have

$$f(x,y,z) = 4xz^3 + 3x^2y^4 + x^3y^2z + g(z)$$

Going back and repeating the same procedure for z, we find that:

$$\frac{\partial f(x,y,z)}{\partial z} = 12xz^2 + x^3y^2 + g'(z) = V_z \quad \text{(which is given)}$$
$$12xz^2 + x^3y^2 + g'(z) = 12xz^2 + x^3y^2 + 8z$$

Again, after simplifying, we find that:

$$g'(z) = 8z \to g(z) = 4z^2 + C$$

Here, C is an arbitrary integration constant. We can now write out the final potential function:

$$f(x,y,z) = 4xz^3 + 3x^2y^4 + x^3y^2z + 4z^2 + C$$

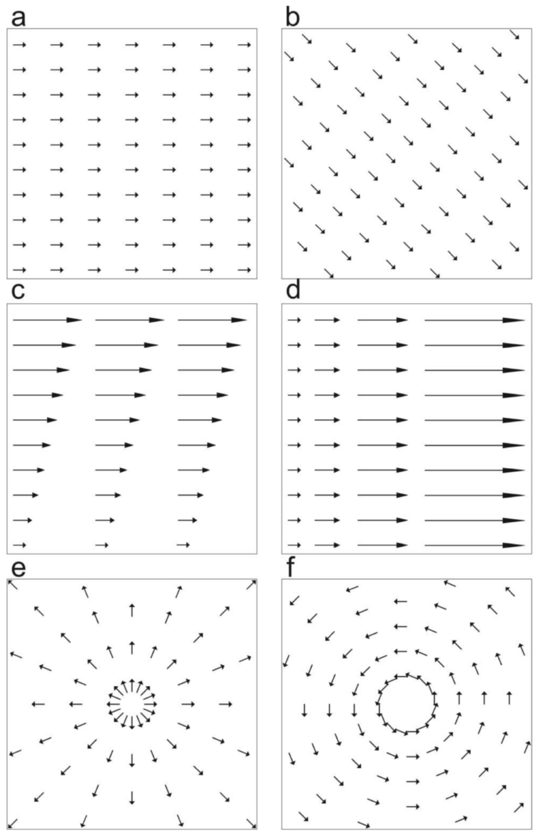

Figure 2: see problem 1.5

1.5 2D vector fields

Problem:
For each of the vector fields in Figure 2, **use physical intuition** to determine if their divergence and curl are zero or nonzero. The length of the arrows represents the magnitude of the field vectors. Arrows are the same size for all figures except for c and d.
Solution:

Vector Field	Divergence	Curl
a	0	0
b	0	0
c	0	Nonzero
d	Nonzero	0
e	Nonzero	0
f	0	Nonzero

From http://betterexplained.com/articles/divergence/:

> Basically, divergence has to do with how a vector field changes its magnitude in the neighborhood of a point. Another way to better understand divergence is to consider that the vector fields represent the flow of a material. The divergence is the "flux density" (i.e. the amount of material that flows through a given area per unit time), of a material entering or leaving a point. If you were to draw a control volume in the vector field (it is helpful to select a control volume with boundaries that are parallel or perpendicular to the direction of the flow), the divergence of the field would be equal to the NET flux of material through the control volume. If what flows into the control volume is equal to what flows out of the control volume, then the divergence of the vector field is zero; otherwise, the divergence of the vector field is nonzero.

From http://betterexplained.com/articles/vector-calculus-understanding-circulation-and-curl/:

Additionally, curl has to do with how a vector field changes its direction in the neighborhood of a point. Again, to gain better intuition for what a curl is, consider that the vector fields represent the flow of a material. Circulation is the amount of "pushing" that the flowing material exerts along a path. The curl is the amount of pushing, twisting, or turning force that the flowing material exerts when you shrink the path down to a single point. Let's use water as an example. Suppose we have a vector field that represents the flow of water, and we want to determine if it has curl or not: is there any twisting or pushing force? To test this, we put a paddle wheel into the water and notice if it turns (the center axis of the paddle is vertical, sticking out of the water (i.e. normal to the plane of the vector field like a revolving door). If the paddle does turn, it means the vector field has curl at that point. If it doesn't turn, then the curl of the vector field is zero at that point. What does it really mean if the paddle turns? Well, it means the water is pushing harder on one side than the other, making it twist. The larger the difference, the more forceful the twist and the bigger the curl. Also, a turning paddle wheel indicates that the field is "uneven" and not symmetric; if the field were even, then it would push on all sides equally and the paddle would not turn at all.

