D. 3.1.2: Recipe: 1D time-dependent diffusion (cartesian)

Solve $\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$ for $0 \leq x \leq L$

with homogeneous BC and given IC $c_0(x)$

step 1: general solution of PDE + BC

1.1 Find base solutions via separation of variables

$$\varphi_n(x,t) = X(x)\,T(t)$$

PDE → 2 ODEs, linked by eigenvalues λ_n

1.1.1: Solve $X''(x) + \lambda X(x) = 0$

From BC determine

- λ_n 'eigenvalues' (nontrivial solutions X_n)
- $X_n(x)$ 'eigenmodes' (typically sines and/or cosines)

1.1.2: Solve $T'(t) + D\lambda\, T(t) = 0$

$$T_n(t) = e^{-D\lambda_n t}$$

$$\Rightarrow \varphi_n(x,t) = X_n(x)\, T_n(t)$$

general solution:

$$c(x,t) = \sum_{n=1}^{\infty} A_n \varphi_n(x,t)$$

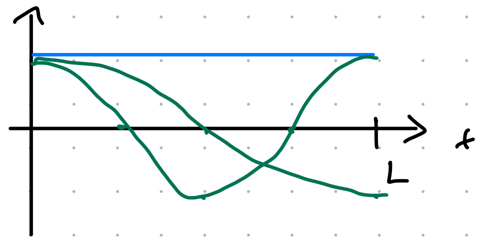
step 2: Determine $\{A_n\}$ via Fourier-'trick':

$$\int_0^L c_0(x) X_m(x)\,dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L X_n(x) X_m(x)\,dx}_{=0\,;\; n \neq m}$$

$$\Rightarrow A_n = \frac{\int_0^L c_0(x) X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

D.3.1.3 other homogeneous BC

$\cdot$ no-flux / no-flux



BC: $\left.\frac{\partial c}{\partial x}\right|_{x=0} = \left.\frac{\partial c}{\partial x}\right|_{x=L} = 0$

- in solution of $X''(x) + \lambda X(x) = 0$

  $\lambda < 0$: only trivial solution

  $\lambda = 0$: $X(x) = \alpha_1 = \text{const}$

  $\lambda > 0$: $X(x) = \alpha_1 \cos(\sqrt{\lambda}\, x) + \alpha_2 \sin(\sqrt{\lambda}\, x)$

  BC: $\alpha_2 = 0$

  $0 = -\alpha_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, L)$; $\alpha_1 = 0$ $\Rightarrow \sin(\sqrt{\lambda}\, L) = 0$

  $\Rightarrow \lambda_n = \left(\frac{\pi n}{L}\right)^2$; $n = 1, 2, 3, \ldots$

$\Rightarrow X_n(x) = \cos\left(\frac{n\pi}{L} x\right)$; $n = 1, 2, 3, \ldots$

$X_0(x) = 1$; $\lambda_0 = 0$

$$\Rightarrow c(x,t) = B_0 \, 1 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{L} x\right)$$

- determine

$$B_n = \frac{\int_0^L c_0(x) \cos\left(\frac{n\pi}{2} x\right)}{\int_0^L \cos^2\left(\frac{n\pi}{2} x\right) dx} = \frac{2}{L} \int_0^L c_0(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

for $n > 0$

$$B_0 = \frac{1}{L} \int_0^L c_0(x)\, dx \quad \text{(mean concentration)}$$

$$\Rightarrow c(x,t) = \frac{1}{L} \int_0^L c_0(x)\, dx$$

$$+ \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L c_0(\tilde{x}) \cos\left(\frac{n\pi}{2} \tilde{x}\right) d\tilde{x} \right] e^{-D\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L} x\right)$$