## D.3.2 Diffusion on an infinite domain (Green's fct. approach)

PDE: $$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \quad ; \quad -\infty < x < \infty$$

BC: $$c \to 0 \quad ; \quad x \to +\infty$$
$$c \to 0 \quad ; \quad x \to -\infty$$

IC: $$c(x, t=0) = c_0(x)$$

Note: sine / cosine expansion approach does <u>not</u> work!

<u>Idea:</u> Try to find ('guess') a solution of PDE + BC

- Material is conserved (no sinks / sources)

$$B := \int_{-\infty}^{\infty} c(x,t)\, dx = \text{const.}$$

- Dimensional analysis (later):

  $\sqrt{Dt}$ is a length (diffusion length scale)

  seek similarity solution as a function of similarity variable $\eta = \frac{x}{\sqrt{Dt}}$

$\Rightarrow$ Form: $$c(x,t) = \frac{B}{\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right)$$

PDE $\to$ ODE for $f(\eta)$

$$\Rightarrow c(x,t) = \frac{B}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

One solution of PDE + BC ( B = 1 )

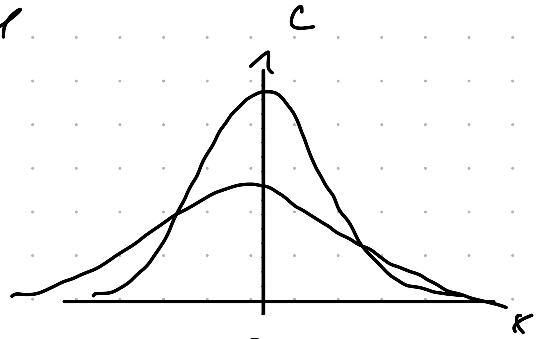
$$C(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

- solves diffusion equation ( ⇝ plug in ✓ )
- satisfies BC

$$x \to \pm\infty : \quad C(x,t) \longrightarrow 0$$

or: $$\lim_{x \to \pm\infty} C(x,t) = 0$$

- $$\int_{-\infty}^{\infty} C(x,t)\, dx = 1$$ normalized




Q: What is the IC ?

$$C(x, t=0) = \lim_{t \to 0} \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

$$= \delta(x)$$

- width $w \sim \sqrt{2Dt}$ diffusive spreading
- peak concentration $\sim \frac{1}{\sqrt{Dt}}$

Dirac delta 'function' / distribution

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \; ; \; \int_{-\infty}^{\infty} \delta(x) g(x)\, dx = g(0) \text{ for all } g(x)$$

Initial condition is a point distribution at $x = 0$

- 'Shift' this solution

$$c(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-\tilde{x})^2}{4Dt}} =: \mathcal{G}(x,t;\tilde{x})$$

is a solution with $\lim\limits_{t\to 0} \mathcal{G}(x,t;\tilde{x}) = \delta(x-\tilde{x})$

point distribution at $x = \tilde{x}$

$\mathcal{G}(x,t;\tilde{x})$ is the '<u>Green's function</u>'

- as before: Use linearity to construct the <u>general solution</u> of PDE + BC

$$c(x,t) = \int_{-\infty}^{\infty} A(\tilde{x})\, \mathcal{G}(x,t;\tilde{x})\, d\tilde{x}$$

- Determine $A(\tilde{x})$ such that $c(x,t=0) = c_0(x)$

$$\lim_{t\to 0} c(x,t) = \int_{-\infty}^{\infty} A(\tilde{x}) \underbrace{\mathcal{G}(x,t\to 0;\tilde{x})}_{=\delta(x-\tilde{x})} d\tilde{x}$$

$$= A(x) \overset{!}{=} c_0(x)$$

$\Rightarrow$ solution satisfying IC:

$$\boxed{c(x,t) = \int_{-\infty}^{\infty} \frac{c_0(x)}{\sqrt{4\pi D t}} e^{-\frac{(x-\tilde{x})^2}{4Dt}} d\tilde{x}}$$