Therefore, we can explain our answers as follows:

- In **a** and **b**, the vector fields have zero divergence and zero curl since they don't change in either magnitude or direction in the neighborhood of any point.
- In **c**, if you draw a rectangular control volume (C.V.), you will see that the horizontal flux of material is the same going into and out of the C.V.; thus, the divergence is zero. If you were to insert a small paddle wheel into the "flow" of the vector field, it would rotate clockwise due to the strength of the vector field being uneven in the y-direction.
- In **d**, if you draw a rectangular control volume, you will see that the horizontal flux of material is larger when it exits the C.V. from the right side, relative to that when it enters from the left side; thus, the divergence is nonzero. If you were to insert a small paddle wheel into the "flow" of the vector field, it would not rotate because the strength of the vector field is symmetric about the direction in which the vector field points.
- In **e**, if you draw a circular control volume centered at the origin, you will see that the vector field only flows out of the C.V.; thus, the divergence is nonzero. If you were to insert a small paddle wheel into the "flow" of the vector field such that its axis of rotation is on the origin, the wheel would not rotate because the strength of the vector field is radially symmetric about the origin.
- Lastly, in **f**, if you draw a circular control volume centered at the origin, you will see that the vector field does not flow through the C.V. since it remains parallel to the boundary; thus, the divergence is zero. If you were to insert a small paddle wheel into the "flow" of the vector field such that its axis of rotation is on the origin, the wheel will rotate (in the counter-clockwise direction) because the flow circulates about the origin.

2 Taylor expansion

Problem:
Calculate the Taylor series for the following functions, up to fourth order, around the given point:

$$(a) \quad f(x) = \cos(3x) \qquad \text{around } x = 0 \ ,$$

$$(b) \quad g(x) = \left(x^2 + \frac{1}{x}\right) e^x \qquad \text{around } x = 1 \ .$$

Solution:
The fourth order Taylor series of $f(x)$ around $x = c$ is given by:

$$p(x;c) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f''''(c)}{4!}(x-c)^4$$

$(a) \quad f(x) = \cos(3x) \quad$ around $x = 0$

$$\cos(3x)\Big|_{x=0} \approx 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4$$

$(b) \quad g(x) = \left(x^2 + \frac{1}{x}\right)e^x \quad$ around $x = 1$

$$\left(x^2 + \frac{1}{x}\right)e^x\bigg|_{x=1} \approx 2e + 3e(x-1) + 4e(x-1)^2 + \frac{11}{6}e(x-1)^3 + \frac{5}{4}e(x-1)^4$$

3 Shear stress

Problem:
A 10 kg block slides down a smooth inclined surface (Figure 3). Determine the terminal velocity of the block if the 0.1 mm gap between the block and the surface contains SAE 30 oil at 15°C ($\mu = 0.38\ Ns/m^2$). Assume the velocity distribution in the gap to be linear, and the area of the block in contact with the oil to be 0.1 m^2.

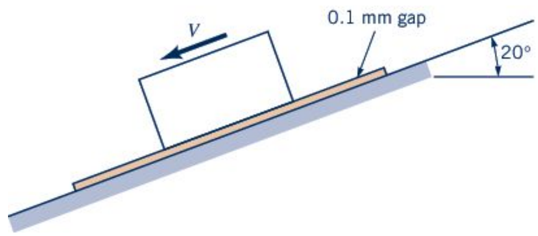

Figure 3: see problem 3

Solution:
Consider the force balance in downhill direction (x-direction). Forces on the block are the x-component of the weight W_x, and the shear force F_τ (see Figure 4). At terminal velocity, the block's acceleration is zero, and the force balance is

$$\sum F_x = W_x + F_\tau = 0.$$

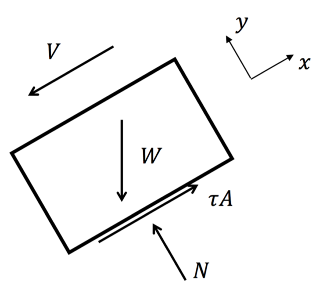

Figure 4: see problem 3

For W_x, we get

$$W_x = -\sin(20°)W,$$

and the force due to the shear stress τ of the oil film is

$$F_\tau = \tau A.$$

Since the oil velocity at the bottom is zero (*no slip-condition*), the stress-strain rate relation is $\tau = \mu\,(V/b)$, where b is film thickness and V is the speed of the block, yielding a force balance

$$\sin(20°)W = \mu \frac{V}{b} A$$

Thus, the terminal velocity is (with $W = mg = 10\ kg \cdot 9.81\ m/s^2$)

$$V = \frac{bW \sin 20°}{\mu A} = \frac{(0.0001m)\,(10kg)\left(9.81\ \frac{m}{s^2}\right)(\sin 20°)}{\left(0.38\ \frac{Ns}{m^2}\right)(0.1m^2)} = 0.0883\ \frac{m}{s}$$