Released: 21.2.2025
Exercise Class: 26.2.2025
Solutions: 26.2.2025

Problem Set 1: Mathematical Reprise

1 Vector fields

1.1 Gradient and Laplacian

Calculate the gradient and the Laplacian of the following scalar fields.

$$
\begin{aligned}
&(a) \qquad f(x,y) = x^3 + y^2(3x - y)\\
&(b) \qquad g(x,y,z) = 4x\sqrt{y} + 2y^2 z\\
&(c) \qquad h(x,y,z) = \frac{y}{x} + \sin(y^2) - \ln\left(2y + e^{z^2}\right)
\end{aligned}
$$

Recall that for a scalar field, $f(x,y,z)$, gradient and Laplacian are defined as:

$$
\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}
$$

1.2 Divergence and curl

Calculate the divergence and the curl of the following vector fields.

$$
\begin{aligned}
&(a) \qquad \mathbf{V} = (x^2 + z)\hat{\mathbf{i}} + (yz - 2x)\hat{\mathbf{j}} + \left(\frac{xy^2}{z^2}\right)\hat{\mathbf{k}}\\
&(b) \qquad \mathbf{V} = \left(\frac{y^3 - z^2}{x}\right)\hat{\mathbf{i}} + \left(xz + \frac{1}{2}\right)\hat{\mathbf{j}} + (2xyz)\,\hat{\mathbf{k}}\\
&(c) \qquad \mathbf{V} = \left(e^{x^2 y}\right)\hat{\mathbf{i}} + \left[\ln\left(\cos\left(\frac{xz}{y}\right)\right)\right]\hat{\mathbf{j}}
\end{aligned}
$$

Recall that for a vector field, $\mathbf{V}(x,y,z)$, divergence and curl are defined as:

$$
\begin{aligned}
\nabla \cdot \mathbf{V} &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\\
\nabla \times \mathbf{V} &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)
\end{aligned}
$$

1.3 Cylindrical coordinates

Consider the scalar field $f(x,y,z) = x + 5zy^2$, which is given in Cartesian coordinates. Calculate the gradient of the scalar field in cylindrical coordinates by
(a) first doing the transformation $f(x,y,z) \rightarrow f(r,\phi,z)$ and then calculating $\mathbf{V}(r,\phi,z) = \nabla f(r,\phi,z)$,
(b) first calculating $\mathbf{V}(x,y,z) = \nabla f(x,y,z)$ and then doing the transformation $\mathbf{V}(x,y,z) \rightarrow \mathbf{V}(r,\phi,z)$.

Recall the coordinate transformation from cylindrical to Cartesian coordinates

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\,\cos(\phi) \\ r\,\sin(\phi) \\ z \end{pmatrix}_{(x,y,z)}
$$

and the transformation matrix

$$(V_r, V_\phi, V_z) = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} .$$

The gradient in cylindrical coordinates is

$$\nabla f(r, \phi, z) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z} \right)_{(r,\phi,z)} .$$

1.4 Potential

A vector field $\mathbf{V} = (V_x, V_y, V_z)$ is given by

$$V_x = 4z^3 + 6xy^4 + 3x^2y^2z ,$$
$$V_y = 12x^2y^3 + 2x^3yz ,$$
$$V_z = 12xz^2 + x^3y^2 + 8z.$$

Integrate the components of the vector field to find the potential function from which it is derived. (Find f such that $\nabla f = \mathbf{V}$.)

1.5 2D vector fields

For each of the vector fields in Figure 1, **use physical intuition** to determine if their divergence and curl are zero or nonzero. The length of the arrows represents the magnitude of the field vectors. Arrows are the same size for all figures except for c and d.

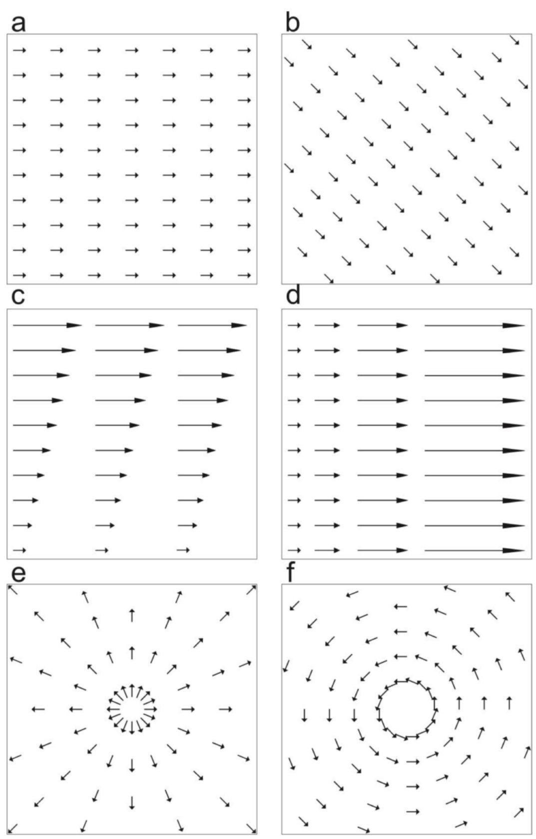

Figure 1: see problem 1.5

2 Taylor expansion

Calculate the Taylor series for the following functions, up to fourth order, around the given point:

$$(a) \quad f(x) = \cos(3x) \qquad \text{around } x = 0 \ ,$$

$$(b) \quad g(x) = \left(x^2 + \frac{1}{x}\right) e^x \qquad \text{around } x = 1 \ .$$

3 Shear stress

A 10 kg block slides down a smooth inclined surface (Figure 2). Determine the terminal velocity of the block if the 0.1 mm gap between the block and the surface contains SAE 30 oil at $15°C$ ($\mu = 0.38 \ Ns/m^2$). Assume the velocity distribution in the gap to be linear, and the area of the block in contact with the oil to be $0.1 \, m^2$.

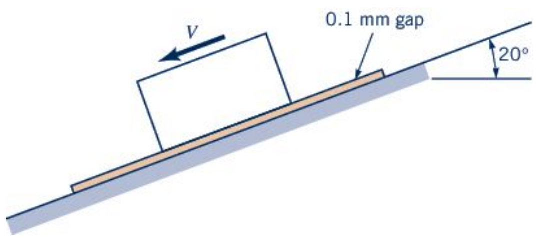

Figure 2: see problem 3